Accelerated stochastic diffusion processes

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We give a purely probabilistic demonstration that all effects of non-random (external, conservative) forces on the diffusion process can be encoded in the Nelson ansatz for the second Newton law. Each random path of the process together with a probabilistic weight carries a phase accumulation (complex valued) weight. Random path summation (integration) of these weights leads to the transition probability density and transition amplitude respectively between two spatial points in a given time interval. The Bohm–Vigier, Fenyes–Nelson–Guerra and Feynman descriptions of the quantum particle behaviours are in fact equivalent.

1. Time reversal in the description of random phenomena

Discussion of time reversal in the general context of stochastic processes is tamed by folklore arguments about the inherent (dissipation) irreversibility of random phenomena. For example [1] the emergence of the pseudo-Fokker–Planck equation (negative definite diffusion matrix) was noticed in quantum optics investigations, but its probabilistic significance refuted. Under the familiar name of the backward Fokker–Planck equation, however, it appears quite naturally [2] in the description of stationary Markovian diffusion in connection with the notion of detailed balance.

Indeed, the time reversal symmetry is not an intrinsic property of a Markovian diffusion: the Markov property itself is time symmetric but Markovian diffusion in general is not [3–6]. Hence it is of some interest to discuss the circumstances under which the concept of time reversal may be relevant for the description of random motions.

Our principal goal is to extract from the probabilistic data information on *how the random process is affected by the action of non-random* (hence external to the process) *force fields*. The forces of interest (conservative) have the property of time reversal invariance whose impact must be seen in stochastic propagation.

Let us confine attention to a fixed time interval $[0, T]$ and consider a classical configuration space path connecting the points $q_0=q(0)$ and $Q_0=q(T)$. Apparently it comes from the Hamilton equations, whose time reversal invariance allows one to consider the new time dependent configuration variables

$$\hat{q}(t)=q(T-t)\,,$$
$$\hat{q}(0)=Q_0 \to \hat{q}(T)=q_0\,. \tag{1}$$

Here $\hat{q}(t)$ draws the same trajectory in $\mathbb{R}^3$ as $q(t)$ but followed in the reverse direction: $\hat{p}(t)=-p(T-t)$, $\hat{p}(0)=-P_0=-p(T)$. Accordingly we deal with the alternative ways of connecting the given configuration space points q_0 and Q_0 by a classical path followed in the time interval $[0, T]$. Markovian diffusion is known to admit the existence of both forward (x, s, y, t) and backward $p_*(x, s, y, t)$, $s\leqslant t$, transition probability densities, where

$$\rho(x, s)p(x, s, y, t)=p_*(x, s, y, t)\rho(y, t)\,, \tag{2}$$

provided $\rho(x, t)$ is the probability distribution of the random variable of the process at time t.

The forward probability density solves the forward Fokker–Planck equation in y, t which we choose in the form

$$\partial_t p=\nu\triangle_y p-\nabla_y(bp)\,. \tag{3}$$

$b(y, t)$ is the forward drift of the process, $\nu\geqslant 0$ is the

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diffusion coefficient (usually a constant). In variables x, s we have solved here the backward diffusion equation

$$\partial_s p = -\nu \triangle_x p - b \nabla_x p \,. \tag{4}$$

The backward probability density $p_*(x, s, y, t)$ solves the backward Fokker–Planck equation in x, s and the forward diffusion equation in y, t:

$$\partial_s p_* = -\nu \triangle_x p - \nabla_x (b_* p_*) \,,$$

$$\partial_t p_* = \nu \triangle_y p - b_* \nabla_y p_* \,. \tag{5}$$

b_* is the backward drift of the process. Since

$$\rho(x, t) = \int \rho(y, 0) p(y, 0, x, t) \, \mathrm{d}y \tag{6}$$

solves

$$\partial_t \rho = \nu \triangle \rho - \nabla (b\rho) \tag{7}$$

in the finite time interval, we have the initial distribution $\rho_0(x) = \rho(x, 0)$ *and the final distribution* $\rho(x, T)$ of the process for the case of forward propagation.

