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Derivation of the quantum potential from realistic Brownian particle motions

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We present in this Letter a detailed analysis of mechanisms by which the phase space Brownian motion of an ensemble of massive particles, in the diffusion regime, is governed by the Schrödinger equation. It is explicitly shown how the pressure of the diffusing ensemble is linked to the quantum potential, known to appear in the Hamilton–Jacobi–Madelung formulation of the corresponding quantum dynamics. The quantum state vector (wave function) corresponds in this picture to the physically real diffusing medium governing the collective evolution of the particle ensemble.

1. Motivation

In the framework of Nelson's reformulation of quantum mechanics [1] the notion of the particle path acquires a well defined meaning on the level of configuration space motions. One can view it as a sample trajectory [2–4] followed by the mass point undergoing the Markovian diffusion process in $\mathbb{R}^3$ with dynamics constrained by the second Newton law in the conditional mean.

While trying to understand this model of quantum phenomena on physically deeper grounds, one is tempted to search for a random phase space propagation, whose configuration space projection would imply stochastic mechanics. This would amount to a phase space derivation of Nelson's mean acceleration formula.

A deep analysis of the links between Einstein–Smoluchowski (configuration space) and Langevin (phase space) descriptions of the Brownian motion was made in the expository reference [5]. As one knows the large friction regime of the Ornstein–Uhlenbeck process allows for the Smoluchowski approximation (e.g. spatial diffusion) in the case of the general external force, albeit with the applicability limited to the Fick law governed cases.

Alternatively, attempts to derive stochastic mechanics in the framework of the so-called stochastic electrodynamics [6,7] indicate that in the Markovian approximation of processes with short correlation times (high friction) one should disregard friction at all to arrive eventually at Nelson's formalism.

It thus seems that the search for a realistic phase space justification of stochastic mechanics ends up with a confusing picture involving both high and low friction limits of Markovian stochastic processes.

Our purpose is to view stochastic mechanics (and eventually quantum mechanics) as a special version of the general problem of random flights [8] which arises in the diffusion approximation of the phase space random motion. To get a clear idea of the links [9] between quantum mechanics and the conventional Brownian motion it seems advantageous to analyse any solvable model in detail. For clarity of discussion (not hampering the generalisations) we shall confine our attention to the case of freely moving Brownian particles in one space dimension.

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2. Brownian motion of a single particle

The general form of the joint probability distribution $W(x, u, t)$ for a freely moving Brownian particle which at $t=0$ begins its motion at $x_0=0$ with an arbitrary velocity u_0 is derived in ref. [8] under the assumption of the maximally symmetric displacement probability law:

$$W(x,u,t)=W(R,S)=[4\pi^2(FG-H^2)]^{-1/2}\times\exp\left(-\frac{GR^2-2HRS+FS^2}{2(FG-H^2)}\right), \quad (1)$$

where $R=x-u_0(1-e^{-\beta t})\beta^{-1}$, $S=u-u_0e^{-\beta t}$ while

$$F=\frac{D}{\beta}(2\beta t-3+4e^{-\beta t}-e^{-2\beta t}),$$

$$G=D\beta(1-e^{-2\beta t}),\quad H=D(1-e^{-\beta t})^2. \quad (2)$$

β is the friction coefficient, D will appear later to be the spatial diffusion constant, $D=k_BT/m\beta$.

The marginal distribution of velocities

$$w(u,t)=\int dx\, W(x,u,t)=(2\pi G)^{-1/2}\exp(-S^2/2G)=w(S) \quad (3)$$

in the large friction regime (alternatively at times t much larger than the relaxation time β^{-1}) takes the conventional form

$$w(u,t)=\left(\frac{m}{2\pi kT}\right)^{1/2}\exp\left(-\frac{mu^2}{2k_BT}\right), \quad (4)$$

characterising diffusions in velocity space.

