RANDOM VERSUS DETERMINISTIC PATHS IN STOCHASTIC MECHANICS

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We discuss the uniqueness conditions for the derivation of Schrödinger wave functions within the Fenyes–Nelson formalism. The primary concept of the transition probability density induces the probabilistic path summation formula with respect to the random (sample) paths of the diffusion process. Bohm–Vigier causal trajectories arise as the mean displacement paths of the underlying process.

Fenyes–Nelson stochastic mechanics [1] is one of the very few [1–4] attempts to reconcile the individual particle trajectory notion with the wave (Schrödinger) theory of quantum phenomena. According to the recipes of ref. [1] to a given solution of the Schrödinger equation one can in principle attribute a stochastic diffusion process satisfying Newton's second law in the mean. The corresponding stochastic differential equation describes a propagation of a point particle through a non-dissipative random medium. Sample paths of the process can be approximately identified with the realistic configuration space paths of (perhaps) physical particles.

For a quantum particle in a conservative force field we have

$$\mathrm{i}\hbar\partial_t\psi(\boldsymbol{x},t)=\frac{\hbar^2}{2m}\triangle\psi(\boldsymbol{x})+V(\boldsymbol{x})\psi(\boldsymbol{x},t)\,, \tag{1}$$

which implies the continuity equation for $\rho=|\psi|^2$,

$$\partial_t\rho=-\operatorname{div}\boldsymbol{j}\,,\quad \boldsymbol{j}=\frac{\hbar}{2m\mathrm{i}}(\bar{\psi}\nabla\psi-\psi\nabla\bar{\psi})\,. \tag{2}$$

In the case of nowhere zero ψ (locally at least, there are existence proofs for singular diffusions), upon the standard substitution $\psi=\exp(R+\mathrm{i}S)$, (2) goes over into

$$\partial_t\rho=\operatorname{div}[-(\hbar/m)\nabla S\rho]=\frac{\hbar}{2m}\triangle\rho-\operatorname{div}\rho\boldsymbol{b}, \tag{3}$$

where

$$\rho=\exp(2R)\,,\quad \boldsymbol{b}=\boldsymbol{u}+\boldsymbol{v}\,,\quad \boldsymbol{u}=(\hbar/m)\nabla R\,,$$
$$v=(\hbar/m)\nabla S\,,\quad \boldsymbol{b}_*=\boldsymbol{v}-\boldsymbol{u}\,, \tag{4}$$

and one more equation

$$\partial_t\rho=-\frac{\hbar}{2m}\triangle\rho-\operatorname{div}\rho\boldsymbol{b}_* \tag{5}$$

is obeyed by ρ. It is identifiable [1] as the backward Fokker–Planck equation, while (3) is the forward one. In the theory of stochastic processes such equations are known to determine the time development of respectively the forward and backward transition probability densities for the diffusion process.

Setting

$$\nu=\hbar/2m\,,\quad \boldsymbol{b}=(\hbar/m)\nabla(R-S)$$

in the forward case we have

$$\partial_t p(\boldsymbol{y},0,\boldsymbol{x},t)=\operatorname{div}_x[\,\nu\nabla_x p(\boldsymbol{y},0,\boldsymbol{x},t)$$
$$-\boldsymbol{b}(\boldsymbol{x},t)p(\boldsymbol{y},0,\boldsymbol{x},t)\,]\,. \tag{6}$$

According to the rules of the Ito stochastic calculus, one can uniquely associate (6) with the stochastic differential equation

$$\mathrm{d}\boldsymbol{X}(t)=\boldsymbol{b}(\boldsymbol{X}(t),t)+\sqrt{2\nu}\,\mathrm{d}\boldsymbol{W}(t)\,, \tag{7}$$

where $\mathrm{d}\boldsymbol{W}(t)$ represents the normalized Wiener noise. $\boldsymbol{X}(t)$ takes values in $\mathbb{R}^3$ as a continuous function of time, and with time passing draws a sto-

chastic trajectory in the configuration space. Given $\rho_0(\boldsymbol{x})=\rho(\boldsymbol{x}, 0)$ and $p(\boldsymbol{y}, 0, \boldsymbol{x}, t)$ solving (6). Apparently

$$\int \mathrm{d}\boldsymbol{y}\, p(\boldsymbol{y}, 0, \boldsymbol{x}, t)\rho_0(\boldsymbol{y})$$

provides a solution of (6) with the initial condition $\rho_0(\boldsymbol{x})$, hence by the uniqueness theorem for the parabolic (Kolmogorov) equation it equals $\rho(\boldsymbol{x}, t)$.