Would we be interested in the stochastic analogue of the previous purely deterministic analysis in the case of the reverse (backward) propagation, then instead of $\rho(x, t)$, $0 \leqslant t \leqslant T$, we need

$$\hat{\rho}(x, t) = \rho(x, T-t) \,, \quad 0 \leqslant t \leqslant T \,. \tag{8}$$

The corresponding propagation comes from

$$\hat{\rho}(x, t) = \rho(x, T-t)$$

$$= \int p_*(x, T-t), y, T-s) \rho(y, T-s) \, \mathrm{d}y$$

$$= \int \hat{p}(y, s, x, t) \hat{\rho}(y, s) \, \mathrm{d}y \,, \tag{9}$$

which implies

$$\partial_t \hat{\rho}(x, t) = -\partial_\tau \rho(x, \tau) = \nu \triangle \hat{\rho} + \nabla (b_* \rho) \,,$$

$$\tau = T - t \,, \quad b_* = b_*(x, T-t) \,. \tag{10}$$

By defining

$$\hat{b}(x, t) = -b_*(x, T-t) \,, \tag{11}$$

we replace (10) by the forward equation

$$\partial_t \hat{\rho}(x, t) = \nu \triangle \hat{\rho}(x, t) - \nabla (\hat{b} \hat{\rho})(x, t) \,, \tag{12}$$

which apart from referring to the forward propagation provides us with the reversal of the original time evolution for $\rho(x, t)$:

$$\hat{\rho}(x, 0) = \rho(x, T) \to \hat{\rho}(x, T) = \rho(x, 0) \,.$$

According to the rules of the Ito calculus [2] the forward Fokker–Planck equation is equivalent to the stochastic differential equation

$$\mathrm{d}X(t) = b(X(t), t) \, \mathrm{d}t + \sqrt{2\nu} \, \mathrm{d}W(t) \,, \tag{13}$$

where the initial values of the random variable $X(t)$ are distributed in $\mathbb{R}^3$ according to $\rho_0(x) = \rho(x, 0)$.

Eq. (13) describes forward propagation of the random variable $X(t)$ undergoing the Wiener process $W(t)$ with the forward drift $b = b(X(t), t)$ and $X(s)$, $s \leqslant t$, independent increments $W(t) - W(s)$. Apparently the same reasoning can be applied to (12) with the result

$$\mathrm{d}\hat{X}(t) = \hat{b}(\hat{X}(t), t) \, \mathrm{d}t + \sqrt{2\nu} \, \mathrm{d}W(t) \to$$

$$\mathrm{d}X(T-t) = -b_*(X(T-t), T-t) \, \mathrm{d}t + \sqrt{2\nu} \, \mathrm{d}W(t) \,.$$

$$0 \leqslant t \leqslant T \,. \tag{14}$$

Notice that to transform $\mathrm{d}W(t)$ to the explicit $T-t$ dependent form, we must define $W(t) = W_*(T-t)$ where $\hat{X}(s) = X(T-s)$, $s \leqslant t$, independent of $W(t) - W(s)$ implies the $X(T-s)$, $s \leqslant t$, independence of $W_*(T-t) - W_*(T-s)$. Here $s \leqslant t \to T-t \leqslant T-s$ hence $W_*(\tau)$ is a Wiener process whose increments $W_*(\tau) - W_*(\sigma)$ are $X(\sigma)$ independent not for $\sigma \leqslant \tau$ but for $\tau \leqslant \sigma$.

The initial values $\hat{X}(0) \in \mathbb{R}^3$ of $\hat{X}(t)$ are distributed according to $\hat{\rho}(x, 0) = \rho(x, T)$. In terms of $\sigma = T-s$, $\tau = T-t$ we have

$$\mathrm{d}X(\tau) = b_*(X(\tau), \tau) \, \mathrm{d}\tau + \sqrt{2\nu} \, \mathrm{d}W_*(\tau) \,,$$

$$\partial_\tau \rho(x, \tau) = -\nu \triangle \rho(s, \tau) - \nabla (b_* \rho)(x, \tau) \,, \tag{15}$$

which is a manifest backward (e.g. pseudo-Fokker–Planck) decoding of the forward propagation (12).

In the above discussion p_* defines the reversal of the random propagation governed by p. Hence for the diffusion of interest we must have guaranteed the existence of both p and p_* in the interval $[0, T]$, but this definitely is not the case [1,2,5–7] for all conceivable Markovian diffusions. Proper limitations on the process (existence criteria) must thus be found.

Remark. The forward process tells us what *will be* the probability distribution at time t, given $\rho(x, 0)$.

The reverse process must not necessarily be viewed as a genuine (realistic, realizable in nature) random propagation in the backward direction. It may be interpreted as telling us that the distribution *was* in the time interval $T-t$ *before* $\rho(x, T)$ has been reached in the course of the forward evolution, i.e. as an artifice to reproduce the past data of the process, given the present.