Analogously the marginal space configuration

$$w(x,t)=\int du\, W(x,u,t)=(2\pi F)^{-1/2}\exp(-R^2/2F)=w(R) \quad (5)$$

in the diffusion regime gives rise to the familiar heat kernel

$$w(x,t)=(4\pi Dt)^{-1/2}\exp(-x^2/4Dt), \quad (6)$$

solving $\partial_t w=D\triangle w$.

Let us now evaluate the first local moment of the joint distribution:

$$\langle u\rangle=\int du\, uW(x,u,t)=w(R)\,[(H/F)R+u_0e^{-\beta t}], \quad (7)$$

which in the diffusion regime reduces to

$$\langle u\rangle=w(x,t)\frac{x}{2t}. \quad (8)$$

The local (configuration space conditioned) moment of W finally reads

$$\langle u\rangle_x=\frac{x}{2t}=-D\frac{\nabla w(x,t)}{w(x,t)}. \quad (9)$$

For the second moment we have

$$\int du\, u^2W(x,u,t)=\int dS\, S^2W(R,S)+2u_0e^{-\beta t}\int dS\, SW(R,S)+u_0^2e^{-2\beta t}\int dS\, W(R,S). \quad (10)$$

In the diffusion regime the leading contribution comes from $\int dS\, S^2W(R,S)$ which can be evaluated by means of handy formulas given in ref. [10]:

$$\int dS\, S^2W(R,S)=\left(\frac{FG-H^2}{F}+\frac{H^2}{F^2}R^2\right)\times(2\pi F)^{-1/2}\exp(-R^2/2F). \quad (11)$$

In the diffusion regime we thus obtain

$$\langle u^2\rangle=\left(\frac{D(2\beta t-1)}{2t}+\frac{x^2}{4t^2}\right)w(x,t) \quad (12)$$

and the configuration space conditioned (local) moment of $W(x, u, t)$ takes the form

$$\langle u^2\rangle_x=(D\beta-D/2t)+\langle u\rangle_x^2. \quad (13)$$

The transport (Fokker–Planck) equation governing the time development of $W(x, u, t)$ reads

$$\partial_t W+u\nabla_x W=\beta\nabla_u(Wu)+q\triangle_u W,\quad q=D\beta^2 \quad (14)$$

and implies the local conservation laws

$$\partial_t w+\nabla(\langle u\rangle_x w)=0,$$

$$\partial_t(\langle u\rangle_x w)+\nabla_x(\langle u^2\rangle_x w)=-\beta\langle u\rangle_x w. \quad (15)$$

Introducing

$$P(x,t)=(\langle u^2\rangle_x-\langle u\rangle_x^2)w(x,t) \quad (16)$$

and assuming that $w(x, t)$ has no zeros, we can isolate the leading contribution to P in:

$$\partial_t w = -\nabla(\langle u\rangle_x w) ,$$

$$(\partial_t + \langle u\rangle_x \nabla)\langle u\rangle_x = -\beta\langle u\rangle_x - \nabla P/w . \qquad (17)$$

In the diffusion regime we have

$$-\frac{\nabla P}{w} = -\beta D\frac{\nabla w}{w} + \frac{D}{2t}\frac{\nabla w}{w} , \qquad (18)$$

which when inserted into the above momentum conservation law gives rise to the *cancellation of the friction term* (!): *our process effectively appears to be frictionless although in fact operating in the high friction regime.*

We have here

$$(\partial_t + \langle u\rangle_x \nabla)\langle u\rangle_x = \frac{D}{2t}\frac{\nabla w}{w} , \qquad (19)$$

where it is instructive to notice that

$$\frac{D}{2t}\frac{\nabla w}{w} = -\frac{1}{w}\nabla P_{\text{osm}} , \quad P_{\text{osm}} = D^2 w \Delta \ln w . \qquad (20)$$

Since $\nabla P_{\text{osm}}/w$ is irrotational we can easily recover the corresponding potential (the general discussion is given in refs. [1,9])

$$Q = Q(x,t) = 2D^2 \frac{\Delta w^{1/2}}{w^{1/2}} ,$$

$$(\partial_t + \langle u\rangle_x \nabla)\langle u\rangle_x = -\frac{1}{w}\nabla P_{\text{osm}} = -\nabla Q . \qquad (21)$$

