Stochastic mechanics and the Kepler problem

Piotr Garbaczewski*
Research Center Bielefeld-Bochum-Stochastics, University of Bielefeld, D-4800 Bielefeld, Germany
(Received 21 August 1985)

The stochastic mechanics of Nelson and Guerra is formulated for the hydrogen atom. We demonstrate that this simple quantum system can be described in terms of three independent Gaussian Markov processes which are driven (controlled) by the classical Kepler problem. It reveals a manifest connection between the classical and quantized versions of the Kepler problem.

I. MOTIVATION

The idea of stochastic quantization, as developed in Refs. 1 and 2, amounts to associating the stochastic processes to quantum states of the dynamical system. The procedure works quite successfully as long as ground states of the simplest models are considered; the determination of the Madelung fluid representation for higher excited states is much more involved. In fact, the stochastic strategy works in full generality for an example of the harmonic oscillator and its most straightforward generalizations (see also the studies of its two-level, Fermi version[3,4]). However, the formulation of the stochastics mechanics for another simple quantum system, that of the hydrogen atom, except for the ground state,[5] is yet to be accomplished. This fact is a bit puzzling since, like many other simple quantum systems, the quantized Kepler problem admits a realization in terms of (a quartet of) harmonic oscillators,[6-11] and should in principle allow for the generalization of the arguments of Refs. 2 and 12. Moreover the concept of related coherent states was introduced in Ref. 11, and the construction of oscillator stochastic processes is most transparent with respect to the coherent basis.

It is our aim to take advantage of the oscillator reconstruction of the Kepler problem, to formulate the stochastic mechanics of the latter. While working with the four-oscillator system, the functions of Madelung density-phase variables $\rho_i(x)$, $S_i(x)$, $i=1,2,3,4$, arise through computing the coherent-state expectation values $(\alpha \mid \hat{A} \mid \alpha)=\mathscr{A}(\rho,S)$ of operator-valued quantities. To recover the hydrogen problem, the constraints must be accounted for.

As we demonstrate in the course of the paper, the (analytic) stochastic mechanics of the problem, if formulated in the Madelung $[\rho(x),S(x)]$ parametrization, is in all respects equivalent to the standard classical mechanics of the singular (constrained) Hamiltonian system, whose phase manifold is parametrized by holomorphic coherent-state labels $(\bar{\alpha},\alpha)$:

$$|\alpha)=\exp\left[\sum_{i=1}^{4}(\alpha_i a_i^* - \bar{\alpha}_i a_i)\right]|0) \, .$$

They are explicitly related to the action-angle variables of the four-oscillator system through $\alpha=\sqrt{J}e^{i\theta}$. As a consequence, we identify the coherent-state domain for the Kepler problem, whose $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ parameters are completely determined in terms of the canonical variables of the standard classical Kepler problem. It allows for the final conclusion that the three independent Gaussian-Markov processes can be associated with the hydrogen atom. Moreover, these processes are driven (controlled in the language of Ref. 12) by the classical Kepler motion. It establishes an apparent link between the quantized and classical versions of the Kepler problem, the connection which could hardly have been seen from the path-integral computation presented in Ref. 6.

II. HYDROGEN ATOM AS THE CONSTRAINED FOUR-OSCILLATOR SYSTEM

The reduced three-dimensional problem with Coulomb forces is described by the Hamiltonian

$$\begin{gathered} H=-\frac{\hbar^2}{2\mu}\nabla^2-\frac{Ze^2}{r^2} \\ r=(x^2+y^2+z^2)^{1/2}, \quad \mathbf{p}=-i\hbar\nabla \, , \end{gathered} \tag{2.1}$$

where μ stands for the reduced mass and Ze denotes the charge of the nucleus. The operator H is known to commute both with the orbital angular momentum operator $\mathbf{L}=\mathbf{r}\times\mathbf{p}$ and the Runge-Lenz vector:

$$\mathbf{M}=-\frac{Ze^2\mathbf{r}}{r}-\frac{\hbar}{2\mu i}(\nabla\times\mathbf{L}-\mathbf{L}\times\nabla) \tag{2.2}$$

whose square reads

$$\mathbf{M}^2=(Ze^2)^2+\frac{2}{\mu}H(\mathbf{L}^2+\hbar^2) \, . \tag{2.3}$$

It is obvious that $[\mathbf{L}^2,\mathbf{M}^2]_-=0$ and, moreover, that the two important identities

$$\mathbf{L}\cdot\mathbf{M}=0=\mathbf{M}\cdot\mathbf{L} \tag{2.4}$$

hold true.