The normalization

$$\int \mathrm{d}\boldsymbol{x}\, \rho(\boldsymbol{x}, t)=1$$

is preserved by virtue of

$$\int \mathrm{d}\boldsymbol{x}\, p(\boldsymbol{y}, 0, \boldsymbol{x}, t)=1 .$$

Let us emphasize that the knowledge of $p(\boldsymbol{y}, 0, \boldsymbol{x}, t)$ does not determine $\rho(\boldsymbol{x}, t)$ unless $\rho_0(\boldsymbol{x})$ is specified. Consequently, given (3) it is rather natural to demand the validity of this equation not only for $\rho(\boldsymbol{x}, t)$ but also for the transition probability densities $p(\boldsymbol{y}, 0, \boldsymbol{x}, t)$ which automatically associates (7) with (1).

Remark 1. The existence and uniqueness of the solution of the stochastic differential equation (7) with the initial condition $\boldsymbol{X}(t_0)=\boldsymbol{x}_0$ are discussed in refs. [5,6]. To have (7) solved in the interval $[t_0, T]$ one usually imposes the smoothness (Lipschitz) condition

$$|\boldsymbol{b}(\boldsymbol{x}, t)-\boldsymbol{b}(\boldsymbol{y}, t)| \leqslant K|\boldsymbol{x}-\boldsymbol{y}|$$

for all $\boldsymbol{x}, \boldsymbol{y}$ and $t \in [t_0, T]$, K being a constant and the growth condition

$$2\nu+|\boldsymbol{b}(\boldsymbol{x}, t)|^2 \leqslant K'(1+|\boldsymbol{x}|^2) .$$

The latter, if fulfilled, guarantees that the solution will not explode for finite times.

As is well known [1–4,7] the Schrödinger equation (1) can be equivalently rewritten as a coupled system of equations, one of which is (3), while another has the familiar Hamilton–Jacobi form:

$$\partial_t S=\frac{\hbar}{2m}(|\nabla R|^2-|\nabla S|^2+\triangle R)-\frac{V}{\hbar} . \tag{8}$$

Let us define the conditional expectation for the stochastic process $\boldsymbol{X}(t)$ solving (7),

$$E_t[f(\boldsymbol{X}(t'))]=E[f(\boldsymbol{X}(t'))|\boldsymbol{X}(t)=\boldsymbol{x}]$$
$$=\int \mathrm{d}\boldsymbol{y}\, p(\boldsymbol{x}, t, \boldsymbol{y}, t')f(\boldsymbol{y}) , \quad t' \geqslant t . \tag{9}$$

In terms of (9) the mean forward and backward derivatives D_+, D_- of the process can be introduced,

$$(\mathrm{D}_\pm f)(\boldsymbol{X}(t), t)$$
$$=\lim_{\Delta t \downarrow 0} E_t\{\pm(1/\Delta t)[f(\boldsymbol{X}(t\pm\Delta t), t\pm\Delta t)$$
$$-f(\boldsymbol{X}(t), t)]\}$$
$$=[\partial_t+\boldsymbol{b}_\pm\cdot\nabla\pm(\hbar/2m)\triangle]f(\boldsymbol{X}(t), t) , \tag{10}$$

such that

$$\mathrm{D}_+\boldsymbol{X}(t)=\boldsymbol{b}_+=\boldsymbol{b} , \quad \mathrm{D}_-\boldsymbol{X}(t)=\boldsymbol{b}_-=\boldsymbol{b}_* , \tag{11}$$

and the following equation holds,

$$\tfrac{1}{2}m(\mathrm{D}_+\mathrm{D}_-+\mathrm{D}_-\mathrm{D}_+)\boldsymbol{X}(t)$$
$$=\tfrac{1}{2}m(\mathrm{D}_+b_-+\mathrm{D}_-b_+)(\boldsymbol{X}(t), t)$$
$$=\hbar\nabla[\partial_t S-(\hbar/2m)(|\nabla R|^2-|\nabla S|^2$$
$$+\triangle R)](\boldsymbol{X}(t), t) . \tag{12}$$