$Q(x, t)$ has the familiar form of the Bohm–Vigier quantum potential (set $\hbar/2m = D$) *except for the opposite sign.*

Irrespective of the sign, however, holds $\nabla w \sim \nabla P_{\text{osm}} = w\nabla Q$. The link of Q and P_{osm} has been known for a long time [9,11,12] in connection with the hydrodynamical models of quantum mechanics, where, *however*, a given particle distribution $\rho(x, t) = |\psi(x, t)|^2$ is associated with

$$P_{\text{q}} = -D^2 \rho \Delta \ln \rho \quad \leftrightarrow \quad Q_{\text{q}} = -2D^2 \frac{\Delta \rho^{1/2}}{\rho^{1/2}} ,$$

$$\partial_t \rho = -\nabla(\rho v) , \quad (\partial_t + v\nabla)v = -\nabla Q_{\text{q}} ,$$

$$v = 2D\nabla \ln(\psi/\rho^{1/2}) , \quad D = \hbar/2m . \qquad (22)$$

3. Brownian evolution of particle ensembles

All previous derivations refer to a single particle suffering random displacements according to the maximally symmetric probability law, which uniquely characterises the random medium.

However [8], it is clearly allowed instead to imagine a very large number of particles starting under the same initial conditions and undergoing Brownian displacements without any mutual interference according to the same probability law.

Such an ensemble can be built as well of sample flights consecutively executed and having flight duration time t: even if not perfectly realisable in practice, such an ensemble can be easily produced by means of a computer simulation.

Once passing to the ensemble picture, we can consistently follow the lore of the kinetic theory of gases and e.g. understand $P(x, t) = \langle u^2\rangle - \langle u\rangle_x^2 w = (D\beta - D/2t)w$ as the *pressure* exerted at point x (on the ensemble average!). This concept can be exclusively attributed to the swarm of particles, where each member independently follows a Brownian motion according to the same displacement probability law.

It is a canonical statement of the *macroscopic* theory of diffusion [8] that if $w(x, t)$ denotes the concentration of diffusing substance at x at time t, then the amount escaping through a given point from the area of larger concentration, per time Δt is given by

$$-D\nabla w(x,t)\,\Delta t = -u(x,t)w(x,t)\,\Delta t . \qquad (23)$$

Here $u(x, t)$ is called [1,5] the osmotic velocity, $u = D\nabla w/w$. In our case apparently $u(x, t) = -x/2t = -\langle u\rangle_x$ and since $\langle u\rangle_x$ is the local velocity of the particle flow, we realise that (23) does refer indeed to the *osmotic transport*. In its course the particle swarm expands at the expense of lowering its concentration by the obvious reason: there are more particles leaving the region of higher density than entering it.

For a deeper understanding of the previously discussed striking affinity between Q and Q_{q} let us consider the particle ensemble, whose initial (at time t_0) spatial distribution reads

$$\rho_0(x) = (\pi\alpha^2)^{-1/2} \exp(-x^2/\alpha^2) . \qquad (24)$$

We assume that our particles have approached this distribution *in the course of the well defined phase* *space evolution*. Irrespective of its detailed nature (Brownian, classical etc.) we can safely admit that a certain mean velocity field is given in parallel to $\rho_0(x)$:

$$\langle u\rangle_x^0 = V(x, t_0) = V_0(x) . \qquad (25)$$

We then address the following problem: what is the probability to reach a Δx neighbourhood of a certain point x at time $t>t_0$, if particles are *conditioned* to emanate from x_0 at time t_0 with the *mean* velocity $V_0(x_0)$, and the propagation is Brownian?