We confine our attention to the bound-state (discrete) spectrum of H, which after accounting for the identity $[H,\mathbf{M}]_-=0$ allows one to write

$$\mathbf{A}=\left[-\frac{\mu}{2H}\right]^{1/2}\mathbf{M} \tag{2.5}$$

and to make use of $\mathbf{A}$ and $\mathbf{L}$ to form generators of the Lie algebra for the spectrum-generating group $SO_4 \sim SU_2 \otimes SU_2 / Z_2$. Indeed, we have

$$\mathbf{J}=(\mathbf{L}+\mathbf{A})/2, \quad \mathbf{K}=(\mathbf{L}-\mathbf{A})/2 \tag{2.6}$$

so that

$$\begin{aligned} &[J_l,J_m]_-=i\epsilon_{lmn}J_n \ , \\ &[J_l,K_m]_-=0 \ , \\ &[K_l,K_m]_-=i\epsilon_{lmn}K_n \ . \end{aligned} \tag{2.7}$$

Consistently, we can diagonalize H in the representation space of the SO_4-group Lie algebra. The representation space is

$$h_0=h^{\otimes 4}=h\otimes h\otimes h\otimes h, \quad h=L^2(R^1) \tag{2.8}$$

and one can generate it from the Fock vacuum by operating with the quartet of harmonic oscillators:

$$\begin{aligned} &Q_i=\left[\frac{2\hbar}{m\omega}\right]^{1/2}(a_i+a_i^*)/2 \ , \\ &P_i=(2m\hbar\omega)^{1/2}(a_i-a_i^*)/2i \ , \\ &[a_i,a_j^*]_-\equiv\delta_{ij} \ , \\ &[a_i,a_j]_-=0=[a_i^*,a_j^*]_- \ , \\ &a_i\,|\,0)=0 \quad \forall i \ . \end{aligned} \tag{2.9}$$

The underlying boson representation of operators (2.7) reads

$$\begin{aligned} &J_1=(a_1^*a_2+a_1a_2^*)\hbar/2 \ , \\ &J_2=(a_1^*a_2-a_1a_2^*)\hbar/2i \ , \\ &J_3=(a_1^*a_1-a_2^*a_2)\hbar/2 \ , \\ &K_1=(a_3^*a_4+a_3a_4^*)\hbar/2i \ , \\ &K_2=(a_3^*a_4-a_3a_4^*)\hbar/2i \ , \\ &K_3=(a_3^*a_3-a_4^*a_4)\hbar/2 \ , \end{aligned} \tag{2.10}$$

so that (2.4) acquires the form

$$\begin{aligned} &n_1+n_2-n_3-n_4=0 \ , \\ &n_i=a_i^*a_i \ , \end{aligned} \tag{2.11}$$

while the operator identity (2.3) after inserting $\mathbf{L}=\mathbf{J}+\mathbf{K}$ and $\mathbf{M}=(-2H/\mu)^{1/2}(\mathbf{J}-\mathbf{K})$ can be resolved with respect to H:

$$H=\frac{\mu Z^2e^4}{2}[\hbar^2+2(\mathbf{J}^2+\mathbf{K}^2)]^{-1} \ , \tag{2.12}$$

where

$$2(\mathbf{J}^2+\mathbf{K}^2)=\frac{\hbar^2}{2}[(n_1+n_2+1)^2+(n_3+n_4+1)^2-2] \ , \tag{2.13}$$

i.e.,

$$H=\frac{\mu Z^2e^4}{\hbar^2}\frac{1}{(n_1+n_2+1)^2+(n_3+n_4+1)^2} \ . \tag{2.14}$$

