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On the statistical origins of the de Broglie–Bohm quantum potential: Brownian motion in a field of force as Bernstein diffusion

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We demonstrate that Brownian motion in the Smoluchowski approximation generates a rich class of Markovian diffusions in the framework of Zambrini's "Euclidean quantum mechanics". The role of dynamical semigroups in this formalism is made explicit.

A mathematical idealisation [1–3] of the individual Brownian particle dynamics, in case of free evolution in the high friction regime, is provided by the configuration space (Wiener) projection of the phase space (Ornstein–Uhlenbeck) process. One deals then with the stochastic differential equation

$$\mathrm{d}X(t)=\sqrt{2D}\,\mathrm{d}W(t)\,,$$
$$X(0)=x_0\in\mathbb{R}^3\,,\quad t\in[0,T]\,,\quad D>0\,,\tag{1}$$

which is a symbolic expression representing an ensemble of possible [3] instantaneous values (sample locations in space), generated by the random noise $W(t)$ according to a definite statistical law. Equations (1) is known (via the stochastic Itô calculus) to imply the Kolmogorov equation for the transition probability density (heat kernel here), i.e. a fundamental law of random displacements of the process, which gives rise to the Fokker–Planck (heat) equation for the time development of the probability distribution of diffusing particles,

$$\partial_t\rho=D\triangle\rho\,,\quad \rho(x,0)=\rho_0(x)\,.\tag{2}$$

Then $\rho(x,t)$ is the probability distribution of the random variable $X(t)$, given the distribution $\rho_0(x)$ of its initial values $X(0)$ in $\mathbb{R}^3$.

By introducing the (irrotational, $\mathrm{rot}\,v=0$) local velocity field,

$$v=-D\frac{\nabla\rho}{\rho}\quad\Rightarrow\quad \partial_t\rho=-\nabla(\rho v)\,,\tag{3}$$

for all conceivable choices of the smooth function $\rho_0(x)$, the heat equation, if combined with assumption (3), inevitably gives rise [4] to the local conservation law (the momentum balance equation in the kinetic theory lore [5])

$$\partial_t v+(v\nabla)v=-\frac{1}{m}\nabla Q\,,$$
$$Q=2mD^2\frac{\triangle\rho^{1/2}}{\rho^{1/2}}\,,\quad v_0=-D\frac{\nabla\rho_0}{\rho_0}\,,\tag{4}$$

where m stands for the hitherto absent (albeit included in the definition of the diffusion constant D via the fluctuation–dissipation theorem) mass parameter of diffusing particles, while the potential Q is recognized to have the standard functional form of the familiar de Broglie–Bohm "quantum potential", except for the opposite [4,5] sign.

In case of an arbitrary non-symmetric distribution $\rho_0(x)$ we have the following property, which is maintained in the course of the diffusion process $(X(t)\in\mathbb{R}^3)$,

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$$\frac{1}{m}\partial_i Q = \frac{1}{\rho}\sum_{j=1}^{3}\partial_j P_{ij}\,, \quad P_{ij} = D^2\rho\partial_i\partial_j \ln\rho\,, \tag{5}$$

where $\nabla \equiv (\partial_1, \partial_2, \partial_3)$ and $i, j = 1, 2, 3$. Apparently, $P_{ij} = \delta_{ij}D^2\rho\triangle\ln\rho$ in the totally isotropic case (see e.g. ref. [5]). The unconventional "pressure" term $(-(1/m)\nabla Q)$ in (4) is a distinctive characteristic of all diffusions derivable (via conditioning as example [6]) from the Brownian motion proper and is a collective, statistical ensemble measure of the momentum transfer per unit of time and per unit of volume: *away* ($-\nabla Q$ corresponds to the conventional Brownian propagation with the obvious tendency of a particle to leave the area of higher concentration) or *towards* ($+\nabla Q$, see e.g. ref. [5]) the infinitesimal surrounding of the given spatial location $x \in \mathbb{R}^3$ at time t, at the very same rate.