The universal maximally symmetric Brownian probability law for spatial displacements ΔR in the diffusion approximation coincides with the heat kernel expression

$$w(\Delta R) = (4\pi D\Delta t)^{-1/2} \times \exp[-(\Delta R)^2/4D\Delta t] . \qquad (26)$$

However, what ΔR is needs to be carefully specified since it depends on the *mean* velocity value at the reference point (source of particles). Apparently, if we start from x_0 with a member of the $V_0(x_0)$ beam then after time Δt the particle will reach the point x,

$$x = x_0 + V_0(x_0)\,\Delta t + \Delta R , \qquad (27)$$

where ΔR is the purely random contribution. Accordingly we must write

$$\Delta R = x - x_0 - V_0(x_0)\,\Delta t , \qquad (27')$$

thus arriving at the distorted [8,2,3] infinitesimal Brownian propagator. This displacement probability law induces [8] the diffusion equation

$$\partial_t w(x, t) = -V_0 \nabla w(x, t) + D\Delta w(x, t) , \qquad (28)$$

which is solved by the Brownian kernel

$$w_{x_0}(x, t) = [4\pi D(t-t_0)]^{-1/2} \times \exp\left(-\frac{[x - x_0 - V_0(x_0)(t-t_0)]^2}{4D(t-t_0)}\right) . \qquad (29)$$

For simplicity we shall disregard t_0 in the above formula, which provides us with the answer to the problem posed before.

It is, however, remarkable that our isolated problem can be easily extended (by varying x_0) to the general issue of the Brownian propagation of initially given particle distributions with non-trivial initial velocity fields. For particles *conditioned* to pass at t_0 an arbitrary point x' with the mean velocity $V_0(x')$ we develop a Brownian evolution *anew* (this situation has very much in common with the problem of repeated measurements in stochastic mechanics [13]).

The corresponding propagation formula (with $t_0=0$) reads

$$\rho(x, t) = \int dx'\, w_{x'}(x, t)\rho_0(x') = (4\pi^2\alpha^2 Dt)^{-1/2} \times \int dx' \exp\left(-\frac{[x - x' - V_0(x')t]^2}{4Dt} - \frac{x'^2}{\alpha^2}\right) . \qquad (30)$$

Consequently the Brownian evolution of the particle ensemble $\rho_0(x) \to \rho(x, t)$ quite sensitively depends on the initial mean velocity field, to be contrasted with the single particle Brownian propagation which is insensitive to the initial particle velocity u_0 in the diffusion regime. Hence, how the ensemble propagates due to random fluctuations depends on the *phase space preparation procedure* which not only fixes $\rho_0(x)$ but supplements it with a mean velocity field $V_0(x)$.

In particular, if the particle ensemble was Brownian prepared (through Brownian evolution from $x_0=0$ and arbitrary initial velocity u_0) then we should set $\alpha^2=4Dt_0$ and the osmotic flow is characterised by $V_0(x)=x/2t_0$.

Let us consider the ensemble preparation procedure giving rise to $\rho_0(x)$ and also to

$$V_0(x) = \gamma 2Dx/\alpha^2 , \qquad (31)$$

with $\gamma \in \mathbb{R}^1$ left unspecified at the moment. The Gaussian propagation integral can be immediately evaluated. We have

$$\rho(x, t) = \frac{\alpha}{\{\pi[\alpha^4 + 4Dt\alpha^2(1+\gamma) + \gamma^2 4D^2t^2]\}^{1/2}} \times \exp\left(-\frac{x^2\alpha^2}{\alpha^4 + 4Dt\alpha^2(1+\gamma) + \gamma^2 4D^2t^2}\right) . \qquad (32)$$

The choice of $\gamma=1$ produces the Brownian diffusion of the Brownian prepared $V_0(x)=x/2t_0=2Dx/\alpha^2$ ensemble, while the choice $\gamma=-1$ apparently leads to (see also ref. [14])

$$\rho(x,t)=\frac{\alpha}{[\pi(\alpha^4+4D^2t^2)]^{1/2}}\times\exp\left(-\frac{x^2\alpha^2}{\alpha^4+4D^2t^2}\right), \qquad (33)$$

which is a well known probability distribution associated with the solution of the Cauchy problem

$$\mathrm{i}\partial_t\psi(x,t)=-D\Delta\psi(x,t),$$
$$\psi(x,0)=(\pi\alpha^2)^{-1/4}\exp(-x^2/2\alpha^2). \qquad (34)$$

