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Relativistic problem of random flights and Nelson's stochastic mechanics

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We extend to special relativity the nonrelativistic arguments by which the phase space Brownian motion of an ensemble of massive particles in the diffusion regime (the problem of random flights) is governed by the Schrödinger equation. The corresponding relativistic dynamics is governed by the Klein–Gordon equation.

A satisfactory relativistic generalisation of Nelson's stochastic mechanics [1,2] (see also refs. [3–7]) still remains an open question despite the numerous attempts [8,9] to solve it. Apparently by imposing the Markov property on diffusions in Minkowski space, one is led either to the violation of causality, or to purely deterministic motion [10–13]. On the other hand it is well known [12,13] that the phase space of a particle is a natural arena for the construction of a relativistic stochastic process. We have given before [1] a phase-space stochastic derivation of Nelson's theory in the case of free motion (see also ref. [3] for a more general discussion), hence it is rather tempting to examine the relativistic extension of the given arguments and eventually to analyse possible links with the relativistic invariant field equations like e.g. the Klein–Gordon or (once random rotations are incorporated into the formalism) the Dirac ones.

In contrast to Minkowski space considerations [8,9,14–17], the passage to relativistic theory is free of conceptual difficulties on the phase-space level (a more extended discussion of this issue can be found in ref. [18]).

We are additionally motivated by our previous attempt [4,7] of placing a stochastic description of spin-$\frac{1}{2}$ in the relativistic setting, which in fact enforces the phase-space approach to stochastic mechanics (and eventually to quantum theory). However, a consistent phase-space derivation (not a formulation! that would place the problem in the Wigner function approach) of Nelson's stochastic mechanics is not available in the literature. Although some partial arguments in this direction can be found in ref. [19] (see also ref. [18] for more references).

Our hunch is that once the physical reality of particle trajectories [1,4,20] is accepted as the essential theoretical input in the quantum context, then there is no escape from the deepened phase-space discussion. Quite in analogy with the analysis of the Einstein–Smoluchowski versus the Ornstein–Uhlenbeck–Kramers description of the Brownian motion, we look for realistic physical phenomena (like e.g. energy–momentum transfers with strictly observed conservation laws) which are responsible for the particle destiny in for example the single particle interference experiments [20].

Before proceeding to a further phase-space discussion, let us mention the apparent problem arising in this approach (raised by a referee of the present paper): regular quantum mechanics suggests that, as long as the

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dynamics is represented by canonical transformations which are linear in position and momentum (i.e. quadratic Hamiltonians, Gaussian case), the phase-space approach is very convenient since the quantization procedure uses nothing but the classical information available on the system. The basic difficulty with the phase-space approach to stochastic mechanics is that even in the simpler Gaussian cases, it is not possible to exhibit a diffusion of Fenyes–Nelson without solving first the associated Schrödinger equation. No algorithm using exclusively the classical information on the system is known. This "strange" feature is due to the fact that the classical potential $V(x)$ is not, as suggested by stochastic mechanics, sufficient to characterise this diffusion. Hidden behind Nelson's formalism is the Bohm–Vigier quantum potential essential for this purpose. This is known for several years. It is however possible to present Nelson's approach without even mentioning the role of this quantum potential. This was the way followed by Nelson himself since 1966 in good faith. But this way drove him into inextricable difficulties regarding the lack of "locality" of stochastic mechanics, a hardly surprising feature for anyone familiar with the effects of the quantum potential.

In fact, motivated by refs. [1,3] we advocate a "physical rehabilitation" of the role of the quantum potential in Nelson's theory. In particular, we wish to understand the two kinds of well-defined diffusions known today as being associated with quantum mechanics, namely the Fenyes–Nelson (in real time) and Schrödinger–Bernstein ones, constructed in 1985 and 1986 [21] in the "Euclidian quantum mechanics" as *two different manifestations of physical phenomena taking place in the real time only*, with the notion of *complementary* (mutually compensating to satisfy the action–reaction principle by particle interacting with the diffusive medium) diffusions involved [3].

To illustrate the concept of complementarity let us exploit the following example due to Vigier: for an analogy one can consider a droplet of ink whose constitutive elements undergo Brownian motion in a diffuse medium (say a bucket of milk). The elements of the medium also undergo stochastic motions. Both fluids interact and undergo separately both drift and osmotic motions in general (these concepts are explicitly borrowed from the theory of osmotic diffusion whose example is provided by Nelson's stochastic mechanics). In equilibrium the milk is tainted. In terms of two types of diffusion (see refs. [1,3]), the ink elements undergo diffusion analysed in quantum terms, while the diffusion of milk elements is governed by most standard diffusion processes respecting the osmotic law of diffusion and it might be irreversible like in the case of the canonical free Brownian motion [1].

