Lėvy-Schrödinger semigroups

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Two classes of stochastic jump-type processes are considered:

- driven by Langevin equation with Lévy noise
- driven by Lévy semigroups

Issues addressed:

- differences in dynamical behavior
- common asymptotic stationary probability densities
- confinement (pdf has a finite number of moments)
- hyper-confinement (all moments in existence)

Special topic

- logarithmic potentials and heavy-tailed invariant pdfs of diffusion-type processes

Inspiration: Targeted stochasticity idea of I. Eliazar and J. Klafter, J. Stat. Phys. **111**, 739, (2003)

Lévy-Driven Langevin Systems: Targeted Stochasticity

$$X(dt) = \underbrace{-f(X(t))\,dt}_{\text{Drift}} + \underbrace{L(dt)}_{\text{Driver}}$$

1. **Evolution:** *What is the Fokker–Planck equation governing the evolution of the pdf of the system's state?*

2. **Steady state:** *In steady state, what is the connection between the system's drift function* f*, driving noise, and stationary pdf?*

3. **Reverse engineering:** *Given a "target" pdf* p*, can we "tailor design" a drift function* f *so that the system's stationary pdf would equal the desired "target" pdf* p*?*

Question: Do we have a guarantee that an invariant density may actually be an asymptotic target ?

Getting started: Brownian motion inspirations

$$\dot{x} = b(x,t) + A(t)$$

$$\langle A(s) \rangle = 0 \qquad \langle A(s)A(s') \rangle = 2D\delta(s - s')$$

$$\partial_t \rho = D\triangle\rho - \nabla\,(b \cdot \rho)\,.$$

Smoluchowski diffusion processes

$$b = \frac{f}{m\beta} = -\frac{1}{m\beta}\nabla V$$

stationary asymptotic regime

$$b = \boxed{b_* = u_* = D\nabla \ln \rho_*\,.}$$

Stationary pdf

$$\rho_*(x) = \exp\left([F_* - V(x)]/k_B T\right) \doteq \exp[2\Phi(x)]$$

$$\rho_*^{1/2} = \exp\Phi \text{ and } b = 2D\nabla\Phi$$

Becoming parabolic - no difference in the ultimate dynamics and asymptotics of the inferred pdf !

$$\rho(x,t) \doteq \theta_*(x,t)\exp[\Phi(x)]\,.$$

$$\partial_t\theta_* = D\Delta\theta_* - \mathcal{V}\theta_*$$

$$\partial_t\theta = -D\Delta\theta + \mathcal{V}\theta$$

$$\rho(x,t) \doteq \theta(x,t)\theta_*(x,t) = \int p(y,s,x,t)\rho(y,s)dy$$

$$\theta = \theta(x) = \exp\Phi(x)$$

$$\mathcal{V}(x) = \frac{1}{2}\left(\frac{b^2}{2D} + \nabla b\right) \quad = D\,\frac{\Delta\rho_*^{1/2}}{\rho_*^{1/2}}\,.$$

$$\theta \sim \rho_*^{1/2}$$

Schrödinger semigroups

$$\theta_*(t) = [\exp(-t\hat{H})\theta_*](0)$$

$$\hat{H} = -D\Delta + \mathcal{V}$$

Note: suitable restrictons upon the semigroup potential need to be respected, to have a positive and continuous semigroup kernel function

$$k(y,s,x,t) = \left(\exp[-(t-s)\hat{H}]\right)(y,x) \quad = \int exp[-\int_s^t \mathcal{V}(X(u),u)du]\, d\mu[s,y \mid t,x]$$

$$\rho(x,t) \doteq \int p(y,s,x,t)\rho(y,s)dy$$

$$k(y,s,x,t) = p(y,s,x,t)\frac{\rho_*^{1/2}(y)}{\rho_*^{1/2}(x)} = p(y,s,x,t)\ \exp[\Phi(y) - \Phi(x)]$$

If $\rho_*(x)$ has the Gibbs form then $\Phi(y) - \Phi(x) = (1/2k_BT)[V(x) - V(y)]$

Ornstein-Uhlenbeck example

$$\hat{H} = -D\Delta + \mathcal{V}$$

$$\mathcal{V}(x) = \frac{\gamma^2 x^2}{4D} - \frac{\gamma}{2} \qquad \hat{H}\rho_*^{1/2} = 0.$$

$$\partial_t \rho = D\Delta\rho - \nabla(b \cdot \rho) \qquad b(x) = -\gamma x,\ \gamma \equiv \kappa/m\beta,\ \kappa > 0.$$

$$\rho_*(x) = \left(\frac{\gamma}{2\pi D}\right)^{1/2} \exp\left(-\frac{\gamma}{2D}x^2\right) = \exp\left(\frac{F_* - V(x)}{k_B T}\right)$$

$$V(x) = \kappa\frac{x^2}{2}$$

Targeted stochasticity in the Gaussian case: given $\rho_*(x)$ we trivially reconstruct (i) semigroup dynamics (ii) Langevin-type dynamics

$$\mathcal{V} = D\frac{\Delta\rho_*^{1/2}}{\rho_*^{1/2}} \qquad b_* = D\nabla \ln \rho_*$$

No difference in the dynamics (i) or (ii) of $\rho_0(x)$ into $\rho_*(x)$

Lévy flights

$$E[\exp(ipX_t)] = \exp[-tF(p)]$$

Lévy-Schrödinger semigroups

$p \to \hat{p} = -i\nabla$

$\hat{H} = F(\hat{p})$

$$[exp(-t\hat{H})f](x) = [exp(-tF(p))\tilde{f}(p)]^{\vee}(x)$$

~ indicates Fourier transform, $\vee$ its inverse

Stable noise and its generator

$$F(p) = \lambda|p|^{\mu} \Rightarrow \hat{H} \doteq \lambda|\Delta|^{\mu/2}$$

$\partial_t \rho = +D\Delta\rho$

$$\partial_t \rho = -\lambda|\Delta|^{\mu/2}\rho$$

Cauchy noise

$$F(p) = \lambda|p| \to \hat{H} = F(\hat{p}) = \lambda|\nabla| \doteq \lambda(-\Delta)^{1/2}$$

First input: Schrödinger's boundary data problem (1932)

Deduce the Markovian interpolation consistent with a given pair of boundary measure data at fixed initial and terminal time instants $t_1 < t_2$; A and B are two Borel sets in R.

$$m(A, B) = \int_A dx \int_B dy\, m(x, y),$$
$$\int_R m(x, y)dy = \rho(x, t_1),$$
$$\int_R m(x, y)dx = \rho(y, t_2),$$

where

$$m(x, y) = f(x)k(x, t_1, y, t_2)g(y)$$

$f(x)$ and $g(y)$ are of the same sign and nonzero, $k(x, s, y, t)$ is an a priori chosen, bounded strictly positive and continuous (dynamical semigroup) kernel, $t_1 \leq s < t \leq t_2$.