2. Notions of velocity and acceleration for random motions

Random trajectories of the diffusion process are continuous but nowhere differentiable. Anyway we can introduce the forward and backward time derivatives of the process $X(t)$ in the appropriately smoothened sense:

$$(\mathrm{D}_+X)(t)\Delta t \sim \int \mathrm{d}y\, p(x,t,y,t+\Delta t)y - x\,.$$

$$(\mathrm{D}_-X)(t)\Delta t \sim x - \int \mathrm{d}z\, p_*(z,t-\Delta t,x,t)z\,, \qquad (16)$$

which in fact provides us with the drifts $(\mathrm{D}_+X)(t)=b$, $(\mathrm{D}_-X)(t)=b_*$ of the process: $b_\# = b_\#(X(t), t)$.

If the diffusion pertains to massive (point) particles, we have a natural physical interpretation of the forward drift as the mean velocity of particles leaving x at time t (along sample paths), while the backward drift can be viewed as the mean velocity of particles approaching (coming to) x at time t. Accordingly $x-b_*(x, t)\,\Delta t$ is the mean position evaluated for incoming particles a time Δt before they will reach x at t. The mean position evaluated along sample paths of outgoing particles, a time Δt after they left x at t is $x+b(x, t)\,\Delta t$.

The mean motion in the interval $[t-\Delta t, t+\Delta t]$ can be viewed as uniform, and the propagation from $x-b_*\Delta t$ through x to $x+b\Delta t$ can be approximated by the uniform motion with velocity $v=v(x, t)=\frac{1}{2}(b+b_*)$ along the line segment $[x-b_*\Delta t, x+b\Delta t]$, accomplished in time $2\Delta t$. v is called the current velocity of the process. This interpolating motion is given by

$$x_\tau = x - b_*\Delta t + \tfrac{1}{2}(b+b_*)\tau$$
$$= x - b_*\Delta t + b_*\tau + \tfrac{1}{2}(b-b_*)\tau, \quad \tau\in[0, 2\Delta t], \qquad (17)$$

where the osmotic velocity of the process $u=u(x, t)=\frac{1}{2}(b-b_*)$ naturally appears.

Let us recall that $b_*(x, t)$ is the mean velocity evaluated for all particles whose destiny is to reach x at time t after the flight time Δt (along sample paths of the process). The abrupt change of $b_*(x, t)$ into $b(x, t)$ at x is of purely stochastic origin: it is the outcome of random fluctuations (scattering at x) which modify the mean velocity of particles into b. The net (mean) drift velocity change at x is $2u=b-b_*$ and this purely stochastic (osmotic) effect is accounted for in the finite difference propagation formula (17): the u deviation from the backward drift b_* accumulates after $2\Delta t$ to the $2u\Delta t$ spatial increment. The definitions (16) of time derivatives for stochastic motion can be extended to arbitrary (smooth) functions $f(X(t), t)$ of the random variable. It implies [3–5] a variety of stochastic accelerations through increments of the drifts,

$$b_*(x,t) \sim b_*(x,t-\Delta t) + (\mathrm{D}_-^2 X)(t)\Delta t\,,$$
$$b_*(x,t+\Delta t) \sim b_*(x,t) + (\mathrm{D}_+\mathrm{D}_- X)(t)\Delta t\,,$$
$$b(x,t) \sim b(x,t-\Delta t) + (\mathrm{D}_-\mathrm{D}_+ X)(t)\Delta t\,,$$
$$b(x,t+\Delta t) \sim b(x,t) + (\mathrm{D}_+^2 X)(t)\Delta t\,,$$
$$\mathrm{D}_- b_\# = (\partial_t + b_*\nabla - \nu\triangle)b_\#\,,$$
$$\mathrm{D}_+ b_\# = (\partial_t + b\nabla + \nu\triangle)b_\#\,. \qquad (18)$$

Related increments of the current velocity are

$$v(t) \sim v(t-\Delta t) + \tfrac{1}{2}(\mathrm{D}_-^2 + \mathrm{D}_-\mathrm{D}_+)X(t)\Delta t\,,$$
$$v(t+\Delta t) \sim v(t) + \tfrac{1}{2}(\mathrm{D}_+^2 + \mathrm{D}_+\mathrm{D}_-)X(t)\Delta t\,, \qquad (19)$$

where

$$v(t+\Delta t) \sim v(t-\Delta t) + (\mathscr{D}_s^2 X)(t)(2\Delta t) \qquad (20)$$

and [7] $\mathscr{D}_s = \frac{1}{2}(\mathrm{D}_+ + \mathrm{D}_-)$,

$$(\mathscr{D}_s^2 X)(t) = [\tfrac{1}{4}(\mathrm{D}_+^2 + \mathrm{D}_-^2)$$
$$+ \tfrac{1}{4}(\mathrm{D}_+\mathrm{D}_- + \mathrm{D}_-\mathrm{D}_+)]X(t) \qquad (21)$$

is called the current acceleration of the process. Notice its time reversal symmetry:

$$\mathscr{D}_s^2 X(t) = \mathscr{D}_s^2 \hat{X}(t) = \mathscr{D}_s^2 X(T-t)\,. \qquad (22)$$