By equating (which is a restriction on the process making it time reversal invariant [7])

$$\tfrac{1}{2}m(\mathrm{D}_+\mathrm{D}_-+\mathrm{D}_-\mathrm{D}_+)\boldsymbol{X}(t)=-\nabla V , \tag{13}$$

the second Newton law of motion is obeyed in the stochastic mean. Apparently we deal here with the gradient form of (9). Since the osmotic $\boldsymbol{u}$ and current $\boldsymbol{v}$ velocities are gradients, it is convenient to rewrite (3) and (8) in terms of them only. Then

$$\partial_t\boldsymbol{u}=-\frac{\hbar}{2m}\triangle\boldsymbol{v}-\nabla(\boldsymbol{v}\cdot\boldsymbol{u}) ,$$

$$\partial_t\boldsymbol{v}=\frac{\hbar}{2m}\triangle\boldsymbol{u}+\tfrac{1}{2}\nabla\boldsymbol{u}^2-\tfrac{1}{2}\nabla\boldsymbol{v}^2-\frac{1}{m}\nabla V \tag{14}$$

may be considered as the starting point for the stochastic analysis, once the initial velocity fields $\boldsymbol{u}(\boldsymbol{x}, t_0)$, $\boldsymbol{v}(\boldsymbol{x}, t_0)$ are chosen and the Cauchy problem (14) is solvable.

Remark 2. Let us emphasize that the causal approach [2–4] exploits directly eqs. (3) and (8). it is precisely the Hamilton–Jacobi form of (8) which allow one to associate certain deterministic motions with the wave equation (1). In the stochastic approach [1,7] the situation is different. The paramount importance of the stochastic differential equation (7) makes here quite a substantial difference between the deterministic and random imple-

mentation of (possibly particle) trajectories associated with solutions of (1).

Eqs. (3), (8) provide us merely with another form of (1), while the equivalence of (14) with (7), (13) is more intricate. On the other hand, by taking the gradients of (3), (8) we recover (14), hence on the mathematical (at least) level a manifest link exists between Schrödinger wave functions and random (diffusive) motions of point particles.

The major problem of stochastic mechanics is then to reveal to which extent wave functions are derivable on purely probabilistic (diffusion processes) grounds.

Apparently it amounts to recovering the potentials upon an assumption that $\boldsymbol{u}(\boldsymbol{x}, t)$, $\boldsymbol{v}(\boldsymbol{x}, t)$ solving (14) are gradient fields. Let $\boldsymbol{u}$, $\boldsymbol{v}$ solve (14) with the initial data $\boldsymbol{u}_0(\boldsymbol{x})=\boldsymbol{u}(\boldsymbol{x}, t_0)$, $\boldsymbol{v}_0(\boldsymbol{x})$. By introducing $\boldsymbol{b}=\boldsymbol{u}+\boldsymbol{v}$ we can pass to the stochastic differential equation (7) which in turn implies (6). Accordingly $\rho(\boldsymbol{x}, t)$ is determined by the choice of $\rho(\boldsymbol{x}, t_0)$. Assuming that $\boldsymbol{u}_0(\boldsymbol{x})$ is the gradient field, we can locally reproduce the potential with accuracy up to the additive constant (e.g. Poincaré lemma). The normalization condition $\int\rho(\boldsymbol{x}, t_0)\,\mathrm{d}\boldsymbol{x}=1$, $\exp(2R_0)=\rho_0$ removes the arbitrariness, hence $\boldsymbol{u}_0(\boldsymbol{x})$ determines $\rho_0(\boldsymbol{x})$ and by (6) $\rho(\boldsymbol{x}, t)$.