There is *no* phase-space theory of osmotic diffusion in the literature, as well as there is *no* phase-space derivation of Markov–Bernstein diffusion which seem to be the most general osmotic diffusions associated with Schrödinger wave mechanics. Therefore, our discussion must remain at the moment partly heuristic, with the goal to identify possible lines of attack for the completion of the phase-space programme.

Let us recall that for a particle with mass m following the relativistic phase-space Brownian motion in the course of which all energy–momentum fluctuations are of purely elastic origin (the problem of random accelerations in special relativity), the particle four-momentum remains on the hyperboloid $p_0^2-\boldsymbol{p}^2=m^2c^2$ for mass m. This means that the pertinent random motion is a diffusion process on the pseudo-Riemannian manifold [13,22,23] selected by demanding p_0 to be positive. In fact it is the well-known Lobachevsky space with a natural Lorentz invariant metric, where one-half of the Laplace–Beltrami operator serves as the diffusion generator.

In the nonrelativistic discussion of ref. [1], while reconciling the Brownian motion of a single particle with that of the particle ensemble, we have exploited certain concepts of the kinetic theory of gases. Elements of the relativistic kinetic theory [24,25] will be utilised below to some extent.

In the nonrelativistic situation, the classical dynamical system (we generalise here the discussion of ref. [1] by passing from $\mathbb{R}^1\times\mathbb{R}^1$ to $\mathbb{R}^3\times\mathbb{R}^3$ and incorporating external conservative forces from the very beginning)

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}=\frac{\boldsymbol{p}}{m},\qquad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}=\boldsymbol{F}=-\nabla V,\qquad V=V(\boldsymbol{x}) \tag{1}$$

is replaced by the white noise Langevin problem with friction. Below we shall admit both signs (±) of the external forces, which is motivated by our previous discussion [1,3] on links of Brownian motion with quantum dynamics and the *resulting concept of complementary diffusions: the relevant physical characteristic is here the reversal of the involved stochastic acceleration.*

$$\frac{d\boldsymbol{x}}{dt}=\frac{\boldsymbol{p}}{m},\qquad \frac{d\boldsymbol{p}}{dt}=\pm\boldsymbol{F}+\boldsymbol{\beta}(t)-\xi\boldsymbol{p}\,,$$

$$\boldsymbol{F}=-\nabla V,\qquad \langle\boldsymbol{\beta}(t)\rangle=0,\qquad \langle\beta_i(t)\beta_j(t')\rangle=\zeta\delta_{ij}\delta(t-t'),\qquad i,j,k=1,2,3\,. \tag{2}$$

It implies the evolution (Fokker–Planck–Kramers) equation for the joint position–momentum distribution associated with the statistical ensemble of diffusing particles:

$$\partial_t\Phi+(\boldsymbol{p}/m)\cdot\nabla_x\Phi\pm\boldsymbol{F}\cdot\nabla_p\Phi=\nabla_p\cdot(\xi\boldsymbol{p}\Phi)+\tfrac{1}{2}\zeta\triangle_p\Phi,\qquad \Phi=\Phi(\boldsymbol{x},\boldsymbol{p},t)\,. \tag{3}$$

Like in the case of the Boltzmann equation, the Cauchy or boundary problem (3) is not easy to solve and information about the statistical features of (2) is usually drawn from the hierarchy of the local conservation laws (moment equations) induced by (3).

By introducing the local moments of $\Phi(\boldsymbol{x},\boldsymbol{p},t)$,

$$\langle f\rangle_x=\frac{1}{w(\boldsymbol{x},t)}\int d^3p\,f(\boldsymbol{p})\Phi(\boldsymbol{x},\boldsymbol{p},t),\qquad w(\boldsymbol{x},t)=\int d^3p\,\Phi(\boldsymbol{x},\boldsymbol{p},t)\,, \tag{4}$$

the first two conservation laws take the form

$$\partial_t w=-\nabla\cdot(w\tilde{\boldsymbol{v}}),\qquad (\partial_t+\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{v}_i=\mp\frac{F_i}{m}-\frac{1}{w}\nabla_j P_{ij}-\xi\tilde{v}_i\,, \tag{5}$$

where the local velocity and the pressure tensor for the flow are given by

$$\tilde{\boldsymbol{v}}=\frac{\langle\boldsymbol{p}\rangle_x}{m},\qquad P_{ij}=\left(\frac{\langle p_i p_j\rangle_x}{m^2}-\tilde{v}_i\tilde{v}_j\right)w\,. \tag{6}$$

It is a crucial observation of ref. [1] (established through analysing the explicit solution for free Brownian propagation) that in the diffusion regime when we pass to the problem of random flights, the dominant contribution to the pressure tensor divergence equals $-\xi\tilde{v}_i w$ *so that the friction term is cancelled.* The remainder of $-(1/w)\nabla_j P_{ij}$ is then surprisingly

$$-\frac{1}{w}\nabla_j P_{ij}^{\mathrm{osm}}=-\nabla_i\tilde{Q}\,, \tag{7}$$

where

$$P_{ij}^{\mathrm{osm}}=D^2 w(\boldsymbol{x},t)\nabla_i\nabla_j\ln w(\boldsymbol{x},t),\qquad \tilde{Q}=2D^2\frac{\Delta w^{1/2}}{w^{1/2}},\qquad \zeta=D\xi^2\,, \tag{8}$$

and $\tilde{Q}(\boldsymbol{x},t)$ *except for the opposite* sign displays the functional dependence on $w(\boldsymbol{x},t)$ which is characteristic for the familiar Bohm–Vigier quantum potential arising in connection with the hydrodynamical (Madelung) description of quantum dynamics.