Hence the spectrum of H in h_0 comes out

$$\begin{aligned} &|\,\psi)\in h_0 \ , \\ &H\,|\,\psi)=-E\,|\,\psi) \ , \\ &(n_1+n_2)\,|\,\psi)=(n_3+n_4)\,|\,\psi)\equiv(N-1)\,|\,\psi) \ , \\ &N=1,2,\ldots \ , \end{aligned} \tag{2.15}$$

so that

$$E=E_N=-\frac{\mu Z^2e^4}{2\hbar^2N^2} \ . \tag{2.16}$$

Be aware that H in (2.14) commutes in h_0 with the four-oscillator Hamiltonian

$$H_0=\hbar\omega\sum_{i=1}^{4}(a_i^*a_i+\tfrac{1}{2}) \ .$$

Among all oscillator eigenstates, the constraint (2.11) selects those appropriate for the quantized Kepler problem.

III. PASSAGE TO THE CLASSICAL KEPLER PROBLEM

Let us observe that if we define

$$\begin{aligned} &H_0=\sum_{k=1}^{4}\nu I_k, \quad \nu=\frac{\omega}{2\pi} \ , \\ &I_k=h(a_k^*a_k+\tfrac{1}{2}) \ , \end{aligned} \tag{3.1}$$

we arrive at the expression for the hydrogen-atom Hamiltonian

$$H=\frac{4\pi^2\mu Z^2e^4}{(I_1+I_2)^2+(I_3+I_4)^2} \ , \tag{3.2}$$

while the constraint reads

$$I_1+I_2=I_3+I_4 \ . \tag{3.3}$$

Both quantum problems H_0 and H are thus written in the manifest action-angle form, where I_i play the role of quantized action variables. Let

$$|\,\alpha)=\exp\left[\sum_{i=1}^{4}(\alpha_ia_i^*-\bar{\alpha}_ia_i)\right]|\,0) \tag{3.4}$$

be the coherent four-oscillator state:

$$a_i\,|\,\alpha)=\alpha_i\,|\,\alpha) \quad \forall i \ . \tag{3.5}$$

We immediately realize that

$$(\alpha\,|\,I_k\,|\,\alpha)=h(\,|\,\alpha_k\,|^2+\tfrac{1}{2})\equiv J_k \ . \tag{3.6}$$

Moreover, to view H as a formal power-series (Taylor) expansion with respect to the operator variables and then to pass to the normal-ordered form of the series (the tree-approximation concept[13] intervenes), we would have obtained

$$(\alpha\,|\,:H:\,|\,\alpha)=\frac{4\pi^2\mu Z^2e^4}{(J_1+J_2)^2+(J_3+J_4)^2} \tag{3.7}$$

which is supplemented by the constraint

$$J_1+J_2=J_3+J_4\ . \tag{3.8}$$

We have here passed to the classical (in the sense of using the commuting ring) description of the quantized (hydrogen) system. The word "classical" does not refer to any $h\to 0$ limit, since the Planck constant is manifestly present in $J_k=h(\,|\alpha_k|^2+\frac{1}{2})$. Nevertheless, our formal tree-approximation device has related a c-number level to the q-number theory, and methods of analytical (classical) mechanics can, in principle, be adopted. Let us note that

$$(\alpha\,|\,H_0\,|\,\alpha)=\nu\sum_{k=1}^{4}J_k\equiv\mathscr{H}_0 \tag{3.9}$$

is defined (via the symplectic structure), to act on the phase manifold equipped with a local (J,θ) parametrization, so that[14]

$$\dot{\theta}_i=\frac{\partial\mathscr{H}_0}{\partial J_i}=\nu\Longrightarrow\theta_i=\nu t+\beta_i \tag{3.10}$$

arises. The same local parametrization enters the singular Hamiltonian system. We admit the negative-energy values; hence it is useful to define

$$-\mathscr{H}\equiv(\alpha\,|\,:H:\,|\,\alpha)\ . \tag{3.11}$$

Then, after accounting for (3.8), we arrive at (see also Ref. 11)

$$\dot{\theta}_j=\dot{\theta}=\frac{\partial\mathscr{H}}{\partial J_1}=\frac{4\pi^2\mu K^2}{(J_1+J_2)^2}=\nu\quad\forall j\ ,\qquad K=Ze^2\ . \tag{3.12}$$