Prescribing $k(x,s,y,t)$ **in advance**, we have functions $f(x), g(y)$ determined uniquely (up to constant factors) ba marginal data, c.f. Beurling, Fortet, Jamison.

By denoting

$$\theta_*(x,t) = \int f(z)k(t_1,z,x,t)dz$$

$$\theta(x,t) = \int k(x,t,z,t_2)g(z)dz$$

it follows that

$$\rho(x,t) = \theta(x,t)\theta_*(x,t) = \int p(y,s,x,t)\rho(y,s)dy,$$

$$p(y,s,x,t) = k(y,s,x,t)\frac{\theta(x,t)}{\theta(y,s)},$$

$$t_1 \le s < t \le t_2$$

Second input: elementary harmonic/functional analysis

Let us consider self-adjoint operators (Hamiltonians) with dense domains in $L^2(R)$, of the form $\hat{H} = F(\hat{p})$, where $\hat{p} = -i\nabla$ and for $-\infty < k < +\infty$, $F = F(k)$ is a real valued, bounded from below, locally integrable function. For $t \geq 0$ we have:

$$exp(-t\hat{H}) = \int_{-\infty}^{+\infty} exp[-tF(k)]dE(k)$$

$dE(k)$ is the spectral measure of $\hat{p}$.

Let us set

$$k_t = \frac{1}{\sqrt{2\pi}}[exp(-tF(p)]^{\vee}$$

then the action of $exp(-t\hat{H})$ can be given in terms of a convolution:

$$exp(-t\hat{H})f = f * k_t$$

where

$$(f * g)(x) := \int_R g(x-z)f(z)dz$$

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If $F(p)$ satisfies the Lévy-Khintchine formula, then k_t is a positive measure for all $t \geq 0$ and we arrive at the simplest (free noise) positivity preserving semigroups.

The integral part of the L-K formula is responsible for random jumps:

$$F(p) = -\int_{-\infty}^{+\infty} [exp(ipy) - 1 - \frac{ipy}{1+y^2}]\nu(dy)$$

$\nu(dy)$ stands for the Lévy measure

Third input: (pseudo) relativistic Hamiltonians

$$F_0(p) = |p|$$

$$F_m(p) = \sqrt{p^2 + m^2} - m\,,\ , m > 0$$

(better known as $H_{cl} = \sqrt{m^2c^4 + c^2p^2} - mc^2$)

Set $\theta(x,t) \equiv 1$, $\theta_*(x,t) \doteq \rho(x,t)$ and thus

$$[exp(-t\hat{H})\rho](x) = \rho(x,t)$$

where $F(p \to -i\nabla) := \hat{H}$ implies

$$F_0(p) \Longrightarrow \partial_t \rho(x,t) = -|\nabla|\rho(x,t)$$

$$F_m(p) \Longrightarrow \partial_t \rho(x,t) = -[\sqrt{-\Delta + m^2} - m]\rho(x,t)$$

(Note: $\partial_t \rho = \Delta \rho$, with $\hat{H} = -\Delta$ derives from the Wiener process)

Fourth input: spectral properties of $\hat{H}$ vs "free noise"

Cauchy semigroup generator: $\hat{H} = |\nabla|$

$$\rho(x,t) = \frac{1}{\pi}\frac{t}{t^2+x^2} \Longrightarrow k^0(y,s,x,t) = \frac{1}{\pi}\frac{t-s}{(t-s)^2+(x-y)^2}]$$

$$0 < s < t$$

$$\langle \exp[ipX(t)] \rangle := \int_R \exp(ipx)\overline{\rho}(x,t)\, dx = \exp[-tF_0(p)] = \exp(-|p|t)$$

Lévy measure needed to evaluate the Lévy-Khintchine integral reads:

$$\nu_0(dy) := lim_{t\downarrow 0}[\frac{1}{t}k^0(0,0,y,t)]dy = \frac{dy}{\pi y^2}$$

Relativistic generator $\sqrt{-\Delta + m^2} - m$, formulas determining the stochastic jump process are much less appealing:

$$\langle \exp[ipX(t)]\rangle := exp[-tF_m(p)] = exp[-t(\sqrt{p^2+m^2} - m)]$$

$$\overline{\rho}(x,t) = \frac{m}{\pi}\frac{t\exp(mt)}{\sqrt{x^2+t^2}}K_1(m\sqrt{x^2+t^2})$$

$$[\exp(-(t-s)F_m(-i\nabla))](x-y) = k^m(y,s,x,t)$$

$$\nu_m(dy) = \frac{m}{\pi|y|}K_1(m|y|)dy$$

$K_1(z)$ is the modified Bessel function of third kind, of order 1.

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$\partial_t \rho = +D\Delta\rho$

$$\partial_t \rho = -\lambda |\Delta|^{\mu/2} \rho$$

Cauchy noise

$$F(p) = \lambda |p| \to \hat{H} = F(\hat{p}) = \lambda |\nabla| \doteq \lambda(-\Delta)^{1/2}$$

Response to external potentials

Langevin scenario

$$\dot{x} = b(x) + A^{\mu}(t) \Longrightarrow \partial_t \rho = -\nabla(b \cdot \rho) - \lambda |\Delta|^{\mu/2} \rho$$

Lévy-Schrödinger semigroups

$$\hat{H}_{\mu} \doteq \lambda |\Delta|^{\mu/2} + \mathcal{V}$$

$$\exp(-t\hat{H}_{\mu})$$

$$\partial_t \rho = -\nabla(b \cdot \rho) - \lambda |\Delta|^{\mu/2} \rho$$

$$b = 2D\nabla\Phi$$

$$b(x) = -\lambda \frac{\int |\Delta|^{\mu/2} \rho_*(x)\, dx}{\rho_*(x)}$$

Targeted stochasticity

Schrödinger's boundary data problem

$$\partial_t \theta_* = -\lambda |\Delta|^{\mu/2} \theta_* - \mathcal{V} \theta_* .$$

$$\partial_t \theta = \lambda |\Delta|^{\mu/2} \theta + \mathcal{V} \theta$$

$$\theta^*(x,t)\theta(x,t) = \rho(x,t)$$

$$\theta_*(x,t) = \rho(x,t) \exp[-\Phi(x)]$$

$$\theta(x) = \exp[\Phi(x)] = \rho_*^{1/2}(x)$$

$$\mathcal{V} = -\lambda \frac{|\Delta|^{\mu/2} \rho_*^{1/2}}{\rho_*^{1/2}}$$

$$\partial_t \rho = \theta \partial_t \theta^* = -\lambda (\exp \Phi) |\Delta|^{\mu/2} [\exp(-\Phi)\rho] - \mathcal{V} \cdot \rho$$