With $\rho(\boldsymbol{x}, t)$ established, we are finally left with eqs. (13) whose integration amounts to solving the Cauchy problem

$$\partial_t s+H(\nabla s, \boldsymbol{x}, t)=0\,,$$

$$s(\boldsymbol{x}, t_0)=s_0(\boldsymbol{x}),\quad \nabla s_0(\boldsymbol{x})=m\boldsymbol{v}_0(\boldsymbol{x}),\quad \boldsymbol{s}=\hbar\boldsymbol{S}, \tag{15}$$

with

$$H(\boldsymbol{p}, \boldsymbol{x}, t)=\frac{\boldsymbol{p}^2}{2m}+U(\boldsymbol{x}, t)\,,$$

$$U(\boldsymbol{x}, t)=V(\boldsymbol{x}, t)-\frac{\hbar^2}{2m}\frac{\triangle\rho^{1/2}}{\rho^{1/2}}\,. \tag{16}$$

Indeed, if we have a solution $s(\boldsymbol{x}, t)$ of (15), then $\nabla s(\boldsymbol{x}, t)$ solves (13), hence (14). By the uniqueness argument for solutions of the Cauchy problem, $\nabla s(\boldsymbol{x}, t)=m\boldsymbol{v}(\boldsymbol{x}, t)$ provides a solution of (14) with $\boldsymbol{v}_0(\boldsymbol{x})=(1/m)\nabla s_0(\boldsymbol{x})$. The only non-uniqueness pertains to the initial data $\nabla s_0(\boldsymbol{x})=m\boldsymbol{v}_0(\boldsymbol{x})$ since in the contractible spatial area $\boldsymbol{v}_0(\boldsymbol{x})$ determines the corresponding potential up to the additive constant.

To see how this arbitrariness can be removed, let us consider the absolute expectation value of (15). Then $\langle\partial_t s\rangle=-\langle H\rangle$ where (integrating by parts [12])

$$\begin{aligned}\langle H\rangle &= \int \mathrm{d}\boldsymbol{x}\,[\tfrac{1}{2}m(\boldsymbol{v}^2-\boldsymbol{u}^2)+V(\boldsymbol{x}, t)\\ &\quad -\tfrac{1}{2}\hbar\,\mathrm{div}\,\boldsymbol{u}]\rho(\boldsymbol{x}, t)\\ &= \int \mathrm{d}\boldsymbol{x}\,[\tfrac{1}{2}m(\boldsymbol{u}^2+\boldsymbol{v}^2)+V(\boldsymbol{x}, t)]\rho(\boldsymbol{x}, t)\\ &= \int \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x}, t)\rho(\boldsymbol{x}, t)\,,\end{aligned} \tag{17}$$

and the assumption of the localizability (e.g. $\langle H\rangle<\infty$) of the total (mean) energy of the diffusion process is necessary to have (15) uniquely solved on the basis of (14). The term

$$\int \mathrm{d}\boldsymbol{x}\,\tfrac{1}{2}m(\boldsymbol{u}^2+\boldsymbol{v}^2)\rho(\boldsymbol{x}, t)$$

is known as the kinetic energy of the diffusion process.

By the continuity equation we have

$$\begin{aligned}\partial_t\langle s\rangle &= \int \mathrm{d}\boldsymbol{x}\,(\partial_t\rho)\,s+\langle\partial_t s\rangle\\ &= m\langle\boldsymbol{v}^2\rangle+\langle\partial_t s\rangle\,.\end{aligned} \tag{18}$$

Hence (15) implies

$$\partial_t\langle s\rangle=m\langle\boldsymbol{v}^2\rangle-\langle H\rangle\,, \tag{19}$$

which admits a unique solution $\langle s\rangle(t)$ for given initial data

$$\langle s\rangle(t^0)=\langle s_0\rangle\,.$$

By making the restriction

$$\langle s_0\rangle=0 \tag{20}$$

we have a guarantee that $\langle s\rangle(t)$ is determined in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$ only,

$$\langle s\rangle(t)=\int_{t_0}^{t}(m\langle\boldsymbol{v}^2\rangle-\langle H\rangle)\,\mathrm{d}t\,. \tag{21}$$

Given an arbitrary integral $s'(\boldsymbol{x}, t)$, $\langle s'_0\rangle\not\equiv 0$ of (15). Then, apparently

$$s(\boldsymbol{x}, t)=s'(\boldsymbol{x}, t)-\langle s'_0\rangle \tag{22}$$

obeys both (20) and (15).

Accordingly Schrödinger wave functions with phases obeying (15), (20) can be set in a one-to-one

correspondence with the diffusion process (7), (13).