Consequently, if we set

$$\mathscr{H}=E=-\frac{2\pi^2\mu K^2}{(J_1+J_2)^2} \tag{3.13}$$

we obtain the standard formula (see Ref. 14)

$$\tau=\frac{2\pi}{\nu}=2\pi a^{3/2}\left[\frac{\mu}{K}\right]^{1/2},\quad a=-\frac{K}{2E} \tag{3.14}$$

for the period of the Kepler orbit.

Let us analyze the singular system (3.8) in more detail. The Poisson brackets read

$$\{\mathscr{A},\mathscr{B}\}=\sum_{i=1}^{4}\left[\frac{\partial\mathscr{A}}{\partial\theta_i}\frac{\partial\mathscr{B}}{\partial J_i}-\frac{\partial\mathscr{B}}{\partial\theta_i}\frac{\partial\mathscr{A}}{\partial J_i}\right] \tag{3.15}$$

so that

$$\dot{\theta}_i=\{\theta,\mathscr{H}\}=\frac{\partial\mathscr{H}}{\partial J_i}\ ,\qquad \dot{J}_i=0\ . \tag{3.16}$$

We write the constraint equation as

$$f(J)=0\Longrightarrow\{\mathscr{H},f\}=0\ . \tag{3.17}$$

In case of constrained systems, the unique choice of the manifold on which motions consistent with constraints can occur is guaranteed by choosing the auxiliary constraint condition

$$g=g(\theta,J)=0 \tag{3.18}$$

such that

$$\{f,g\}\neq 0\ . \tag{3.19}$$

Our choice is

$$g=g(\theta)=\theta_4=0 \tag{3.20}$$

so that

$$\{f,g\}=-\frac{\partial g}{\partial\theta_4}\frac{\partial f}{\partial J_4}=-\frac{\partial f}{\partial J_4}=1\ . \tag{3.21}$$

It is well known that a passage to the new canonical variables is possible, so that the dynamics of the problem is given in terms of the three independent canonical pairs.

To arrive at the six-dimensional phase manifold consistent with constraints we must make an appropriate canonical transformation and then account for constraints. The underlying transformation follows from the generating function:

$$\begin{aligned}&F_2=F_2(q,P,t)\ ,\\ &p_i=\frac{\partial F_2}{\partial q_i}\ ,\\ &Q_i=\frac{\partial F_2}{\partial P_i}\ ,\\ &\mathscr{H}\to\mathscr{H}+\frac{\partial F_2}{\partial t}\end{aligned} \tag{3.22}$$

provided we choose $F_2=F_2(q,P)$ as follows:

$$\begin{aligned}&F_2=\theta_1(J_1+J_2)+(\theta_2-\theta_1)J_2+\theta_3J_3+\theta_4J_4\ ,\\ &q_i=\theta_i,\ \ p_i=J_i\ ,\\ &P_1=J_1+J_2,\ \ P_2=J_2,\ \ P_3=J_3,\ \ P_4=J_4\ .\end{aligned} \tag{3.23}$$

Hence

$$\begin{aligned}&Q_1=\theta_1,\ \ Q_2=\theta_2-\theta_1\ ,\\ &Q_3=\theta_3,\ \ Q_4=\theta_4\ .\end{aligned} \tag{3.24}$$