Cauchy driver

$$(|\nabla| f)(x) = -\frac{1}{\pi} \int \frac{f(z) - f(x)}{|z - x|^2} dz$$

Ornstein-Uhlenbeck-Cauchy process

$$\partial_t \rho = -\lambda |\nabla| \rho + \nabla[(\gamma x)\rho]$$

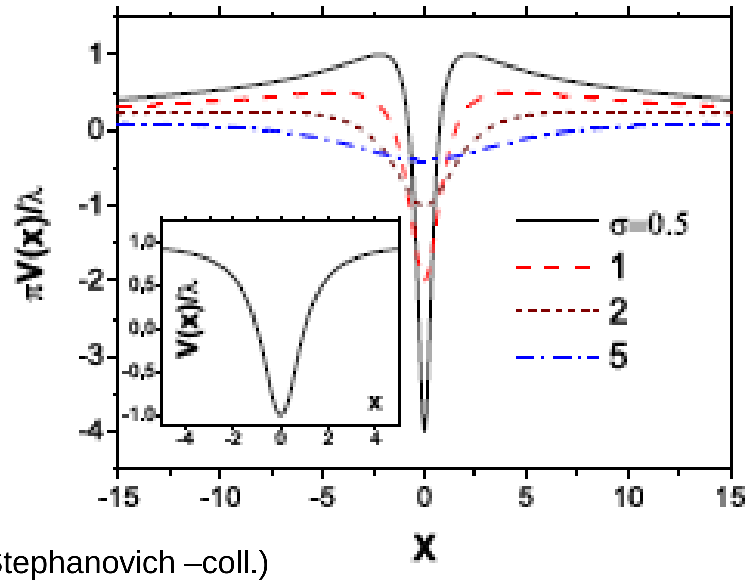
$$\rho_*(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)}$$

Invariant density vs semigroup potential

$$\mathcal{V} = -\lambda \frac{|\Delta|^{\mu/2} \rho_*^{1/2}}{\rho_*^{1/2}}$$

$$\mathcal{V}(x) = \frac{\lambda}{\pi} \left[-\frac{2}{\sqrt{a}} + \frac{x}{a} \ln \frac{\sqrt{a} + x}{\sqrt{a} - x} \right]$$

$$a = \sigma^2 + x^2$$



(V. Stephanovich –coll.)

Confined Cauchy noise

Invariant density

$$\rho_*(x) = \frac{2}{\pi} \frac{1}{(1+x^2)^2}$$

Langevin scenario

$$\partial_t \rho_* = 0 = -\nabla(b\,\rho_*) - \gamma|\nabla|\rho_*$$

Hint: targeted stochasticity

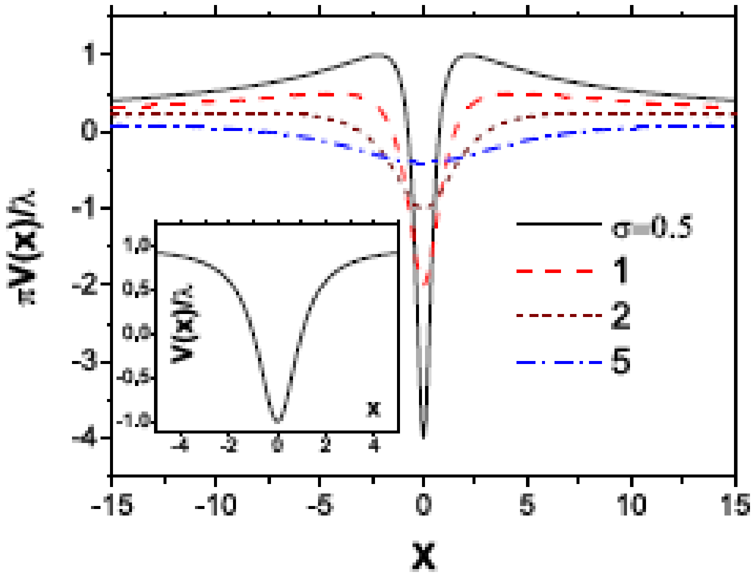
$$b(x) = -\gamma\,(x^2 + 3x)$$

semigroup route

$$\hat{H}_\mu \doteq \lambda|\Delta|^{\mu/2} + \mathcal{V}$$

$$|\Delta|^{1/2} \doteq |\nabla|$$

$$\mathcal{V}(x) = \lambda \frac{x^2 - 1}{x^2 + 1}$$



(V. Stephanovich –coll.)

Confinement hierarchy

$$\rho_*(x) = \frac{\Gamma(\alpha)}{\sqrt{\pi}\Gamma(\alpha - 1/2))} \frac{1}{(1+x^2)^\alpha} \qquad \alpha > 1/2.$$

Let us consider

$$\rho_*(x) = \frac{16}{5\pi} \frac{1}{(1+x^2)^4}$$

Dynamical semigroup reconstruction

$$\mathcal{V} = -\lambda \frac{|\Delta|^{\mu/2} \rho_*^{1/2}}{\rho_*^{1/2}}$$

$$\mathcal{V}(x) = \frac{\gamma}{2(1+x^2)} (x^4 + 6x^2 - 3)$$

Langevin drift reconstruction

Hint: targeted stochasticity

$$b(x) = -\frac{\gamma}{\rho_*(x)} \int (|\nabla|\rho_*)(x)\, dx$$

$$b(x) = -\frac{\gamma x}{16} (5x^6 + 21x^4 + 35x^2 + 35)$$

$$\partial_t \rho = -\nabla(b \cdot \rho) - \lambda |\Delta|^{\mu/2} \rho$$

$$\partial_t \rho_* = 0 = -\nabla(b\, \rho_*) - \gamma |\nabla| \rho_*$$

Bimodal density (Dubkov et al.)