The mapping $\{\boldsymbol{u}, \boldsymbol{v}\} \leftrightarrow \{\rho, S\}$ was investigated by us locally (in a contractible area). However, its extensions can be fruitfully studied by viewing the Schrödinger equation as the linearization of the coupled nonlinear system (14). Then, with all previous reservations concerning the uniqueness of the map, once $\rho(\boldsymbol{x}, t)$, $S(\boldsymbol{x}, t)$ are obtained, we can introduce the diffusion processes for which $\{\rho, S\}$ is a pair of potentials implementing the gradient fields $\{\boldsymbol{u}, \boldsymbol{v}\}$. This route which is quite customary in the literature views the Schrödinger equation as an efficient tool to integrate the Hamilton–Jacobi equation (15), otherwise a highly non-trivial problem even for the simplest examples, see e.g. ref. [11] for an exhaustive discussion.

Apart from the presence of the contribution $-(\hbar^2/2m)\triangle\rho^{1/2}/\rho^{1/2}$, eq. (15) can be considered within the traditional Hamilton–Jacobi formalism [8,9]. Once the three-parameter family $s(\boldsymbol{x}, \boldsymbol{\alpha}, t)$ of solutions of (15) is found, such that $0 \neq \det(\partial^2 s/\partial x_i \partial \alpha_i)$ then automatically solutions of canonical Hamilton equations

$$\dot{q}_i = \frac{\partial}{\partial p_i} H(\boldsymbol{p}, \boldsymbol{q}, t) , \quad \dot{p}_i = -\frac{\partial}{\partial q_i} H(\boldsymbol{p}, \boldsymbol{q}, t) ,$$
$$i = 1, 2, 3 , \tag{23}$$

are generated by the standard identities

$$\frac{\partial s(\boldsymbol{q}, \boldsymbol{\alpha}, t)}{\partial \alpha_i} = \beta_i , \quad i = 1, 2, 3 , \tag{24}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are the integration constants ((20) should be obeyed), which allows one to derive functions $q_i(t, \boldsymbol{\alpha}, \boldsymbol{\beta})$. The identities $p_i = \partial s/\partial q_i$ after inserting $q_i(t)$ allow us to recover $p_i(t) = p_i(t, \boldsymbol{\alpha}, \boldsymbol{\beta})$. In this way the deterministic trajectories of the causal approach do appear in the framework of stochastic mechanics. However, it is the diffusion process (7), (13) which underlies (15), and highly irregular (non-differentiable) sample paths of the process are the primary notions in the formalism.

Let $\boldsymbol{X}(t)$ be the stochastic process (7), (13). The stochastic analogue of the ordinary time derivative was introduced in (9)–(11). It allows the introduction of the symmetric and antisymmetric stochastic derivatives

$$\mathscr{D}_s = \tfrac{1}{2}(D_+ + D_-) , \quad \mathscr{D}_a = \tfrac{1}{2}(D_+ - D_-) ,$$
$$(\mathscr{D}_s f)(\boldsymbol{X}(t), t) = (\partial_t + \boldsymbol{v}\cdot\nabla) f(\boldsymbol{X}(t), t) ,$$
$$(\mathscr{D}_a f)(\boldsymbol{X}(t), t) = \left(\frac{\hbar}{2m}\triangle + \boldsymbol{u}\cdot\nabla\right) f(\boldsymbol{X}(t), t) . \tag{25}$$

In particular

$$\mathscr{D}_s \boldsymbol{X}(t) = \boldsymbol{v}(t) , \quad \mathscr{D}_s^2 \boldsymbol{X}(t) = \mathscr{D}_s \boldsymbol{v} = \partial_t \boldsymbol{v} + \tfrac{1}{2}\nabla \boldsymbol{v}^2 ,$$
$$\mathscr{D}_a \boldsymbol{X}(t) = \boldsymbol{u}(t) , \quad \mathscr{D}_a^2 \boldsymbol{X}(t) = \mathscr{D}_a \boldsymbol{u} = \frac{\hbar}{2m}\triangle \boldsymbol{u} + \tfrac{1}{2}\nabla \boldsymbol{u}^2 , \tag{26}$$

and by (13)

$$(\mathscr{D}_s^2 - \mathscr{D}_a^2)\boldsymbol{X}(t) = -\frac{1}{m}\nabla V(\boldsymbol{X}(t)) . \tag{27}$$