Because of the auxiliary condition, we have $Q_4=0$ while

$$P_4=P_1-P_3\ , \tag{3.25}$$

so that the Hamiltonian in the new variables reads

$$\mathscr{H}=-\frac{2\pi^2\mu Z^2e^4}{P_1{}^2}\ . \tag{3.26}$$

We have thus arrived at the formula (9.75) of Ref. 14, for the three-dimensional (classical) Kepler problem, which by means of the standard analysis (involving a passage from Cartesian to spherical coordinates) follows from the Hamiltonian

$$\mathscr{H}=\frac{1}{2\mu}\left[p_r{}^2+\frac{p_\theta{}^2}{r^2}+\frac{p_\phi{}^2}{r^2\sin^2\theta}\right]-\frac{K}{r}\ . \tag{3.27}$$

Indeed, upon making a canonical transformation $(P,Q)\rightarrow(J,w)$ (see Ref. 14)

$$\begin{aligned}
P_1&=J_\phi+J_\theta+J_r\ ,\\
P_2&=J_\phi+J_\theta\ ,\\
P_3&=J_r\ ,\\
Q_3&=w_r\ ,\\
Q_2&=w_\theta-w_r\ ,\\
Q_1&=w_\phi-w_\theta\ ,
\end{aligned} \tag{3.28}$$

we recover the spherical action-angle version of (3.27):

$$\mathscr{H}=-\frac{2\pi^2\mu K^2}{(J_r+J_\theta+J_\phi)^2}\ . \tag{3.29}$$

We have thus accomplished the reduction of the four-dimensional (oscillating) motion to the three-dimensional Kepler case, whose other realization is provided by the Kustaanheimo-Stiefel transformation.[6–11,15] Anticipating further discussion, let us notice that with respect to the oscillator version (3.2) of the quantized problem, the constraints (3.3) select from the coherent state domain these coherent states only for which (3.8) holds true.

The four-oscillator coherent states were parametrized by complex variables $\alpha_i,\bar{\alpha}_i$, $i=1,2,3,4$ which are related to the (J,θ) variables via the formulas (see, e.g., at the definition of J_k):

$$\alpha_k=(J_k)^{1/2}\exp(i\theta_k),\quad \bar{\alpha}_k=(J_k)^{1/2}\exp(-i\theta_k)\ . \tag{3.30}$$

It provides us with a canonical transformation:

$$(J,\theta)\leftrightarrow(\bar{\alpha},\alpha)\ ,$$

$$\{\mathscr{A},\mathscr{B}\}_{\theta,J}=\{\mathscr{A},\mathscr{B}\}_{\alpha,\bar{\alpha}} \equiv i\sum_{k=1}^{4}\left[\frac{\partial\mathscr{A}}{\partial\alpha_k}\frac{\partial\mathscr{B}}{\partial\bar{\alpha}_k}-\frac{\partial\mathscr{B}}{\partial\alpha_k}\frac{\partial A}{\partial\bar{\alpha}_k}\right]. \tag{3.31}$$

Since after accounting for constraints, we pass from (J,θ) to (P,Q) the formulas (3.15)–(3.26) allow us to rewrite the holomorphic parameters $\alpha_i,\bar{\alpha}_i$ in terms of the independent canonical variables:

$$\begin{aligned}
\alpha_1&=(P_1-P_2)^{1/2}\exp(iQ_1)\ ,\\
\alpha_2&=(P_2)^{1/2}\exp[i(Q_2+Q_1)]\ ,\\
\alpha_3&=(P_3)^{1/2}\exp(iQ_3)\ ,\\
\alpha_4&=(P_1-P_2)^{1/2}\ .
\end{aligned} \tag{3.32}$$

It means that we have arrived at the coherent state domain for the quantized Kepler problem which displays an explicit parametrization in terms of the phase-space variables of the classical Kepler problem. Let us observe that the above analysis implies the following transformation of the four-oscillator Hamiltonian:

$$\mathscr{H}_0=\nu\sum_{i=1}^{4}J_i\rightarrow\mathscr{H}_0=2\nu P_1\ .$$

VI. STOCHASTIC DESCRIPTION OF THE QUANTUM KEPLER PROBLEM

According to Ref. 2 (see also the brief description given in Ref. 4) each harmonic-oscillator coherent state induces a corresponding stochastic process which is driven (controlled) by the corresponding classical system. For the single oscillator Hamiltonian