Indeed

$$\psi(x,t)=\int \mathrm{d}x'\ G(x-x',t)\psi(x',0)$$
$$=(4\pi\mathrm{i}Dt)^{-1/2}\int \mathrm{d}x'\ \exp\left(-\frac{(x-x')^2}{4\mathrm{i}Dt}\right)\psi(x',0)$$
$$=(\alpha^2/\pi)^{1/4}(\alpha^2+2\mathrm{i}Dt)^{-1/2}\times\exp\left(-\frac{x^2}{2(\alpha^2+2\mathrm{i}Dt)}\right) \qquad (35)$$

and $|\psi(x,t)|^2=\rho(x,t)$ in agreement with (33).

The initial (reversed osmotic) velocity field $\langle u\rangle_x=-2Dx/\alpha^2$ is not conspicuously present in (35) but its time evolution is provided by Nelson's osmotic velocity formula [1]

$$u(x,t)=D\nabla\ln\rho,$$
$$u(x,0)=-\frac{2Dx}{\alpha^2}\ \rightarrow\ u(x,t)=-\frac{2D\alpha^2x}{\alpha^4+4D^2t^2}. \qquad (36)$$

By (35) we have here developed a particle current with the velocity (called [1,5] current velocity)

$$v(x,t)=2D\nabla\ln(\psi/\rho^{1/2}),$$
$$v(x,0)=0\ \rightarrow\ v(x,t)=\frac{4D^2tx}{\alpha^4+4D^2t^2}, \qquad (37)$$

which together with (36) yields

$$\frac{m}{2}\int \mathrm{d}x\,\rho(x,t)(u^2+v^2)(x,t)=\text{const},$$

compare e.g. ref. [12] where the hydrodynamical discussion is given and ref. [1] for a discussion from the stochastic mechanics viewpoint. The current velocity solves the transport equation

$$(\partial_t+v\nabla)v=-\nabla Q_q=\nabla[2D^2(\Delta\rho^{1/2})/\rho^{1/2}]. \qquad (38)$$

Q_q refers to the osmotic diffusion process against the initial flow.

Remark 1. Eq. (34) is a conventional Schrödinger equation, if we set $D=\hbar/2M$. Here [4] we can identify M with the mass m of diffusing particles, which fixes the diffusion constant. However, we can as well choose D to be the universal constant and then introduce an effective mass $M=\hbar/2D$.

Remark 2. The Brownian prepared ensemble evolution with $\gamma=1$ converges to the $\gamma=-1$ evolution for larger times. Indeed then the term $8Dt\alpha^2$ can be neglected against $4D^2t^2$.

Remark 3. If the ensemble preparation procedure is tuned to the background random field to provide $\rho_0(x)$ with the exact reversal of the osmotic flow, then the Brownian evolution of such an ensemble is governed by the Schrödinger equation. This amounts to analysing the diffusion process in terms of average (collective) flows. One flow is directed towards the main concentration, another is born due to the Brownian fluctuations and transports particles away from the main concentration, hence a diffusion process against the initially given flow appears.

The process respects the energy conservation law in the mean:

$$\frac{m}{2}\int \mathrm{d}x\,\rho(x,t)\ (u^2+v^2)(x,t)=\text{const}.$$

In the course of the Brownian propagation two competing flows combine into the mean drift, which asymptotically is dominated by the outgoing current. The quantum potential Q_q appears to be a mathematical encoding of such a diffusion process.