In addition

$$\mathscr{D}_s^2 = \mathscr{D}_0 + \tfrac{1}{4}(D_+ D_- + D_- D_+) ,$$
$$\mathscr{D}_a^2 = \mathscr{D}_0 - \tfrac{1}{4}(D_+ D_- + D_- D_+) , \tag{28}$$

and we may consistently call $\mathscr{D}_s^2 \boldsymbol{X}(t)$ the current acceleration and $-\mathscr{D}_a^2 \boldsymbol{X}(t)$ the osmotic deceleration. Here $\mathscr{D}_0 = \frac{1}{4}(D_+^2 + D_-^2)$. Given $\boldsymbol{X}(t)$, we define $\boldsymbol{P}(t)$:

$$\boldsymbol{P}(t) = m\mathscr{D}_s \boldsymbol{X}(t) = m\boldsymbol{v}(\boldsymbol{X}(t), t) . \tag{29}$$

With $H(\boldsymbol{p}, \boldsymbol{q}, t)$ given by (15) we define

$$H(\boldsymbol{X}(t), \boldsymbol{P}(t), t) = \frac{1}{2m}\boldsymbol{P}^2(t) + U(\boldsymbol{X}(t), t)$$
$$= \boldsymbol{P}\cdot\mathscr{D}_s \boldsymbol{X}(t) - L ,$$
$$L = L(\boldsymbol{X}(t), \mathscr{D}_s \boldsymbol{X}(t), t)$$
$$= \tfrac{1}{2}m[\mathscr{D}_s \boldsymbol{X}(t)]^2 - U(\boldsymbol{X}(t), t) . \tag{30}$$

Then because of (26) the canonical stochastic system (compare e.g. also refs. [10,11]) emerges,

$$\mathscr{D}_s \boldsymbol{X}(t) = \frac{\partial H}{\partial \boldsymbol{P}(t)} , \quad \mathscr{D}_s \boldsymbol{P}(t) = -\frac{\partial H}{\partial \boldsymbol{X}(t)} , \tag{31}$$

which is an obvious generalization of the purely deterministic system (23) to the theory of random motions. In addition to (7) $\boldsymbol{X}(t)$ satisfies the reverse (backward diffusion) stochastic differential equation

$$\mathrm{d}_* \boldsymbol{X}(t) = \boldsymbol{b}_*(\boldsymbol{X}(t), t)\,\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}\boldsymbol{W}_*(t) . \tag{32}$$

Technically it refers to the independence of incre-

ments $\boldsymbol{W}(t)-\boldsymbol{W}(s)$ on $\boldsymbol{X}(t)$ for $s\leqslant t$, while $\boldsymbol{W}_*(t)-\boldsymbol{W}_*(s)$ for $s\geqslant t$.

Remark 3. (7) leads to the forward Kolmogorov equation with solutions $p(\boldsymbol{x}, s, \boldsymbol{y}, t)$, $s\leqslant t$. Eq. (32) leads to the backward Kolmogorov equation with solutions

$$\rho(\boldsymbol{y}, t)p_*(\boldsymbol{x}, s, \boldsymbol{y}, t)=p(\boldsymbol{x}, s, \boldsymbol{y}, t)\rho(\boldsymbol{x}, s)$$
$$\Rightarrow \rho(\boldsymbol{x}, s)=\int \mathrm{d}\boldsymbol{y}\, p_*(\boldsymbol{x}, s, \boldsymbol{y}, t)\rho(\boldsymbol{y}, t)\,,$$
$$s\leqslant t\,. \tag{33}$$

Let us observe that the stochastic increments $\mathrm{d}\boldsymbol{X}(t)$, $\mathrm{d}_*\boldsymbol{X}(t)$ arise as limits of finite increments

$$\Delta\boldsymbol{X}(t)=\boldsymbol{X}(t+\Delta t)-\boldsymbol{X}(t)\,,$$
$$\Delta_*\boldsymbol{X}(t)=\boldsymbol{X}(t)-\boldsymbol{X}(t-\Delta t)\,.$$