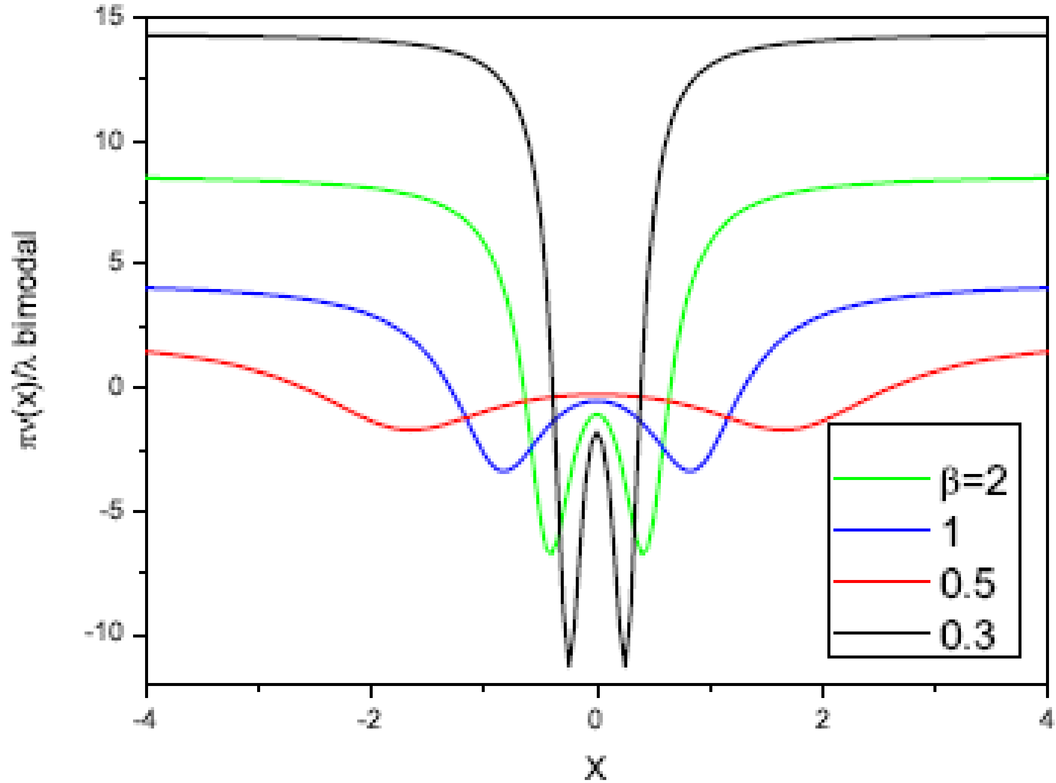
$$\partial_t \rho_* = 0 = -\nabla(b\,\rho_*) - \gamma|\nabla|\rho_*$$

$$\rho_* = \frac{\beta^3}{\pi(x^4 - x^2\beta^2 + \beta^4)}$$

$$b = -\gamma x^3,\ \beta = \lambda/\gamma$$

$$\mathcal{V} = -\lambda\,\frac{|\Delta|^{\mu/2}\,\rho_*^{1/2}}{\rho_*^{1/2}}$$

$$\nu(x) = \frac{\lambda}{\pi}\sqrt{x^4 - \beta^2 x^2 + \beta^4}\int_{-\infty}^{\infty}\frac{dy}{y^2}\left[\frac{1}{\sqrt{(x+y)^4 - \beta^2(x+y)^2 + \beta^4}} - \frac{1}{\sqrt{x^4 - \beta^2 x^2 + \beta^4}}\right]$$



(V. Stephanovich –coll.)

Langevin

$$\partial_t \rho = -\nabla \left(-\frac{\nabla V}{m\beta} \rho \right) - \lambda |\Delta|^{\mu/2} \rho$$

„Topological"

$$\partial_t \rho = \theta \partial_t \theta^* = -\lambda (\exp \Phi) |\Delta|^{\mu/2} [\exp(-\Phi)\rho] - \mathcal{V} \cdot \rho$$

A discord and its analysis in terms of a targeted stochasticity

(i) choose a functional form of $V(x)$ and thus the drift of the Langevin-type process,

(ii) infer an invariant density ρ_* that is compatible with the fractional Fokker-Planck equation

$$\exp[\Phi(x)] = \rho_*^{1/2}(x)$$

(iii) given ρ_*, deduce the corresponding Feynman-Kac (e.g. dynamical semigroup) potential $\mathcal{V}$

$$\mathcal{V} = -\lambda \frac{|\Delta|^{\mu/2} \rho_*^{1/2}}{\rho_*^{1/2}}$$

(iv) use $\mathcal{V}$ and verify whether the "topologically induced dynamics" is affine to

(v) check an asymptotic behavior of $\rho(x,t)$ in both scenarios

(vi) repeat the procedure in reverse by starting from (iii) and then deduce the drift for the Langevin equation with Lévy noise

Targeted stochasticity in the **time domain**

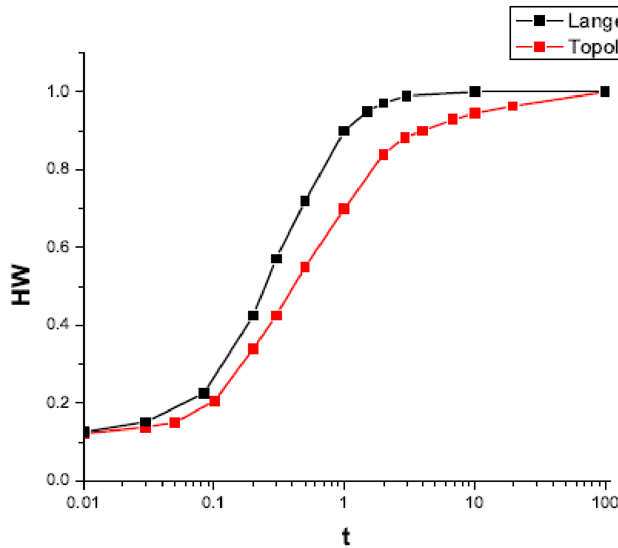


FIG. 1: Temporal behavior of the half-maximum width (HW): for the OUC process in Langevin-driven and semigroup-driven (topological) processes. Motions begin from common initial data $\rho(x, t = 0) = \delta(x)$ and end up at a common pdf (20) for $\sigma = 1$.

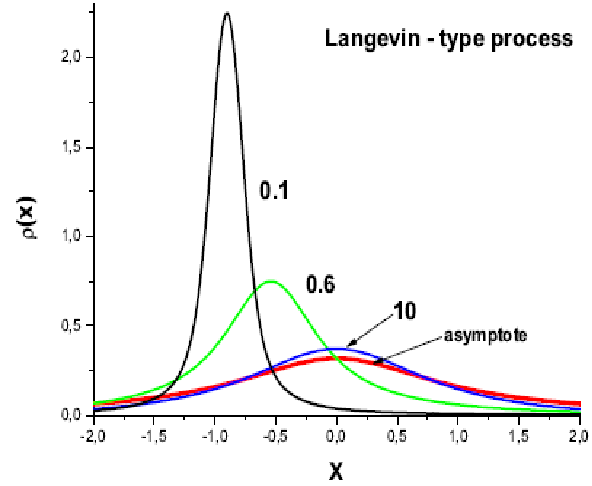


FIG. 2: Time evolution of Langevin-driven pdf $\rho_L(x,t)$ beginning from the initial data $\rho_L(x, t = 0) = \delta(x + 1)$ and ending at the pdf (20) (shown as "asymptote" in the figure) for $\sigma = 1$. Figures near curves correspond to t values.