The conventional Brownian dynamics is a very special solution of the general Cauchy problem composed of the mass conservation law (2) and the momentum balance equation (4) with the initial data $\rho_0(x)$, $v_0(x)$ in principle unrelated, in contrast to assumption (3). Then we arrive [6–8] at the rich family of Markovian diffusions, all of which are the *descendants* of the Brownian motion, the Brownian motion itself included.

To be more specific, let us consider the boundary probability distributions $\rho_0(x) = \rho(x, 0)$, $\rho_T(x) = \rho(x, T)$ for a stochastic diffusion process in $\mathbb{R}^3$, confined to the time interval $[0, T] \ni t$. We realise that the dynamical semigroup operator $\exp(tD\triangle)$ provides us with the *probabilistic semigroup transition mechanism*, in the sense that the strictly positive semigroup (heat in our case) kernel is given,

$$h(y, 0, x, t) = (4\pi Dt)^{-1/2}\exp[-(x-y)^2/4Dt]$$
$$= [\exp(tD\triangle)](y, x)\,. \tag{6}$$

Following Schrödinger [6–9], we look for the joint probability distribution

$$m(x, y) = \theta_*(x, 0)h(x, 0, y, T)\,\theta(y, T)\,, \tag{7}$$

whose marginals

$$\int dx\, m(x, y) = \rho_T(y)\,,$$
$$\int dy\, m(x, y) = \rho_0(x)\,, \tag{8}$$

coincide with the previously prescribed boundary data for random propagation in the interval $[0, T]$. It is clear that for arbitrarily chosen (not necessarily disjoint) areas A and B in $\mathbb{R}^3$, the probability to find in B a particle which originated from A at time 0 and was subject to the random (Brownian, e.g. Wiener) perturbations in the whole run of duration T, reads

$$m(\mathrm{A}, \mathrm{B}) = \int_{\mathrm{A}} dx \int_{\mathrm{B}} dy\, m(x, y)\,. \tag{9}$$

The existence of $m(x, y)$ is guaranteed by the major mathematical demonstration [9] that the solution of the Schrödinger system (7), (8) in terms of positive functions $\theta_*(x, 0)$ and $\theta(y, T)$ is unique and can always be found.

With the data $\theta_*(x)$ and $\theta_T(x)$ we can construct respectively the forward and backward diffusive propagation by means of kernel (6),

$$\partial_t\theta_* = D\triangle\theta_*\,, \quad \partial_t\theta = -D\triangle\theta\,,$$
$$\theta_*(x, 0) = \theta_{*0}(x)\,, \quad \theta(x, T) = \theta_T(x)\,,$$
$$t \in [0, T]\,, \tag{10}$$

where

$$\theta_*(x, t) = \int h(y, 0, x, t)\theta_{*0}(y)\, dy\,,$$

$$\theta(x, t) = \int h(x, t, y, T)\theta_T(y)\, dy, \quad 0 \leqslant t \leqslant T. \tag{11}$$

The local conservation laws (2) and (4) are satisfied by

$$\rho(x, t) = (\theta\theta_*)(x, t)\,,$$
$$v(x, t) = D\nabla\ln(\theta/\theta_*)\,(x, t)\,,$$
$$x \in \mathbb{R}^3, \quad t \in [0, T]\,. \tag{12}$$

Complete statistical information about the *most likely* way the individual particles propagate, is provided by the transition density $p(y, s, x, t) = h(y, s, x, t)\theta(x, t)/\theta(y, s)$ which solves the Kolmogorov (Fokker–Planck) equation associated with the (individual particle motion recipe) stochastic differential equation

$$dX(t) = b(X(t), t)\, dt + \sqrt{2D}\, dW(t)\,,$$

$$b(x,t)=(u+v)(x,t)\,,\quad u(x,t)=D\frac{\nabla\rho}{\rho}\,, \tag{13}$$

see e.g. refs. [7,6,10,11]. Notice that in notation (8)–(12) the standard Brownian motion is found in a trivial way by substituting $\theta_*=\rho(x,t)$, $\theta=1$ for all times $t\in[0,T]$.