By invoking (9)–(11) we realize that

$$E_t[\,\Delta\boldsymbol{X}(t)\,]\simeq\boldsymbol{b}(\boldsymbol{X}(t), t)\Delta t\,,$$
$$E_t[\,\Delta_*\boldsymbol{X}(t)\,]\simeq\boldsymbol{b}_*(\boldsymbol{X}(t), t)\Delta t\,,$$
$$E_t[\,\tfrac{1}{2}\Delta\boldsymbol{X}(t)+\tfrac{1}{2}\Delta_*\boldsymbol{X}(t)\,]\simeq\boldsymbol{v}(\boldsymbol{X}(t), t)\Delta t\,. \tag{34}$$

Given an initial point $\boldsymbol{q}_0=\boldsymbol{x}$. Let us consider all sample paths of the diffusion process, which originate from $\boldsymbol{x}$ at time t_0. The average over all sample locations achieved after time Δt is $\boldsymbol{x}+\boldsymbol{b}(\boldsymbol{X}(t_0), t_0)\Delta t$ as a result of the forward drift and $\boldsymbol{x}+\boldsymbol{b}_*(\boldsymbol{X}(t_0), t_0)\Delta t$ as a consequence of the backward drift. Their arithmetic average defines the mean displacement taking us from $\boldsymbol{x}=\boldsymbol{q}(t_0)$ to $\boldsymbol{q}=\boldsymbol{q}(t_0+\Delta t)$,

$$\boldsymbol{q}_1=\boldsymbol{q}(t_0+\Delta t)=\boldsymbol{q}_0+\boldsymbol{v}(\boldsymbol{q}_0, t_0)\,\Delta t\,. \tag{35}$$

A repetition of this procedure for the next interval Δt gives us $\boldsymbol{q}_2=\boldsymbol{q}(t_2)=\boldsymbol{q}(t_1+\Delta t)$, and so on up to the final time instant $T=t_0+N\Delta t$,

$$\boldsymbol{q}_N=\boldsymbol{q}(T)=\boldsymbol{q}(T-\Delta t)+\boldsymbol{v}(\boldsymbol{q}(T-\Delta t), T-\Delta t)\Delta t\,. \tag{36}$$

We have constructed a finite difference scheme: a chain of N mean displacements which interpolate between initial $\boldsymbol{q}_0=\boldsymbol{x}$ and certain $\boldsymbol{q}(T)$ in a smooth way. Formulas (23) prove the existence of a continuously differentiable continuum limit $(T-t_0)/N=\Delta T\to 0$ of the above finite difference procedure. As a consequence deterministic trajectories of the causal approach [2–4] are not alien objects in the stochastic framework. Their probabilistic origin can be summarized as follows: they provide a continual interpolation between points $\boldsymbol{q}(t_0)=\boldsymbol{x}$ and $\boldsymbol{q}(T)=\boldsymbol{x}'$ in terms of (infinitesimal) mean displacements from each instantaneous location achieved. In this sense they encode certain qualitative information about how the random field makes particles propagate.

Random trajectories of the conditioned Wiener process are known [13] to provide a natural probabilistic background for Feynman's path concept, which is condensed in the Feynman–Kac formula. Its particular case is the path integral expression for the heat equation kernel, whose stochastic mechanics implementation was discussed in ref. [14]. Indeed, the simplest example of the smooth Markovian diffusion is the Wiener process with the transition probability density solving the heat equation:

$$p(\boldsymbol{y}, 0, \boldsymbol{x}, t)=\nu\Delta_x p(\boldsymbol{y}, 0, \boldsymbol{x}, t)\,,$$
$$p(\boldsymbol{y}, 0, \boldsymbol{x}, t)=(4\pi\nu t)^{-3/2}\exp\left(-\frac{|\boldsymbol{y}-\boldsymbol{x}|^2}{4\nu t}\right). \tag{37}$$

The corresponding stochastic differential equation reads $\mathrm{d}\boldsymbol{x}(t)=(2\nu)^{1/2}\mathrm{d}\boldsymbol{W}(t)$.