Remark 4. The initial phase space distribution to which remark 3 applies can be immediately obtained from the Brownian solution (1) if we specialise it to time t_0 and formally replace H by $-H$. Keeping in mind that we are interested in the $t_0\gg\beta^{-1}$ regime and setting $t_0=\alpha^2/4D$ the form of $W_0(x,u)$ reads

$$W_0(x,u)=(2\pi^2D\beta\alpha^2)^{-1/2}\times\exp\left(-\frac{4Dx^2+8Dxu+2\alpha^2u^2}{4D\beta\alpha^2}\right). \qquad (39)$$

By (3), (5) one easily obtains $\rho_0(x)$, (26) and $\langle u\rangle_x^0=D\nabla\rho_0/\rho_0$ to be compared with (9).

4. Brownian evolution of particle ensembles: phase space picture

To complete the arguments of section 3 we shall present a parallel phase space description of the Brownian ensemble propagation, for the case $V_0(x)=-(2D/\alpha^2)x$. The more general choice (31) can be easily investigated by following the pattern.

The initial phase space distribution (39) is to be propagated by the Brownian kernel of the form (1), except for another choice of R and S. Namely we need

$$R=x-x_0-V_0(x_0)t-u_0(1-e^{-\beta t})\beta^{-1}\,,$$
$$S=u-V_0(x_0)-u_0 e^{-\beta t}\,. \qquad (40)$$

We consider

$$\int dx_0\, du_0\, W(x,u,t;x_0,u_0)\, W_0(x_0,u_0) \doteq W(x,u,t)\,. \qquad (41)$$

In the diffusion regime the u_0 contributions can be neglected, and the effective du_0 integration yields

$$W(x,u,t)=\int dx_0\, W(x,u,t;x_0)\rho_0(x_0)\,. \qquad (42)$$

Apparently (compare e.g. (29), (30))

$$\int du\, W(x,u,t)=\int du \int dx_0\, W(x,u,t;x_0)\rho_0(x_0)$$
$$=\int dx_0\, w_{x_0}(x,t)\rho_0(x_0)=\rho(x,t)\,. \qquad (43)$$

We can evaluate the local expectation value with respect to u by the same interchange of the integration order procedure:

$$\langle u\rangle=\int du\, uW(x,u,t)$$
$$=\int dx_0 \int du\, uW(x,u,t;x_0)\rho_0(x_0)$$
$$=\int dx_0\, V_0(x_0)\, w_{x_0}(x,t)\rho_0(x_0)\,. \qquad (44)$$

Setting $V_0(x_0)=-(2D/\alpha^2)x_0$ we can proceed further to arrive at

$$\langle u\rangle=-\frac{2D}{\alpha^2}(4\pi^2\alpha^2Dt)^{-1/2}\int dx_0\, x_0 \times\exp\left(-\frac{[x-x_0+(2D/\alpha^2)x_0t]^2}{4Dt}-\frac{x_0^2}{\alpha^2}\right)$$
$$=-\frac{2Dx(\alpha^2-2Dt)}{\alpha^4+4D^2t^2}\rho(x,t)\,, \qquad (45)$$

which implies

$$\langle u\rangle_x=\langle u\rangle/\rho(x,t)=u(x,t)+v(x,t)\,, \qquad (46)$$

where u and v are given by (36), (37) respectively.

In Nelson's notation [1] $\langle u\rangle_x=b(x,t)$ coincides with the forward drift of the Markovian diffusion process.

With the explicit form of ρ, v, u, and b in our hands we can straightforwardly derive the conservation laws respected by our diffusion. The following holds,

$$\partial_t\rho=D\triangle\rho-\nabla(\rho b)\,,$$
$$\partial_t b+b\nabla b=0 \qquad (47)$$

and simultaneously

$$\partial_t\rho=-\nabla(\rho v)\,,$$
$$\partial_t v+v\nabla v=-\nabla Q_q\,, \qquad (48)$$

which completely characterises the diffusion process. We would like to emphasise the classical (Liouville) propagation formula for $b(x,t)=\langle u\rangle_x$ in the absence of external forces. We have thus arrived at the notion of the quantum potential (Q_q) governed diffusion process, by means of purely phase space (Brownian) arguments. The striking affinity between Q and Q_q governed diffusions (cf. section 2) becomes here transparent, if we make a time substitution

$$4D\tau=\alpha^2+4D^2t^2/\alpha^2\,, \qquad (49)$$

which amounts to changing the clock ($t\to\tau(t)$) according to which the stochastic process is executed.