Accordingly, the momentum balance equation in the diffusion regime takes the form of the *frictionless* problem

$$(\partial_t+\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{v}_i=-\nabla_i\left(\tilde{Q}\pm\frac{V}{m}\right), \tag{9}$$

where both $\tilde{Q}$ and V might appear with signs opposite to what would conventionally happen in the case of the classical potential V and (Bohm–Vigier) quantum potential Q.

At this point we shall invoke the *complementarity* hypothesis [1] (whose validity was explicitly verified for the free Brownian diffusion) that, given the diffusion problem (5), (9), automatically defines the complementary diffusion-against-the-flow problem

$$\partial_t\rho = -\nabla\cdot(\rho \boldsymbol{v}), \qquad (\partial_t + \boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla\left(\mp\frac{V}{m} - Q\right), \tag{10}$$

where

$$Q_q = -Q = -2D^2\frac{\Delta\rho^{1/2}}{\rho^{1/2}}. \tag{11}$$

In ref. [1], in the context of free Brownian propagation, we established the existence of a nontrivial link between these two (e.g. complementary) diffusions: through scaling the current velocity by the time dependent factor $s=s(t)$ we introduce a new local clock for the diffusion and in addition map (9) into (10), (11). The compensating (exactly opposite) effects of complementary diffusions become then manifest. Since such velocity scaling appears in the study [26] of ergodic systems in thermal equilibrium with the reservoir, the above mapping might have quite deep roots worth a further analysis.

Let us stress that complementary diffusions in fact appear quite naturally in the discussion of Markov–Bernstein processes [21] but we augment the original Zambrini discussion by demanding the existence of the mapping linking (9) with (10) in general. The two types of diffusion then no longer deserve an independent existence and once we know (5), (9), the diffusion (10) is automatically defined and in reverse, both taking place in real time.

Let us pass to the relativistic generalisation of the above observations. In the finite difference approximation with small time increments we can evaluate the net change of Φ due to noise as follows:

$$\Phi(\boldsymbol{x}+(\boldsymbol{p}/m)\Delta t, \boldsymbol{p}\pm\boldsymbol{F}\Delta t, t+\Delta t) - \Phi(\boldsymbol{x},\boldsymbol{p},t) \simeq (\tfrac{1}{2}\zeta\Delta_p\Phi)\Delta t + \xi\nabla\cdot(\boldsymbol{p}\Phi)\Delta t. \tag{12}$$

In the case of the relativistic invariant [24,25] phase-space distribution $f(x, p)$, the role of Δt is taken by the proper time increment $c\Delta t = \gamma\Delta\tau$ with $\gamma = (1-v^2/c^2)^{-1/2}$, $\boldsymbol{v}=\boldsymbol{p}/m$. Then (12) should be replaced by

$$f(x+(p/m)\Delta\tau, p\pm F\Delta\tau) - f(x,p) \simeq (\text{noise term})\,\Delta\tau, \tag{13}$$

where

$$x=(ct,\boldsymbol{x}), \qquad p=(p_0,\boldsymbol{p}), \qquad p_\mu p^\mu = p_0^2 - \boldsymbol{p}^2 = m^2c^2,$$
$$p = mcu = m\mathrm{d}x/\mathrm{d}\tau, \qquad p_0 = mc_\gamma, \qquad \boldsymbol{p} = m_\gamma \boldsymbol{v}, \qquad F = \mathrm{d}p/\mathrm{d}\tau, \qquad F_\mu u^\mu = 0. \tag{14}$$

Apparently the noise term in (13) must be a relativistic invariant expression coming from the assumption that a Markovian random walk is taking place on the hyperboloid for mass m. As mentioned previously it amounts to studying diffusion on the pseudo-Riemannian differential manifold with the one-half Laplace–Beltrami operator [27] as the generator of diffusion.