Dynamics in the OUC process with: $$\rho_*(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)}$$

Targeted stochasticity in the **time domain (confined noise)**

Invariant density

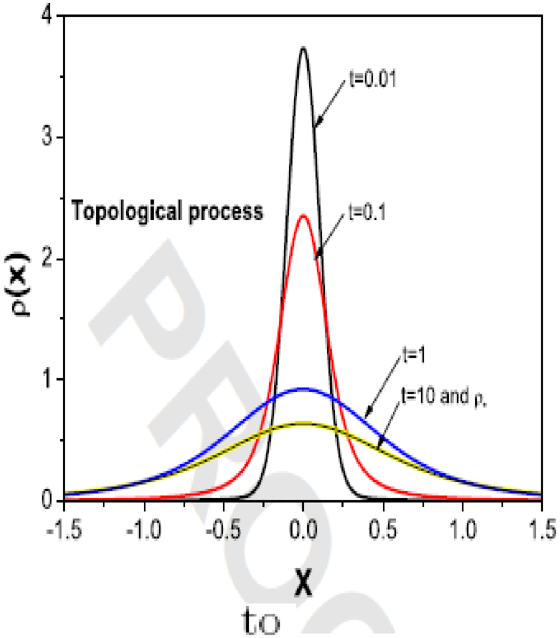
$$\rho_*(x) = \frac{2}{\pi}\frac{1}{(1+x^2)^2}$$

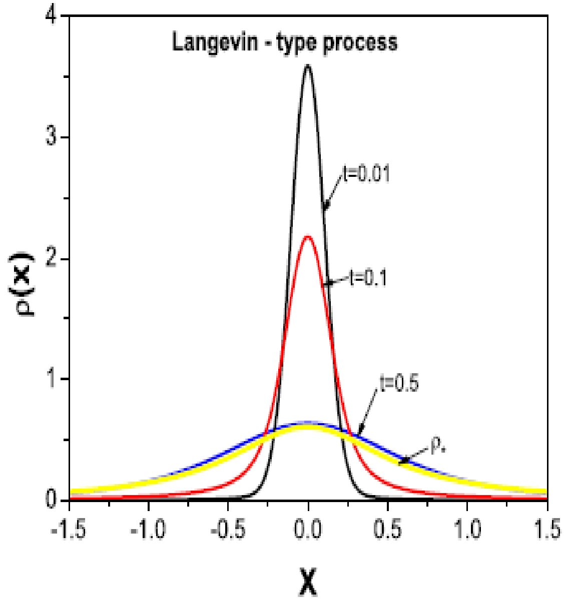
Langevin drift

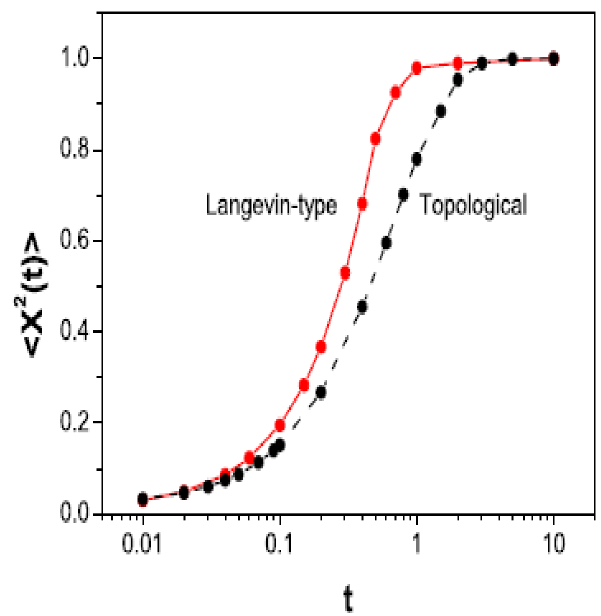
$$b(x) = -\frac{\gamma x}{8}(x^2+3)$$

Semigroup potential

$$\mathcal{V}(x) := \lambda \frac{x^2-1}{x^2+1}$$







Note: PG and R. Olkiewicz, J .Math. Phys. 40, 1057, (1999)

Corollary 2:

a) The Schrödinger boundary-data and interpolation problem (3)–(6) admits a class of unique solutions in terms of Markov stochastic processes, for each concrete choice of the (Feynman–Kac) kernel function that is determined by the Cauchy generator plus a locally bounded, positive and measurable potential function.

(b) The pertinent processes are of the jump-type and arise as suitable limits of *step* processes. In particular, the uniform in time $t \in [0,T]$ convergence in distribution to the perturbed Cauchy process X_t^V is established, when the potential function is bounded.

Remark 1: The developed techniques can be used to investigate the existence issue (including that of the step process approximation) of more general jump-type processes, in particular those related to the quantum evolution with relativistic Hamiltonians.[5,30]

Remark 2: In the present paper, to simplify calculations and to make formulas more transparent, we have considered processes associated with the Cauchy generator (and thus with the α-stable symmetric process as a major tool) in space dimension 1. A glance at the construction of solutions of the Schrödinger problem makes clear that the previous limitations are inessential. In fact, we could consider any $\alpha \in (0,2)$-symmetric stable processes on R^n, for arbitrary $n \geq 1$, and secure the strict positivity and joint continuity in space variables of the corresponding transition density. Such properties for $n \geq 2$ and for potentials from the Kato class $K_{n,\alpha}$ were established in a very recent publication, Ref. 31, Theorems 3.3 and 3.5. However, an issue of sample path properties and of step-process approximations must be settled separately.