$$H_0=\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2\ , \tag{4.1}$$

one finds that its coherent state [if to make an explicit use of the Schrödinger representation of the canonical commutation relation (CCR) algebra] takes the form

$$\begin{aligned}
&\psi(x,t)=(2\pi\sigma)^{-1/4}\exp\left[-\frac{1}{4\sigma}[x-q_c(t)]^2+\frac{i}{\hbar}xp_c(t)-\frac{i}{2\hbar}p_c(t)q_c(t)-i\frac{\omega}{2}t\right]\\
&\langle\psi,q\psi\rangle=q_c(t),\quad \langle\psi,(q-\langle\psi,q\psi\rangle)^2\psi\rangle=\sigma\ .
\end{aligned} \tag{4.2}$$

$\psi(x,t)$ induces a corresponding stochastic process:

$$dq(t)=\left[\frac{1}{m}p_c(t)-\omega[q(t)-q_c(t)]\right]dt+dw(t) \tag{4.3}$$

which is characterized by the density-phase variables of the Madelung fluid:

$$\begin{aligned}
\rho(x,t)&=(2\pi\sigma)^{-1/2}\exp\left[-\frac{1}{2\sigma}[x-q_c(t)]^2\right],\\
S(x,t)&=xp_c(t)-\tfrac{1}{2}p_c(t)q_c(t)-\tfrac{1}{2}\hbar\omega t\ .
\end{aligned} \tag{4.4}$$

According to Ref. 12, if we rewrite (4.3) as

$$dq(t)=v_+(q(t),t)dt+dw(t) \tag{4.5}$$

and denote

$$v_\pm(x,t)=\frac{1}{m}p_c(t)\mp\omega[x-q_c(t)]\ , \tag{4.6}$$

we arrive at the following formulas:

$$v=\frac{1}{2}(v_+ + v_-)=\frac{1}{m}\nabla S\,,$$
$$u=\frac{1}{2}(v_+ - v_-)=\frac{\hbar}{2m}\frac{\nabla\rho}{\rho}\,. \tag{4.7}$$

Moreover, we have

$$q_c=\int dx\, x\rho(x)=q_c(\rho)\,,$$
$$p_c=\int \rho(x)\nabla S(x)dx=p_c(\rho,S)\,. \tag{4.8}$$

If we insert (4.8) into

$$\alpha=\frac{1}{(2\hbar\omega)^{1/2}}\left[\omega q_c+\frac{ip_c}{\sqrt{m}}\right],$$
$$\bar{\alpha}=\frac{1}{(2\hbar\omega)^{1/2}}\left[\omega q_c-\frac{ip_c}{\sqrt{m}}\right], \tag{4.9}$$

we get an explicit dependence of holomorphic (coherent state) parameters $\alpha,\bar{\alpha}$ on the Madelung fluid variables $\rho(x),S(x)$, $\alpha=\alpha(\rho,S)$, $\bar{\alpha}=\bar{\alpha}(\rho,S)$.

The passage to the Madelung fluid description is an essence of the Nelson-Guerra's stochastic mechanics (see Ref. 12) since there is a natural symplectic structure associated with (ρ,S). For any two functions on the phase manifold of the Madelung fluid, the respective two-form implies the Poisson brackets:

$$\{\mathscr{A},\mathscr{B}\}=\int dx\left[\frac{\partial\mathscr{A}}{\partial\rho(x)}\frac{\partial\mathscr{B}}{\partial S(x)}-\frac{\partial\mathscr{B}}{\partial\rho(x)}\frac{\partial\mathscr{A}}{\partial S(x)}\right]. \tag{4.10}$$

On the other hand, since the whole (ρ,S) dependence is coded in the coherent states (4.2), we can introduce the following definition of any $\mathscr{A}(\rho,S)$. Let $\hat{A}=A(a^*,a)$ be an operator defined in the harmonic-oscillator Hilbert space. Then let

$$\mathscr{A}(\rho,S)=(\alpha\mid\hat{A}\mid\alpha)=\langle\psi,\hat{A},\psi\rangle$$
$$=\int dx\,\bar{\psi}(x)(\hat{A}\psi)(x)\,. \tag{4.11}$$