Let us set $p_0>0$ on $p_\mu p^\mu = m^2c^2$ and introduce the hyperbolic parametrisation [23]

$$p_\mu p^\mu = R^2, \qquad p_0 = R\,\mathrm{ch}\,r, \qquad p^1 = R\,\mathrm{sh}\,r\sin\theta\cos\phi, \qquad p^2 = R\,\mathrm{sh}\,r\sin\theta\sin\phi, \qquad p^3 = R\,\mathrm{sh}\,r\cos\theta,$$
$$0\leqslant R<\infty, \qquad 0\leqslant r<\infty, \qquad 0\leqslant\theta<\pi, \qquad 0\leqslant\phi<2\pi. \tag{15}$$

It is instructive to notice that (15) emerges from the general polar parametrisation [28] $\{R, \upsilon_1, \upsilon_2, \upsilon_3\}$ upon formal identifications $\upsilon_1 = \mathrm{i}r$, $\upsilon_2=\theta$, $\upsilon_3=\phi$, where i is the imaginary unit effecting the map $\cos \mathrm{i}r = \mathrm{ch}\,r$, $\sin\mathrm{i}r = \mathrm{i}\,\mathrm{sh}\,r$.

By adopting the general formulas of ref. [28] to the hyperbolic case, we find that the Lorentz invariant differential operator $-\Box_p = \Delta_p - \partial^2/\partial p_0^2$ has a natural decomposition

$$-\Box_p = -\frac{1}{R^3}\frac{\partial}{\partial R}R^3\frac{\partial}{\partial R} + \frac{1}{R^2}\Delta_{\mathrm{LB}}, \tag{16}$$

where the Laplace–Beltrami operator $\triangle_{\rm LB}$ on the three dimensional $\{r, \theta, \phi\}$ manifold $u_\mu u^\mu = 1$ reads (compare e.g. also ref. [13])

$$\triangle_{\rm LB} = \frac{1}{\text{sh}^2 r}\left(\frac{\partial}{\partial r}\,\text{sh}^2 r\,\frac{\partial}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\,\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right). \tag{17}$$

The invariant volume element on $u_\mu u^\mu = 1$ is given by

$$\text{sh}^2 r \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi = \mathrm{d}^3 u/u^0, \qquad u_0 = (1+\boldsymbol{u}^2)^{1/2}. \tag{18}$$

Remark 1. Given a representation $g \to L(g)$ of the Lorentz group on the domain of $\triangle_{\rm LB}$, we have the following invariance property: $L(g)\triangle_{\rm LB} = \triangle_{\rm LB} L(g)$ where $L(g)f(\xi) = f(g^{-1}\xi)$.

By referring to the hyperbolic coordinates (7) we immediately realise that an explicit dependence on R of any function $f(p)$ is automatically eliminated once we pass to functions depending on p/R instead of p. Then

$$f(x, p) \to f(x, p/mc) = f(x, u), \qquad -\Box_p f(x, u) = \frac{1}{m^2 c^2}\triangle_{\rm LB} f(x, u). \tag{19}$$

The above property comes from the coordinate independent definition

$$\triangle_{\rm LB} f = \frac{1}{g}\frac{\partial}{\partial x^k}\left(g^{ik} g \frac{\partial f}{\partial x^i}\right), \qquad x^1 = r, \qquad x^2 = \phi, \qquad x^3 = \theta, \qquad g = [\det(g_{ij})]^{1/2}, \qquad (g^{ik})^{-1} = (g_{ik}),$$

$$g_{ij} = 0 \quad (i \neq j), \qquad g_{11} = 1, \qquad g_{22} = \text{sh}^2 r, \qquad g_{33} = \frac{\text{sh}^2 r}{\sin^2\theta}. \tag{20}$$

The Lorentz invariant analogue of the friction term needs some care. Apparently, we can at once introduce the coordinate independent object $\text{div}[B(\xi)f]$, where the divergence of a given vector field on the manifold M reads

$$\text{div}_{\rm M} X = \frac{1}{g}\frac{\partial}{\partial x^i}(g X^i), \qquad X^i = B^i(\xi, u) f(x, u). \tag{21}$$

It is however useful to recall that the nonrelativistic velocity increment leading to (12) has the form $\Delta \boldsymbol{v} = -[\xi \boldsymbol{v} - (\boldsymbol{F}/m)]\Delta t + \boldsymbol{B}(\Delta t)$, with $\boldsymbol{B}(\Delta t)$ representing the white noise contribution. When passing to special relativity, we need $u^\mu u_\mu = 1$ to be satisfied, hence the general (infinitesimal) four-velocity increment must obey $u^\mu \Delta u_\mu = 0$.