Levy processes in inhomogeneous media

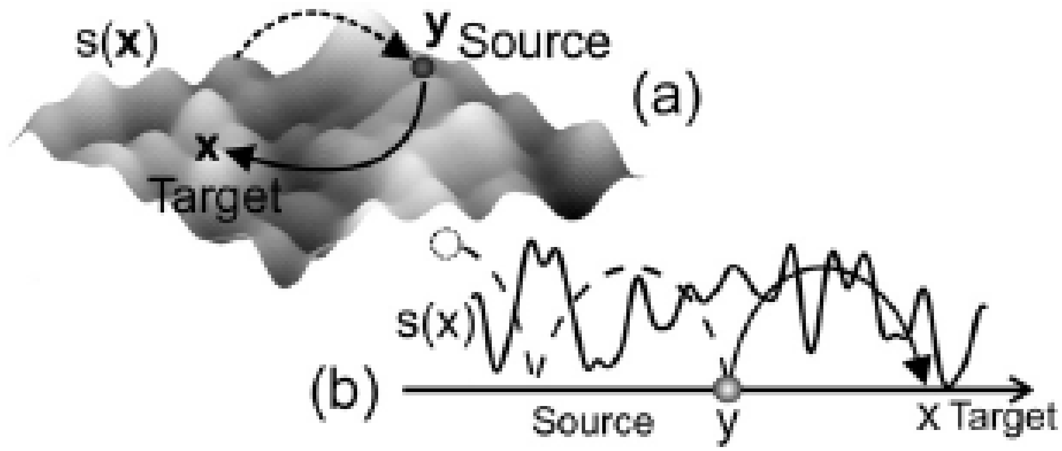


Figure 1. Random walk processes in inhomogeneous salience fields $s(x)$ in two (a) and one (b) dimensions. Source and target locations of a random jump are denoted by y and x, respectively.

The Belik and Brockmann (2007) attractivity or salience field s(x) is identified with an invariant pdf $\rho_*(x)$, but in the Gibbs form !

Topologically induced jump-type processes, an outline

$$\partial_t \rho = -\lambda |\Delta|^{\mu/2} \rho$$

$$(|\Delta|^{\mu/2} f)(x) = -\frac{\Gamma(\mu+1)\sin(\pi\mu/2)}{\pi} \int \frac{f(z)-f(x)}{|z-x|^{1+\mu}}\, dz$$

$$\partial_t \rho(x) = \int [w(x|z)\rho(z) - w(z|x)\rho(x)]\nu_\mu(dz)$$

The jump rate is an even function, $w(x|z) = w(x|z)$

we replace the jump rate $w(x|y) \sim 1/|x-y|^{1+\mu}$

by the expression $w_\phi(x|y) \sim \dfrac{\exp[\Phi(x) - \Phi(y)]}{|x-y|^{1+\mu}}$

$$w_\phi(x|z) \neq w_\phi(z|x)$$

$$(1/\lambda)\partial_t \rho = |\Delta|_{\Phi}^{\mu/2} f = -\exp(\Phi)\,|\Delta|^{\mu/2}[\exp(-\Phi)\rho] + \rho\exp(-\Phi)|\Delta|^{\mu/2}\exp(\Phi)$$

Whatever potential $\Phi(x)$ has been chosen (up to a normalization factor), then formally $\rho_*(x) = \exp(2\Phi(x))$ is a stationary solution

if for a pre-determined $\rho_* = \exp(2\Phi)$, there exists the semigroup potential $\mathcal{V}$ the dynamics belongs to the semigroup framework.

Rewriting the stationary pdf ρ_* as $\rho_*(x) = (1/Z)\exp(-V_*(x)/k_B T)$ We get

$$\partial_t \rho = -\exp(-\kappa V_*/2)\,|\Delta|^{\mu/2}\exp(\kappa V_*/2)\rho + \rho\exp(\kappa V_*/2)|\Delta|^{\mu/2}\exp(-\kappa V_*/2), \;\; \kappa = 1/k_B T.$$

The transport equation has the previous, semigroup-driven form !

$$\partial_t \rho = \theta \partial_t \theta^* = -\lambda(\exp\Phi)|\Delta|^{\mu/2}[\exp(-\Phi)\rho] - \mathcal{V}\cdot\rho$$

Cauchy driver: Gibbsian versus non-Gibbsian asymptotics

$$\partial_t \rho = -\exp(-\kappa V_*/2)\,|\Delta|^{\mu/2}\exp(\kappa V_*/2)\rho + \rho\exp(\kappa V_*/2)|\Delta|^{\mu/2}\exp(-\kappa V_*/2)$$

We scale away dimensional units and consider **typical Gibbsian** forms of $V_*(x) \equiv \Phi(x) = x^4 - 2x^2 +$ 1 and $\Phi \equiv V_*(x) = x^2$

Hyper-confinement

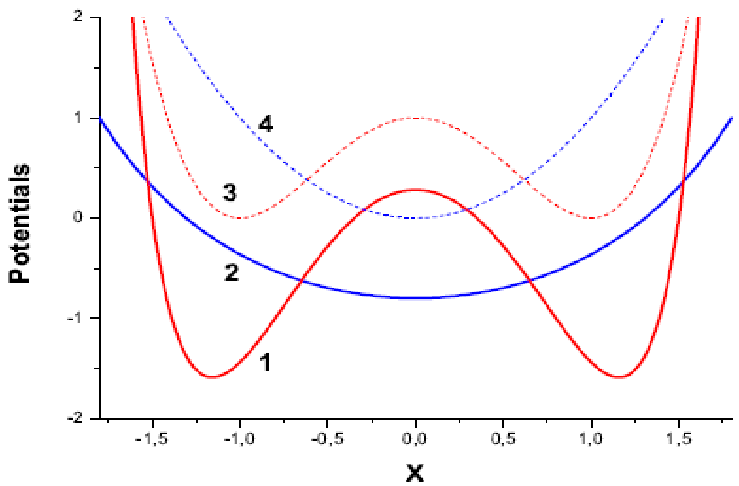


FIG. 4: The coordinate dependence of the semigroup potential $\mathcal{V}(x)$ (curves 1 and 2), corresponding to $V_*(x) = x^4 - 2x^2 + 1$ (curve 3) and $V_*(x) = x^2$ (curve 4), respectively. Curves 3 and 4 are shown for a comparison with, strikingly similar in shape, semigroup potential curves 1 and 2