Our previous discussion was entirely devoted to the free Brownian evolution, and it is quite natural to address the issue of the effects of external force fields on the random propagation. In the high friction regime, like in case of (2), we should consider the Brownian motion in a field of force, in the Smoluchowski approximation [1–3,12].

The Fokker–Planck equation governing the time development of the spatial probability distribution in case of phase space noise with high friction, in the Smoluchowski form reads

$$\partial_t\rho=D\triangle\rho-\nabla(b\rho)\,,$$

$$b(x,t)=\frac{1}{\beta}F(x)\,,\quad \rho_0(x)=\rho(x,0)\,, \tag{14}$$

where β is the friction constant and the external force we assume to be conservative,

$$F(x)=-\nabla\Phi(x)\,. \tag{15}$$

It is well known [12,13] that the substitution

$$\rho(x,t)=\theta_*(x,t)\exp[-\Phi(x)/2D\beta]\,, \tag{16}$$

converts the Fokker–Planck equation (14) into the generalised diffusion equation for $\theta_*(x,t)$ (our notation is motivated by that of refs. [6,10] and formulas (8)–(12)),

$$\partial_t\theta_*=D\triangle\theta_*-\frac{V(x)}{2mD}\theta_*\,, \tag{17}$$

where (the mass m was here introduced per force, but with the very concrete purpose of embedding our discussion in the formalism of the "Euclidean quantum mechanics" [7])

$$V(x)=\frac{m}{\beta}(F^2/2\beta+D\nabla F)\,. \tag{18}$$

Since F^2, D, β are positive, a sufficient condition for the auxiliary potential $V(x)$ to be bounded from below (its continuity is taken for granted) is that the source term $g(x)$ in the familiar Poisson equation

$$\nabla F=-\triangle\Phi=g \tag{19}$$

is bounded from below: $g(x)>-c$, $c>0$, c is finite. Under this condition of boundedness, we know [7,10,11] that eq. (17) defines the fundamental *semigroup transition mechanism* underlying the Smoluchowski diffusion. Indeed, by (17) we have in hands the well defined semigroup operator $\exp[-t(-D\triangle+V/2mD)]$, whose integral kernel is a strictly positive solution of (17) with the initial condition $\lim_{t\to0}h(y,0,x,t)=\delta(y-x)$.

The kernel is defined by the Feynman–Kac formula (in terms of the conditional Wiener measure, which sets an obvious link with the Brownian propagation)

$$\begin{aligned}
&h(y,s,x,t)=h(V;y,s,x,t)\\
&\quad=\lim_{n\to\infty}\frac{1}{(4\pi D\Delta t)^{3n/2}}\int dx_1\ldots\int dx_{n-1}\exp\bigg[-\frac{x_1^2}{4D\Delta t}\\
&\quad-\sum_{j=1}^{n-1}\bigg(\frac{1}{2mD}V(y+x_j,t_j)\Delta t+\frac{(x_{j+1}-x_j)^2}{4D\Delta t}\bigg)\bigg]\,,\\
&\quad\Delta t=(t-s)/n\,,\quad t_j=j\Delta t\,,\quad j=0,1,2,\ldots,n\,,\\
&\quad t_0=s\,,\quad t_n=t\,,\quad X(tj)=x_j\,,\quad x_0=y\,,\\
&\quad x_n=x\,,\quad s\leqslant t\,.
\end{aligned} \tag{20}$$