Obviously we expect $F^\mu u_\mu = 0$ to hold true for the Minkowski force, therefore the analogue $B^\mu(\xi, u)$ of the friction term $\xi \boldsymbol{u}$ must obey $B^\mu u_\mu = 0$ as well. A convenient choice of $B^\mu(\xi, u)$ is suggested by the general construction of the Minkowski (Lorentz) force, and we set

$$B^\mu(\xi, u) = B^{\mu\nu}(\xi, u) u_\nu, \qquad B^{\mu\nu} = -B^{\nu\mu}, \qquad B^{0i} = -B^{i0} = \xi v^i, \tag{22}$$

other components of the antisymmetric tensor $B^{\mu\nu}$ vanish. Then $B^\mu = B^{\mu\nu} u_\nu$ can be written as

$$B^0 = \xi \gamma \boldsymbol{v}^2/c, \qquad \boldsymbol{B} = \xi \gamma \boldsymbol{v}, \qquad \gamma = (1 - \boldsymbol{v}^2/c^2)^{-1/2}, \tag{23}$$

so that $B^\mu u_\mu = 0$ and the correct nonrelativistic limit is guaranteed in the manifestly relativistic invariant diffusion equation generalising (3)

$$c u^\mu \partial_\mu f(x, u) \pm \frac{F^\mu}{mc}\frac{\partial}{\partial u^\mu} f(x, u) = \frac{\zeta}{2m^2 c^2}\triangle_{\rm LB} f(x, u) + \frac{\partial}{\partial u^\mu}(B^\mu f)(x, u). \tag{24}$$

The random noise amplitude ζ and the friction coefficient ξ are relativistic scalars and appear in conformity with the nonrelativistic formulas (2)–(4).

Remark 2. Eq. (3) can be recovered from (24) in the nonrelativistic regime $|\boldsymbol{v}| \ll c$. By consulting refs. [24,25] one learns that the left-hand-side of (22) gets transformed into that of (3). The strong operator limit properties of the Laplace–Beltrami operator for $|\boldsymbol{v}|/c \to 0$ can be deduced from the heuristic argument $\mathrm{sh}\, r \simeq |\boldsymbol{v}|/c \ll 1$, hence $r \simeq |\boldsymbol{v}|/c$. Since then $\mathrm{ch}\, r \simeq 1$ we realise that $\triangle_{\mathrm{LB}} \to m^2c^2\triangle_p$. The $\mathbb{R}^3$ Laplacian arises explicitly in the spherical coordinates.

Like in the nonrelativistic case, instead of trying to solve the diffusion equation (24) on its own, we shall pass to the associated conservation laws for the local (configuration space conditioned) moments of the joint distribution $f(x, u)$.

Analogously to procedures effected with respect to the relativistic Boltzmann equation [24,25], we shall multiply both sides of (24) with appropriate polynomials in the four-velocities and then integrate them with respect to the invariant Riemann measure measure on $u_\mu u^\mu = 1$, $u_0 > 0$.

Eq. (24) differs from the relativistic Boltzmann equation in its right-hand-side where the binary collision term is replaced by the diffusion term, hence usual arguments about the collision invariants do not apply immediately. All necessary integration formulas pertaining to the left-hand-side of (24) can be directly borrowed from ref. [24]. We must only handle the respective integrals involving the diffusion term. For this purpose it is useful to invoke prop. 2.1, ch. X.2 of ref. [27]. Let M be a pseudo-Riemannian manifold and $\triangle$ the Laplace–Beltrami operator on M. $\triangle$ is a symmetric operator, that is

$$\int_{\mathrm{M}} W(x)(\triangle v)(x)\, \mathrm{d}x = \int_{\mathrm{M}} (\triangle w)(x) v(x)\, \mathrm{d}x\,, \tag{25}$$

if $\mathrm{d}x$ is the Riemannian measure on M, and w is infinitely differentiable, while v is a differentiable function of compact support. The support restriction can be relaxed, since for (25) to hold we need the divergence integral to vanish,

$$\int_{\mathrm{M}} \mathrm{div}(w\, \mathrm{grad}\, v - v\, \mathrm{grad}\, w)\, \mathrm{d}x = 0\,. \tag{26}$$

It can certainly be guaranteed if v decreases sufficiently rapidly outside a given compact (in our case with $r \to \infty$).

To specialise the above arguments to (24) we set $\mathrm{d}x = \mathrm{d}^3u/u_0$, $v = f(x, u)$ and next $w = 1$ or $w = mu^\mu$.

In case of $w = 1$, we have

$$\int \triangle f(x, u) \frac{\mathrm{d}^3u}{u_0} = 0\,. \tag{27}$$

For $w = mu^\mu$ we deal with

$$m \int u^\mu \triangle f(x, u) \frac{\mathrm{d}^3u}{u_0} = m \int (\triangle u^\mu) f(x, u) \frac{\mathrm{d}^3u}{u^0}\,. \tag{28}$$

To evaluate (28) we need only to know [10] that

$$-\frac{\zeta}{2m^2c^2} \triangle_{\mathrm{LB}} f(x, u) = a^{\mu\nu} \frac{\partial^2 f}{\partial u^\mu \partial u^\nu}\,, \tag{29}$$

since accordingly

$$\triangle_{\mathrm{LB}} u^{\mu}=0 \tag{30}$$

and thus (28) vanishes like the corresponding collision invariant expectation value in the kinetic theory of gases. We mention that $w=u^{\rho}u^{\sigma}$ would imply the integral (28) not to vanish.