Non-Gibbsian alternative

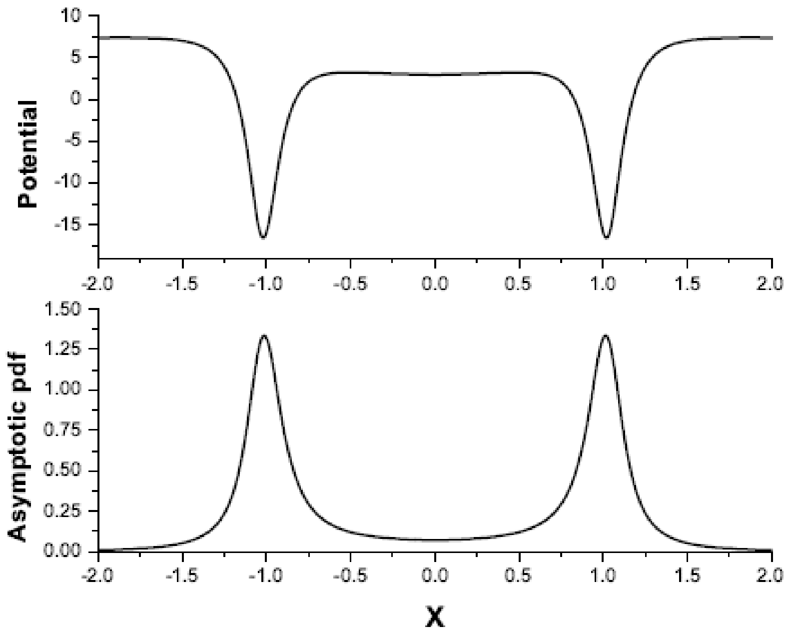
Remark 2: Would we have followed the standard Langevin modeling for the Cauchy driver, with the external force potential $V_*(x) = x^4 - 2x^2 + 1$ and the resultant drift $-\nabla V_* = b$, an invariant pdf of the corresponding fractional Fokker-Planck equation would have the form:

$$\rho_*(x) = \frac{2a(a^2+b^2)}{\pi} \frac{1}{(a^2+b^2)^2 + 2(a^2-b^2)x^2 + x^4}$$

$a \simeq 0.118366$ and $b \simeq 1.0208$

$\mathcal{V}(x)$

$\rho_*(x)$



Cauchy oscillator

$$\hat{H}_{1/2} \equiv \lambda|\nabla| + \left(\frac{\kappa}{2}x^2 - \mathcal{V}_0\right)$$

$$\hat{H} = -D\Delta + \left(\frac{\gamma^2 x^2}{4D} - \frac{\gamma}{2}\right)$$

direct reconstruction route:

$$\left(\frac{\kappa}{2}x^2 - \mathcal{V}_0\right)\rho_*^{1/2} = -\lambda\,|\nabla|\,\rho_*^{1/2}$$

$\tilde{f}(p)$ the Fourier transform of $f = \rho_*^{1/2}(x)$

$$-\frac{\kappa}{2}\Delta_p \tilde{f} + \gamma|p|\tilde{f} = \mathcal{V}_0\tilde{f}$$

$k = (p - \sigma)/\zeta$ $\quad \psi(k) = \tilde{f}(p)$ $\quad \sigma = \mathcal{V}_0/\gamma$ $\quad \zeta = (\kappa/2\gamma)^{1/3}$

$$\frac{d^2\psi(k)}{dk^2} = |k|\psi(k)$$

A unique normalized ground state function of

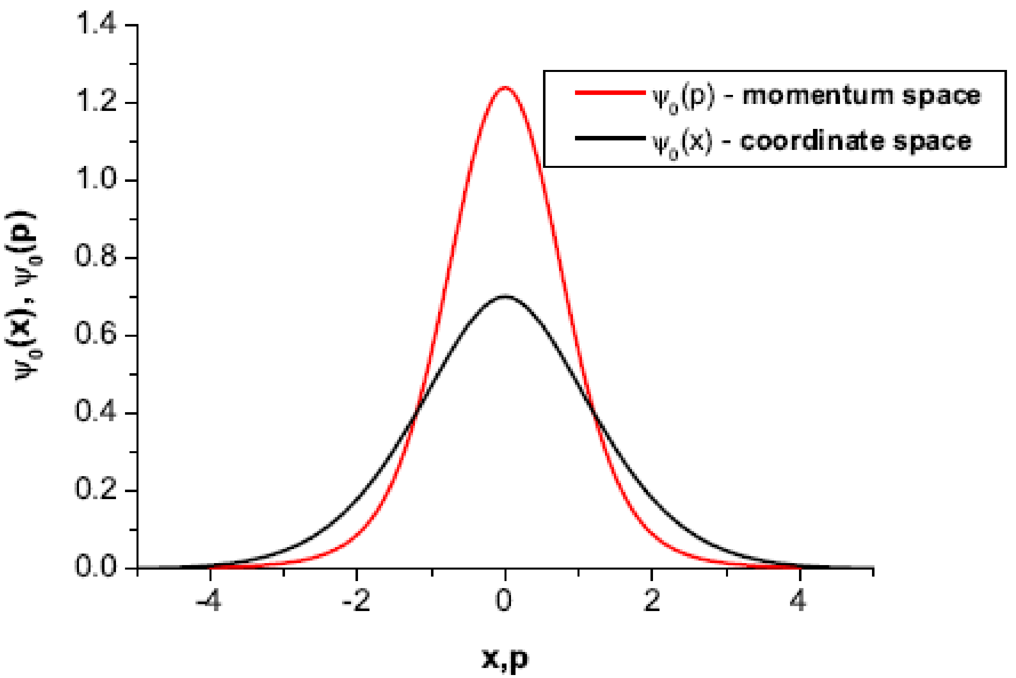
$$\frac{d^2\psi(k)}{dk^2} = |k|\psi(k)$$

is composed of two Airy pieces

that are glued together at the first zero y_0 of the Airy function derivative:

$$\psi_0(k) = A_0 \begin{cases} \mathrm{Ai}(-y_0 + k), & k > 0 \\ \mathrm{Ai}(-y_0 - k), & k < 0, \end{cases}$$

$$A_0 = \left[\mathrm{Ai}(-y_0)\sqrt{2y_0}\right]^{-1}, \quad y_0 \approx 1.01879297$$



To transform the ground state solution back to coordinate space, we evaluate the inverse Fourier transformation of the ground state solution (28), see Appendix C for details. This yields the following real ground state wave function $f(x) \to \psi_0(x)$

$$\psi_0(x) = \frac{A_0}{\pi} \int_{-y_0}^{\infty} \mathrm{Ai}(t) \cos x(t + y_0) dt = \rho_*^{1/2}(x), \quad (29)$$

which determines an invariant pdf $\rho_*(x)$ of the direct engineering problem (25) as follows:

$$\rho_*(x) = \left(\frac{A_0}{\pi}\right)^2 \left[\int_{-y_0}^{\infty} \mathrm{Ai}(t) \cos x(t + y_0) dt\right]^2 \quad (30)$$

$$\equiv \left(\frac{A_0}{\pi}\right)^2 \int_{-y_0}^{\infty} dt \int_{-y_0}^{\infty} dt_1 \mathrm{Ai}(t)\mathrm{Ai}(t_1) \times$$