Obviously, since we parametrize coherent states by means of $\bar{\alpha},\alpha$, we can write, as well,

$$\mathscr{A}(\rho,S)=\mathscr{A}(\alpha,\bar{\alpha})\,. \tag{4.12}$$

At this point, let us come back to the notation (3.30), but restricted to the single-oscillator case (it is a matter of simplicity only). We denote

$$\alpha=\sqrt{J}\exp(i\theta),\quad \bar{\alpha}=\sqrt{J}\exp(-i\theta)\,, \tag{4.13}$$

and define

$$J=\int dx\,J(x)f(x),\quad \theta=\int dx\,\theta(x)f(x)\,, \tag{4.14}$$

where $f(x)$ is a real function obeying the normalization condition

$$2\int_{R^1}f(x)^2dx=1\,. \tag{4.15}$$

It implies

$$\{\alpha,\bar{\alpha}\}_{J,\theta}=i\int dx\left[\frac{\delta\alpha}{\delta J(x)}\frac{\delta\bar{\alpha}}{\delta\theta(x)}-\frac{\delta\bar{\alpha}}{\delta J(x)}\frac{\delta\alpha}{\delta\theta(x)}\right]$$
$$=2\int f(x)^2dx=1 \tag{4.16}$$

and, furthermore, the conclusion that for any two functions of $\alpha,\bar{\alpha}$, we obtain

$$\{\mathscr{A},\mathscr{B}\}_{J,\theta}=i\int dx\left[\left[\frac{\partial\mathscr{A}}{\partial\alpha}\frac{\delta\alpha}{\delta J(x)}+\frac{\partial\mathscr{A}}{\partial J}\frac{\delta\bar{\alpha}}{\delta J(x)}\right]\left[\frac{\partial\mathscr{B}}{\partial\alpha}\frac{\delta\alpha}{\delta\theta(x)}+\frac{\partial\mathscr{B}}{\partial\bar{\alpha}}\frac{\delta\bar{\alpha}}{\delta\theta(x)}\right]-(\mathscr{A}\to\mathscr{B})(\mathscr{B}\to\mathscr{A})\right]$$
$$=\frac{\partial\mathscr{A}}{\partial\alpha}\frac{\partial\mathscr{B}}{\partial\bar{\alpha}}\{\alpha,\bar{\alpha}\}_{J,\theta}+\frac{\partial\mathscr{A}}{\partial\bar{\alpha}}\frac{\partial\mathscr{B}}{\partial\alpha}\{\bar{\alpha},\alpha\}_{J,\theta}=\{\mathscr{A},\mathscr{B}\}_{\alpha,\bar{\alpha}}\,. \tag{4.17}$$

Consequently, once we have the (holomorphic) canonical parametrization $\alpha,\bar{\alpha}$, a passage to the continuous $J(x),\theta(x)$ one is immediate. The next change of variables from $J(x),\theta(x)$ to $\rho(x),S(x)$ is a canonical transformation again, which clearly demonstrates an important property

$$\{\mathscr{A},\mathscr{B}\}_{\alpha,\bar{\alpha}}=\{\mathscr{A},\mathscr{B}\}_{\rho,S}\,. \tag{4.18}$$

Since in the present case, $\alpha,\bar{\alpha}$ are related via (4.13) to the action-angle variable of the harmonic oscillator, we have completed the relation (4.18) by

$$\{\mathscr{A},\mathscr{B}\}_{\rho,S}=\frac{\partial\mathscr{A}}{\partial J}\frac{\partial\mathscr{B}}{\partial\theta}-\frac{\partial\mathscr{A}}{\partial\theta}\frac{\partial\mathscr{B}}{\partial J} \tag{4.19}$$

which translates the symplectic structure of the classical harmonic system into that of the Madelung fluid, so that to the pair, (J,θ) of canonical variables, we have related the fluid variables $\rho(x),S(x)$ and thus the corresponding stochastic process.