The conservation laws (moment equations) induced by (24) do not significantly differ from those known in the standard relativistic kinetic theory [24,25]. Since for a differentiable function $w(u)$ there holds [24]

$$\int wu^{\mu}\frac{\partial f}{\partial x^{\mu}}\frac{\mathrm{d}^3u}{u_0}=\frac{\partial}{\partial x^{\mu}}\int fwu^{\mu}\frac{\mathrm{d}^3u}{u_0}-\int fu^{\mu}\frac{\partial w}{\partial x^{\mu}}\frac{\mathrm{d}^3u}{u_0}, \qquad \pm\int wF^{\mu}\frac{\partial f}{\partial u^{\mu}}\frac{\mathrm{d}^3u}{u_0}=\mp\int fF^{\mu}\frac{\partial w}{\partial u^{\mu}}\frac{\mathrm{d}^3u}{u_0} \tag{31}$$

and in addition to (30) we have

$$\int\frac{\partial}{\partial u^{\mu}}(B^{\mu}f)\frac{\mathrm{d}^3u}{u^0}=0\,,$$

the choice of $w=1$ implies

$$\partial_{\mu}\int u^{\mu}f(x,u)\frac{\mathrm{d}^3u}{u_0}=0\,, \tag{32}$$

while for $w=mu^{\mu}$ we get

$$\partial_{\nu}\int mc^2u^{\mu}u^{\nu}f(x,u)\frac{\mathrm{d}^3u}{u_0}=\int(\pm F^{\mu}-B^{\mu})f(x,u)\frac{\mathrm{d}^3u}{u_0}\,. \tag{33}$$

Notice that if F^{μ} is independent of u we get

$$\int F^{\mu}f(x,u)\frac{\mathrm{d}^3u}{u_0}=F^{\mu}\rho\,, \tag{34}$$

where $\rho(x)$ is a relativistic invariant quantity describing a reduced (space–time) probability distribution of particles diffusing in phase-space. If $F^{\mu}=F^{\mu}(x,u)$ we shall consider the simplest example linear in u of $F^{\mu}=u_{\nu}F^{\mu\nu}$ with $F^{\mu\nu}=F^{\mu\nu}(x)$ and then

$$\int F^{\mu}f(x,u)\frac{\mathrm{d}^3u}{u_0}=\rho F^{\mu\nu}v_{\nu}, \qquad v_{\nu}(x)=\frac{1}{\rho}\int u_{\nu}f(x,u)\frac{\mathrm{d}^3u}{u_0}\,. \tag{35}$$

With the notion of the configuration space conditioned local moment $v_{\nu}(x)$, we can rewrite (32) as the continuity equation

$$\partial_{\mu}(\rho v^{\mu})=0\,, \tag{36}$$

and (33) as

$$\partial_{\nu}T^{\mu\nu}=\pm\rho F^{\mu}-\rho b^{\mu}, \quad \text{or} \quad \pm\rho F^{\mu\nu}v_{\nu}-\rho b^{\mu}\,, \qquad b^{\mu}=\frac{1}{\rho(x)}\int\frac{\mathrm{d}^3u}{u_0}B^{\mu}(\xi,u)f(x,u)=b^{\mu}(x)\,. \tag{37}$$

Remark 3. The Minkowski space conditioned local moment $v_{\mu}(x)$ of $f(x,u)$ is not a genuine four-velocity unless normalised, $v_{\mu}\to v_{\mu}/(v_{\mu}v^{\mu})^{1/2}$. This normalisation is only one [25] of the ways utilised in the literature to introduce a satisfactory relativistic analogue of the hydrodynamical velocity of the flow.

The mean energy–momentum tensor $T^{\mu\nu}$ can be decomposed as follows (to be compared with the nonrelativistic formula),

$$T^{\mu\nu}=mc^2\rho v^\mu v^\nu+\int f(x,u)mc^2(u^\mu-v^\mu)(u^\nu-v^\nu)\frac{d^3u}{u_0}=mc^2(\rho v^\mu v^\nu+p^{\mu\nu}), \tag{38}$$

where $p^{\mu\nu}$ is called the stress tensor (pressure in the nonrelativistic case). Then (37) is transformed accordingly upon employing the continuity equation (36),

$$mc^2[v^\nu\partial_\nu v^\mu+(1/\rho)\partial_\nu p^{\mu\nu}]=\pm F^\mu-b^\mu, \quad \text{or} \quad \pm F^{\mu\nu}v_\nu-b^\mu. \tag{39}$$

At this point we shall directly exploit the nonrelativistic lesson (1)–(9) and *assume* that in the diffusion regime ($\Delta\tau\gg c/\xi$ when we pass to the relativistic problem of random flights) the remainder of $(1/\rho)\partial_\nu P^{\mu\nu}+b^\mu$ is the straightforward relativistic generalisation of (7), (8):

$$p^{\mu\nu}_{\rm osm}=(D/c)^2\rho\partial_\mu\partial_\nu\ln\rho, \qquad \zeta=D\xi^2. \tag{40}$$