$$\times \cos x(t + y_0) \cos x(t_1 + y_0).$$

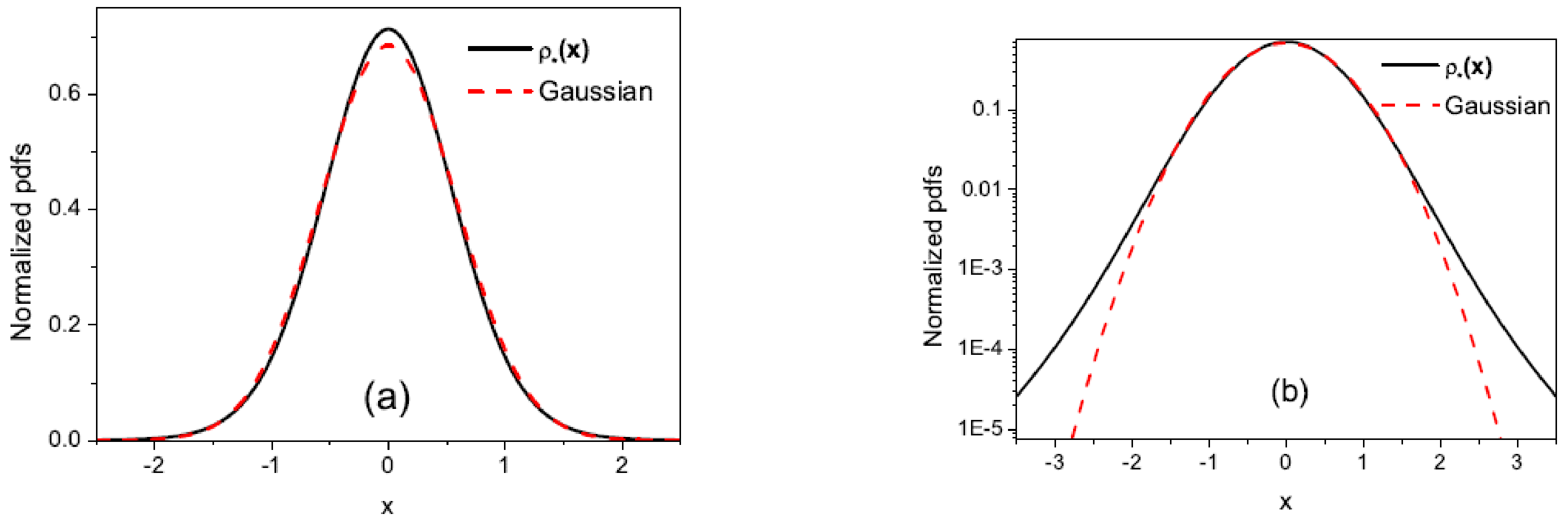


FIG. 7: Normalized invariant pdf (30) (full line) for the quadratic semigroup potential. The Gaussian function, centered at $x = 0$ and with the same variance $\sigma^2 = 0.339598$ is shown for comparison. Panel (a) shows functions in linear scale, while panel (b) shows them in logarithmic scale to better visualize their different behavior.

Reverse engineering for the Cauchy oscillator ground state pdf

For a given ρ_* the definition of a drift function $b(x)$ (we put either $\lambda = 1$ or define $b \to b/\lambda$) is:

$$b(x) = -\frac{1}{\rho_*(x)} \int [|\nabla|\rho_*(x)] dx \equiv$$

$$\frac{1}{\pi \rho_*(x)} \int dx \int_{-\infty}^{\infty} \frac{\rho_*(x+y) - \rho_*(x)}{y^2} dy \, .$$

Inserting $\rho_*(x)$, Eq. (30), we get

$$b(x) = -\frac{\int_{-y_0}^{\infty} \mathrm{Ai}(t) \sin x(t+y_0) dt}{\int_{-y_0}^{\infty} \mathrm{Ai}(t) \cos x(t+y_0) dt} .$$

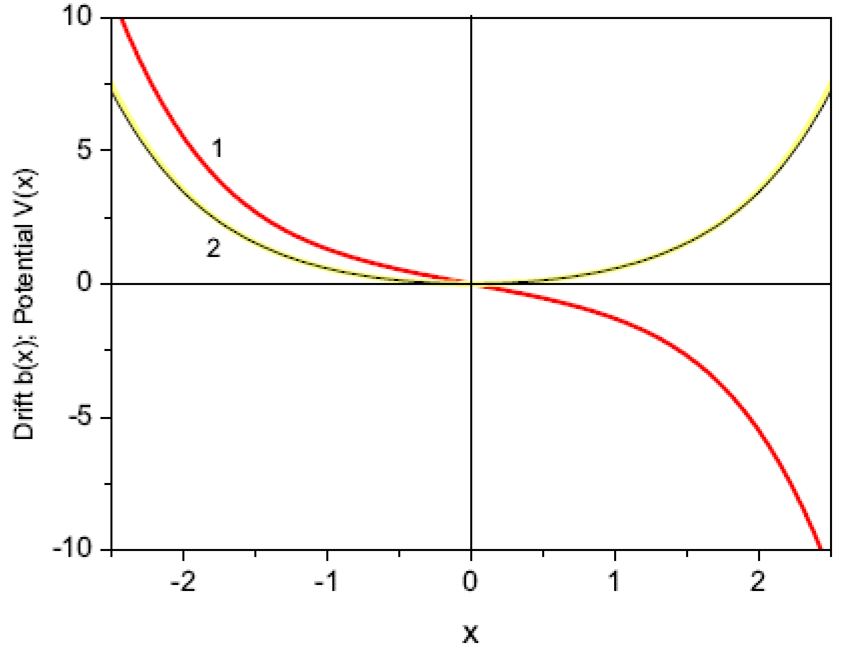


FIG. 8: Langevin - type drift $b(x)$ (curve 1) and its (force) potential $V(x)$ (curve 2), that give rise to an invariant density (30).

Cauchy oscillator ground state pdf is non-Gibbsian

Resume:

We have extended the targeted stochasticity problem of Ref. [7] to the above semigroup-driven (topological) Lévy processes, which are widely used in literature to model various systems, like polymers, glasses and complex networks. Our departure point was as follows: having an invariant pdf $\rho_*(x)$, recover not only the Langevin drift $b(x)$ and potential $V(x) = -\int b(x)dx$, but also the potential $\mathcal{V}(x)$ of the corresponding topological (semigroup) Lévy process, being attributed to the same invariant pdf.

Furthermore, we have relaxed a common pdf requirement and have reformulated the targeted stochasticity problem as a task of reproducing a suitable contractive semigroup, given an invariant pdf, with the Lévy (specifically, Cauchy) driver in action. We have shown, that the semigroup modeling provides much stronger confining properties than the standard Langevin one, such that the resultant asymptotic pdf may have all moments.

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Lévy flights in confining potentials
P. G. and **Vladimir** Stephanovich

Lévy flights in inhomogeneous environments

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