We realize that the analytic features of the stochastic mechanics are an exact reformulation of the corresponding classical mechanics, see (4.18). The above discussion can be immediately generalized to the four-oscillator case, provided we make some minor modifications such as

$$\rho(x),S(x)\to\rho_i(x),S_i(x)\,,$$
$$J,\theta\to J_i,\theta_i,\alpha,\bar{\alpha}\to\alpha_i,\bar{\alpha}_i\,,$$
$$\{\mathscr{A},\mathscr{B}\}_{\rho,S}=\sum_{i=1}^{4}\left[\frac{\partial\mathscr{A}}{\partial J_i}\frac{\partial\mathscr{B}}{\partial\theta_i}-\frac{\partial\mathscr{A}}{\partial\theta_i}\frac{\partial\mathscr{B}}{\partial J_i}\right], \tag{4.20}$$

and consistently,[2] take into account that the quartet of harmonic oscillators induces a quartet of independent Gaussian Markov processes. It is quite amusing to see that the analysis of Sec. III, which results in specifying the six-dimensional $(P_i,Q_i,f=1,2,3)$ manifold on which motions consistent with constraints occur, automatically enforces the parallel reductions with respect to $(\alpha,\bar{\alpha}),(J,\theta),(\rho,S)$ parametrizations, which is due to (4.17)–(4.19).

Here, because of (3.32), the coherent-state expectation value of an operator reads

$$\begin{aligned}\mathscr{A}=(\alpha\mid\hat{A}\mid\alpha)&=\mathscr{A}(\alpha_1,\alpha_2,\alpha_3,\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3)\\&=\mathscr{A}(P_1,P_2,P_3,Q_1,Q_2,Q_3)\\&=\mathscr{A}(\rho_1,\rho_2,\rho_3,S_1,S_2,S_3)\;,\end{aligned}\tag{4.21}$$

where the independent canonical variables are explicitly displayed. Consequently the three independent Gaussian Markov processes can be attributed to the quantized Kepler problem, which conforms well with the general analysis of Ref. 16, showing that it is possible to express the random process in R^n, $n=3$ through a random process in R^N, $N=4$. The number of involved independent Wiener processes is specified by n and N, respectively.

ACKNOWLEDGMENTS

I would like to express my warm thanks to Professor S. Albeverio, Professor Ph. Blanchard, and Professor L. Streit for an invitation to the Research Center Bielefeld-Bochum-Stochastics where the present paper was written.

*Permanent address: Institute of Theoretical Physics, University of Wroclaw, PL-50-205 Wroclaw, Poland.

[1]E. Nelson, Phys. Rev. **150**, 1079 (1966).
[2]F. Guerra, Phys. Rep. **77**, 263 (1981).
[3]G. F. De Angelis, D. de Falco, and F. Guerra, Phys. Rev. D **23**, 1747 (1981).
[4]P. Garbaczewski, Bielefeld Report No. 62, 1985 (unpublished).
[5]A. Truman and J. T. Lewis, Dublin Report No. DIAS-STO-84-51, 1984 (unpublished).
[6]R. Ho and A. Inomata, Phys. Rev. Lett. **48**, 231 (1981).
[7]A. C. Chen, J. Math. Phys. **23**, 413 (1982).
[8]T. Iwai, J. Math. Phys. **23**, 1093 (1982).
[9]M. Kibler and T. Negadi, J. Phys. A **16**, 4265 (1983).
[10]A. C. Chen and M. Kibler, Phys. Rev. A **31**, 3960 (1985).
[11]C. Gerry, J. Phys. A **17**, L737 (1984).
[12]F. Guerra and R. Marra, Phys. Rev. D **28**, 1916 (1983).
[13]P. Garbaczewski, *Classical and Quantum Field Theory of Exactly Soluble Nonlinear Systems* (World Scientific, Singapore, 1985).
[14]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass., 1969).
[15]A. O. Barut, C. Schneider, and R. Wilson, J. Math. Phys. **20**, 2245 (1979).
[16]Ph. Blanchard and M. Sirugue, J. Math. Phys. **22**, 1372 (1981).