It is trivial to check that $h(y,s,x,t)$ propagates $\theta_{*0}(x)$ into a solution of (17),

$$\begin{aligned}
&\theta_{*0}(x)=\rho_0(x)\exp[\Phi(x)/2D\beta]\quad\to\\
&\quad\theta_*(x,t)=\int h(y,0,x,t)\theta_*(y,0)\,dy\,,
\end{aligned} \tag{21}$$

while apparently,

$$\begin{aligned}
&\theta(x,t)=\exp[-\Phi(x)/2D\beta]\\
&\quad=\int h(x,t,y,T)\theta_T(y)\,dy=\theta_T(x)\,,
\end{aligned} \tag{22}$$

for all $t\in[0,T]$. Indeed $\theta(x,t)$ (22) solves

$$\partial_t\theta=-D\triangle\theta+\frac{V}{2mD}\theta\,, \tag{23}$$

where $\partial_t\theta=0$ and

$$D\triangle\theta=\bigg(\frac{(\nabla\Phi)^2}{4D\beta^2}-\frac{\triangle\Phi}{2\beta}\bigg)\theta=\frac{V}{2mD}\theta\,, \tag{24}$$

as it should be. Since the deterministic evolution governed by the Smoluchowski equation gives rise to a definite terminal (in the interval $[0, T]$) outcome $\rho_T(x)$, given $\rho_0(x)$, a straightforward inspection demonstrates that the Schrödinger system (7), (8) is solved by $\theta_{*0}(x)$ (21) and $\theta_T(x)$ (22) with the kernel $h(V; y, s, x, t)$. As a consequence, we have completely specified the unique Markov–Bernstein diffusion interpolating between $\rho_0(x)$ and $\rho_T(x)$, which is identical with the Smoluchowski diffusion itself. We know here [7,10] the transition probability density (e.g. the law of random displacements modified by the presence of external force fields)

$$p(y, s, x, t) = h(y, s, x, t)\,\frac{\theta(x, t)}{\theta(y, s)}\,, \tag{25}$$

which is responsible for the *most likely* particle propagation scenario. We have also automatically satisfied [7,10] the local conservation laws

$$\partial_t\rho = -\nabla(\rho v)\,, \quad \partial_t v + (v\nabla)v = \frac{1}{m}\nabla(V - Q)\,,$$

$$\rho(x, 0) = \rho_0(x), \quad v(x, 0) = v_0(x)\,, \tag{26}$$

where $\rho(x, t)$, $v(x, t)$ are defined by formula (12). Notice that in the detailed derivation, the above momentum balance equation does not appear directly, but in the indirect way by taking the gradient of the much weaker (Hamilton–Jacobi) identity

$$V - Q = 2mD[\partial_t S + D(\nabla S)^2]\,,$$

$$S(x, t) = \tfrac{1}{2}\ln(\theta/\theta_*)\,. \tag{27}$$

In our case, apparently,

$$v(x, t) = D\nabla(-\Phi/D\beta - \ln\rho) = -\frac{1}{\beta}\nabla\Phi - D\frac{\nabla\rho}{\rho} \rightarrow$$

$$\partial_t\rho = \nabla[(1/\beta)(\nabla\Phi)\rho] + D\triangle\rho\,, \tag{28}$$

to be compared with the Smoluchowski equation.

The above discussion admits various generalisations. For example, by choosing a definite (reference) Smoluchowski force potential and then the auxiliary (induced) one V, we have fixed the strictly positive kernel $h(V; y, s, x, t)$. By playing with different choices of the boundary data ρ_0, ρ_T (unrelated to those initially considered) and seeking a solution of the Schrödinger system (7), (8), we can generate a rich class of (conditional) random motions, all of which are governed by conservation laws (26) with the potential V. However, their forward drifts $b(x, t)$ would have a functional form completely divorced from the simple Smoluchowski expression.

We can as well start form the general Cauchy problem (26) with completely arbitrary V (except for being continuous and bounded from below). Then the corresponding Smoluchowski problem can be reproduced only if the potential allows one to decouple from eq. (18) the force field F. The task can be formidable, since even in the simplest, one-dimensional case, (18) becomes the well known Riccati equation [14].

Let us emphasise that our analysis heavily relies on the phase space (Langevin) formulation of Brownian motion as the problem of random accelerations in the presence of friction [1–3]. It lends support to the conjecture that the quantum mechanical looking evolution of statistical particle ensembles (nonlinear Nelson diffusions, which according to ref. [15] are governed by the *Brownian recoil principle*) are derivable from the phase space random motions as well.

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