Consider the family $\{\mathrm{I}_j\subset\mathbb{R}^3, j=1, 2, ..., n\}$ of Borel sets and set $t=(n+1)\Delta t$. We introduce the finite difference (discrete) approximation of random paths as follows:

$$\{\boldsymbol{X}(s): \boldsymbol{X}(0)=\boldsymbol{x}_0, \boldsymbol{X}(t)=\boldsymbol{x}_\mathrm{f}\,,$$
$$\boldsymbol{X}(j\Delta t)\in\mathrm{I}_j, j=1, 2, ..., n\}\,, \tag{38}$$

with the initial $\boldsymbol{x}_0$ and final $\boldsymbol{x}_\mathrm{f}$ points fixed. The cylinder sets (38) are measurable, i.e. the transition probability density conditioned by (38) reads

$$\int_{\mathrm{I}_1}\mathrm{d}\boldsymbol{x}_1 ..., \int_{\mathrm{I}_n}\mathrm{d}\boldsymbol{x}_n\, p(\boldsymbol{x}_0, 0, \boldsymbol{x}_1, \Delta t)$$
$$\times p(\boldsymbol{x}_1, \Delta t, \boldsymbol{x}_2, 2\Delta t)...p(\boldsymbol{x}_n, n\Delta t, \boldsymbol{x}_\mathrm{f}, t)$$
$$=\int_{\mathrm{I}_1}\mathrm{d}\boldsymbol{x}_1 ... \int_{\mathrm{I}_n}\mathrm{d}\boldsymbol{x}\prod_{j=1}^{n+1}\left\{[4\pi\nu(t_j-t_{j-1})]^{-3/2}\right.$$
$$\left.\times\exp\left(-\frac{|\boldsymbol{x}_j-\boldsymbol{x}_{j-1}|^2}{4\pi\nu(t_j-t_{j-1})}\right)\right\}=p_n(\boldsymbol{x}_0, 0, \boldsymbol{x}_\mathrm{f}, t)\,,$$
$$t_0=0,\quad t_{n+1}=t\,,\quad \boldsymbol{x}_{n+1}=\boldsymbol{x}_\mathrm{f}\,. \tag{39}$$

The so introduced conditional Wiener measure on

the cylinder set explicitly attributes the real positive probability weight to each discrete approximant $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \boldsymbol{x}_f)$ of the random path. The measure (38) is known to be countably additive on all cylinder subsets of the set of all continuous trajectories defined on $[0, t]$ and conditioned to connect fixed points $\boldsymbol{x}_0$ and $\boldsymbol{x}_f$.

A formal replacement of all I_j by $\mathbb{R}^3$ followed by the $n\to\infty$ limit in the partition $\Delta t = t/(n+1)$ gives us a complete transition probability density,

$$p(\boldsymbol{x}_0, 0, \boldsymbol{x}_f, t) = \lim_{n\to\infty} \{p_n(\boldsymbol{x}_0, 0, \boldsymbol{x}_f, t) \,|\, I_j = \mathbb{R}^3,\ 1 \leqslant j \leqslant n\}\,. \qquad (40)$$

One must at this point remember that neither of the n-dependent factors in (39) taken separately admits a well defined $n\to\infty$ limit, although the whole expression does.

We have discussed a particular example of the stochastic process, but since the Wiener process can be used to construct more complicated diffusions, the extension of previous arguments to less trivial cases is possible. Attempts in this direction were undertaken recently [15,16]. In order not to repeat the arguments of ref. [16] where the Girsanov theorem (ref. [5], ch. III) was exploited, let us take advantage of the fact that our partitioning of the interval $[0, t]$ requires Δt to be very small. The general formula for small time transition probability densities is available (ref. [17], ch. 4.7). Adapted to our notation (cf. eqs. (6), (7)) it reads

$$p(\boldsymbol{y}, s, \boldsymbol{x}, s+\Delta t) = (4\pi\nu\Delta t)^{-3/2} \times \exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{b}(\boldsymbol{y}, s)\Delta t|^2}{4\nu\Delta t}\right). \qquad (41)$$

By inserting (41) to (39) and performing the limiting operations outlined before, we immediately arrive (up to notation adjustments) at the conclusions of both refs. [15] and [16].

Remark 4. Numerous objections were raised against the physical relevance of stochastic mechanics, but the most essential of them were found [18,19] to come from the misunderstanding of relationships between quantum and stochastic observables. A new criticism has been launched recently [20]. Both the discussion of the present paper and detailed reviews [7,18] indicate that the time-reversal invariance is a principal feature of stochastic mechanics and thus the Bohm–Vigier theory as well. It is not shared by irreversible stochastic processes so typical for dissipative phenomena of standard statistical physics.

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