Eq. (49) implies (the notations of section 2 are strictly followed)

$$\rho(x,t)=w(x,\tau)=(4\pi D\tau)^{-1/2}\exp(-x^2/4D\tau)\,,$$
$$\langle u\rangle_x=x/2\tau=-u(x,t)\,, \qquad (50)$$

where apparently

$$\partial_\tau w=-\nabla(\langle u\rangle_x w)\,,$$
$$(\partial_\tau+\langle u\rangle_x\nabla)\langle u\rangle_x$$
$$=-\nabla Q(x,\tau)=\nabla Q_q(x,t) \qquad (51)$$

and

$$Q(x,\tau)=2D^2\frac{\triangle w^{1/2}}{w^{1/2}}(x,\tau)=-Q_q(x,t)\,,$$

$$P_{\rm osm}(x,\tau) = (D^2 w \Delta \ln w)(x,\tau) = -P_{\rm q}(x,t)\,. \quad (52)$$

By setting $4D\tau_0=\alpha^2$ we relate (50) with the initial distribution (24). The following holds,

$$w(x,\tau) = \int dx'\, w(x-x', \tau-\tau_0) w(x',\tau_0)\,, \quad (53)$$

which is a propagation formula characteristic for free diffusion.

Hence, while $Q_{\rm q}(x,t)$ governs the Brownian diffusion against-the-flow, within a specifically prepared (conditioning!) particle ensemble, it appears that $Q(x,\tau) = -Q_{\rm q}(x,t)$ governs a conventional, unrestricted (no conditioning) diffusion process of section 2, albeit with respect to another clock, (49). The phase space implementation is obvious.

Diffusion processes (48) and (51) are here *inseparably connected* and live simultaneously, providing a suggestive illustration of the action–reaction principle. The "quantum pressure" $P_{\rm q}(t)$ is exactly balanced by the osmotic pressure $P_{\rm osm}(\tau(t))$, the osmotic flow $\langle u \rangle_x(x,\tau)$ balances the remainder $u(x,t)$ of the initial particle flow. Together with the conservation law $\int dx\,(u^2+v^2)\rho(x,t) = \text{const}$ it seems to establish a link with the ideas of ref. [15], where particle–medium interaction was acting both ways (action–reaction) with a strictly observed energy–momentum conservation law at each minute scattering event.

5. Conclusions

In the above context of the essentially Brownian implementation of the quantum mechanical time development, let us stress the fundamental importance of both theoretical and experimental investigation of the particle trajectory concept in the realm of quantum theory [16–18]. In our discussion particle trajectories ("hidden variables") primarily arise as phase space Brownian paths, hence they refer to the non-trivial energy–momentum exchanges between the particle and the surrounding diffusing medium; the quantum potential refers thus in the diffusion regime to a quite realistic physical phenomenon of random accelerations. Consequently any idea of "ghost information" or "empty wave" propagation is inconsistent with this picture. Quantum particles really move in a "hidden thermostat", to recall de Broglie's famous suggestion. [19].

On the other hand we give further support to the idea that quantum mechanical particles are transported along realistic space time trajectories with realistic momenta in plain opposition to the Copenhagen convention.

Quantum state vectors (wave functions) correspond in this model to real physical phenomena which are governed by the background random field. They encode the mean (collective) data of the diffusion process. By (26)–(29) one can here introduce a concept of the stochastic control fields. Indeed, in the description of particle ensembles we have finally arrived at the field of locally defined displacement probability laws, which control the diffusion in the infinitesimal neighborhood of each given point. One can thus tell that $\psi(x,t)$ is related to the specific control field (state of the diffusing medium) from which the quantum mechanical statistics (e.g. Born postulate) does originate.

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