Let us specify F^μ to be the Lorentz force affecting particles with charge e electromagnetically:

$$F^\mu=\frac{e}{c}F^{\mu\nu}v_\nu=\frac{e}{c}(\partial^\mu A^\nu-\partial^\nu A^\mu)v_\nu, \qquad \partial_\mu A^\mu=0, \qquad \pm e=\mp|e|\to\pm F=\mp\frac{|e|}{c}F^{\mu\nu}v_\nu \tag{41}$$

and assume that the following gradient field can be defined,

$$\partial^\mu S=mcv^\mu+(e/c)A^\mu. \tag{42}$$

With these definitions, (39) takes the following form,

$$\frac{1}{m}\left(\partial^\nu S-\frac{e}{c}A^\nu\right)\partial_\nu\left(\partial^\mu S-\frac{e}{c}A^\mu\right)+m(U^\nu\partial_\nu U^\mu+D\Box U^\mu)=\frac{e}{c}(\partial^\mu A^\nu-\partial^\nu A^\mu)\left(\partial_\nu S-\frac{e}{c}A_\nu\right), \tag{43}$$

where

$$U^\mu(x)=D\partial^\mu\ln\rho \tag{44}$$

generalises the nonrelativistic notion of the osmotic velocity [1,2]. Apparently (43) coincides with eq. 57 of ref. [8] (with the reservation that another metric $u_\mu u^\mu=-1$ is used there), if we only set $D=\hbar/2m$ and *with the choice of particles with charge* $e=-|e|$ (electrons) to be acted upon by the Lorentz force, *we change the sign* of the osmotic term $m(U^\mu\partial_\nu U^\mu+D\Box U^\mu)$ *into the opposite.*

Evidently, we are here enforced to consider the previously discussed (albeit on the nonrelativistic level) problem of the relationship between the two, apparently primordial, diffusion problems (9) and (10). To make the issue more explicit, let us study both cases $\mp(1/\rho)\nabla_\nu P^{\mu\nu}_{\rm osm}$ simultaneously, while adjusting the $\mp$ sign choice to our charge ($\pm e=\mp|e|$) convention implying the emergence of the force term $\pm F^\mu$,

$$\frac{1}{m}\left(\partial^\nu S\mp\frac{e}{c}A^\nu\right)\partial_\nu\left(\partial^\mu S\mp\frac{e}{c}A^\mu\right)\mp\left(\frac{1}{\rho}\partial_\mu P^{\mu\nu}_{\rm osm}\right)mc^2=\pm\frac{e}{c}F^{\mu\nu}\left(\partial_\nu S\mp\frac{e}{c}A_\nu\right)=F^\mu(\pm e)=\pm F^\mu. \tag{45}$$

This establishes a straightforward correspondence with (9) and (10). By employing the Lorentz gauge condition and the fact that $\partial_\mu S$ and $\partial_\mu\ln\rho$ are gradient fields, we can [8] integrate (45) with respect to x_μ with the result

$$\frac{1}{m}\left(\partial_\mu S\mp\frac{e}{c}A_\mu\right)\left(\partial^\mu S\mp\frac{e}{c}A^\mu\right)\mp 2mD^2[(\partial^\mu\ln\rho)(\partial_\mu\ln\rho)\times\tfrac{1}{2}+\Box\ln\rho]=M_\mp, \tag{46}$$

where $M_\mp$ is an integration constant.

By neglecting the noise (set formally $D=0$) we end up with a classical problem, which in the free motion case yields (see e.g. also ref. [8])

$$M_\pm=M=mc^2v^\mu v_\mu=mc^2. \tag{47}$$

Let us now pay attention to the role of the choice of the sign with which the potential term

$$Q_{\mp}=\mp 2D^2\Box\rho^{1/2}/\rho^{1/2}=\mp 2D^2[\Box\ln\rho+\tfrac{1}{2}(\partial^{\mu}\ln\rho)(\partial_{\mu}\ln\rho)]\,,\qquad \partial^{\nu}Q_{\mp}=\mp\frac{1}{\rho}\,\partial_{\mu}P^{\mu\nu}_{\mathrm{osm}} \tag{48}$$

appears in (43)–(46).

According to our previous considerations [1] held on the nonrelativistic level, Q_+ controls the standard Brownian diffusion, while Q_- is immediately identifiable as the Bohm–Vigier quantum potential and by ref. [1] controls the Brownian diffusion-against-the-flow. The latter provides an equivalent stochastic description of the Schrödinger dynamics.

Let us observe that a complex function

$$\psi=\exp(R+\mathrm{i}S/\hbar)=\rho^{1/2}\exp(\mathrm{i}S/\hbar) \tag{49}$$

satisfies (we deal with the positive energy case $p_0=mc\gamma>0$)

$$\left(\partial^{\mu}\pm\frac{\mathrm{i}e}{\hbar c}A^{\mu}\right)\left(\partial_{\mu}\pm\frac{\mathrm{i}e}{\hbar c}A_{\mu}\right)\psi=\left[\frac{\Box\rho^{1/2}}{\rho^{1/2}}-\frac{1}{\hbar^2}\left(\partial_{\nu}S\pm\frac{e}{c}A_{\nu}\right)\left(\partial^{\mu}S\pm\frac{e}{c}A^{\mu}\right)\right]\psi=-\frac{mM}{\hbar^2}\psi=-\frac{m^2c^2}{\hbar^2}\psi\,,$$

$$\partial_{\mu}(\rho v^{\mu}_{\pm})=0,\qquad v^{\mu}_{\pm}=\partial^{\mu}S\pm\frac{e}{c}A^{\mu}\,. \tag{50}$$

Hence

$$\left[\left(\partial^{\mu}\pm\frac{\mathrm{i}e}{\hbar c}A^{\mu}\right)\left(\partial_{\mu}\pm\frac{\mathrm{i}e}{\hbar c}A_{\mu}\right)+\frac{m^2c^2}{\hbar^2}\right]\psi=0 \tag{51}$$

is the energy–momentum balance equation when Q_- enters the game. By introducing the *real* function θ^* (we conform to the notations of ref. [21]

$$\theta^*=\exp(\bar{R}-\bar{S}/\hbar)=\bar{\rho}^{1/2}\exp(\bar{S}/\hbar)\,, \tag{52}$$

we obtain

$$\left(\partial^{\mu}\mp\frac{e}{\hbar c}A^{\mu}\right)\left(\partial_{\mu}\mp\frac{e}{\hbar c}A_{\mu}\right)\theta^*=\left[\frac{\Box\bar{\rho}^{1/2}}{\bar{\rho}^{1/2}}+\frac{1}{\hbar^2}\left(\partial_{\nu}\bar{S}\pm\frac{e}{c}A_{\nu}\right)\left(\partial^{\nu}\bar{S}\pm\frac{e}{c}A^{\nu}\right)\right]\theta^*=\frac{mM}{\hbar^2}\theta^*=\frac{m^2c^2}{\hbar^2}\theta^*\,,$$

$$\partial_{\mu}(\bar{\rho}\bar{v}^{\mu}_{\pm})=0,\qquad \bar{v}^{\mu}_{\pm}=\partial^{\mu}\bar{S}\pm\frac{e}{c}A^{\mu}\,. \tag{53}$$

Hence with Q_+ (e.g. with the relativistic generalisation of the standard Brownian diffusion) in hands, we arrive at the generalised theory of heat transport with the energy–momentum balance equation

$$\left[-\left(\partial^{\mu}\mp\frac{e}{\hbar c}A^{\mu}\right)\left(\partial_{\mu}\mp\frac{e}{\hbar c}A_{\mu}\right)+\frac{m^2c^2}{\hbar^2}\right]\theta^*=0\,. \tag{54}$$

By passing to the proper time Schrödinger equation and investigating its stationary (in τ) solutions with

$$\psi(x,\tau)=\exp(-\mathrm{i}mc^2\tau/2\hbar)\rho^{1/2}\exp(\mathrm{i}S/\hbar)\,, \tag{55}$$

we recover $(D=\hbar/2m)$

$$\mathrm{i}\partial_{\tau}\psi=-D\left(\partial^{\mu}\pm\frac{\mathrm{i}e}{\hbar c}A^{\mu}\right)\left(\partial_{\mu}\pm\frac{\mathrm{i}e}{\hbar c}A_{\mu}\right)\psi \tag{56}$$

or, respectively, its thermal (heat transport) version

$$\partial_\tau \theta^* = D\left(\partial^\mu \mp \frac{e}{\hbar c} A^\mu\right)\left(\partial_\mu \mp \frac{e}{\hbar c} A_\mu\right)\theta^* . \tag{57}$$

Zambrini's generalised heat transport equation [21]

$$\partial_t \theta^* = D\triangle\theta^* - \frac{V}{2mD}\theta^* \tag{58}$$

defines the nonrelativistic Q_+ Brownian diffusion, which is complementary to the Q_- (i.e. Schrödinger) one

$$\mathrm{i}\partial_t \psi = -D\triangle\psi + \frac{V}{2mD}\psi \tag{59}$$

in the framework of the theory of Markov–Bernstein processes.

The complementarity hypothesis in the relativistic setting allows one to form pairs of complementary diffusions characterised by the choice of the sign for the charge and for the osmotic potential Q. They are $\{+e, Q_-\}$ with $\{-e, Q_+\}$ and $\{-e, Q_-\}$ with $\{+e, Q_+\}$.

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