Increments of the osmotic velocity,

$$u(t) \sim u(t-\Delta t) + \tfrac{1}{2}(\mathrm{D}_-\mathrm{D}_+ - \mathrm{D}_-^2)X(t)\Delta t\,,$$

$$u(t+\Delta t) \sim u(t) + \tfrac{1}{2}(D_+^2 - D_+ D_-)X(t)\Delta t \qquad (23)$$

combine into the average deviation velocity about x at t:

$$\tfrac{1}{2}[u(t+\Delta t)+u(t-\Delta t)] = u(t) + (\mathscr{D}_a^2 X)(t)(2\Delta t)\,, \qquad (24)$$

where $\mathscr{D}_a = \frac{1}{2}(D_+ - D_-)$,

$$(\mathscr{D}_a^2 X)(t) = [\tfrac{1}{4}(D_+^2 + D_-^2) - \tfrac{1}{4}(D_+ D_- + D_- D_+)]X(t) \qquad (25)$$

is the time reversal invariant osmotic acceleration of the process. The meaning of these accelerations becomes transparent, if we observe that during the time interval $[t-\Delta t, t]$ the surrounding of x has travelled not only by sample paths terminating in x at time t, but also by those which originated from x at time $t-\Delta t$. The respective contributions to the mean velocity of particles flying about x in the interval $[t-\Delta t, t]$ are $b_*(x, t)$ and $b(x, t-\Delta t)$. They combine into the average velocity of particles

$$\begin{aligned} v_-(t) &= \tfrac{1}{2}[b_*(x,t) + b(x, t-\Delta t)] \\ &= v(t) - \tfrac{1}{2}(D_- D_+ X)(t)\Delta t \\ &= v(t-\Delta t) + \tfrac{1}{2}(D_-^2 X)(t)\Delta t\,. \end{aligned} \qquad (26)$$

Analogously for the interval $[t, t+\Delta t]$:

$$\begin{aligned} v_+(t) &= \tfrac{1}{2}[b(x,t) + b_*(x, t+\Delta t)] \\ &= v(t) + \tfrac{1}{2}(D_+ D_- X)(t)\Delta t \\ &= v(t+\Delta t) - \tfrac{1}{2}(D_+^2 X)(t)\Delta t\,, \end{aligned} \qquad (27)$$

where paths outgoing from x at t coexist with paths whose destiny is to reach x at $t+\Delta t$.

The random admixture of the outgoing paths to the incoming flow implies the net change of $(b-b_*)(x, t)$ into

$$\begin{aligned} v_+(t) - v_-(t) &= \tfrac{1}{4}(D_+ D_- + D_- D_+)X(t)(2\Delta t) \\ &= [v(t+\Delta t) - v(t-\Delta t)] \\ &\quad - \tfrac{1}{2}(D_+^2 + D_-^2)X(t)(2\Delta t)\,, \end{aligned} \qquad (28)$$

where one more time reversal invariant acceleration enters the scene:

$$\tfrac{1}{2}(\mathscr{D}_s^2 - \mathscr{D}_a^2)X(t) = \tfrac{1}{2}(D_+ D_- + D_- D_+)X(t)\,. \qquad (29)$$

Analogously we can evaluate how deviations from the forward flow $b_*(x, t)$ in time intervals $[t-\Delta t, t]$ and $[t, t+\Delta t]$ are modified by the admixture of the outgoing (scattered as a result of random fluctuations) paths:

$$\begin{aligned} u_-(t) &= -[b_*(x,t) - b(x, t-\Delta t)] \\ &= 2u(t) - (D_- D_+ X)(t)\Delta t\,, \\ u_+(t) &= -[b_*(x, t+\Delta t) - b(x,t)] \\ &= 2u(t) - (D_+ D_- X)(t)\Delta t\,. \end{aligned} \qquad (30)$$

The average deviation from the forward flow in the interval $[t-\Delta t, t+\Delta t]$ is given by

$$\begin{aligned} &\tfrac{1}{2}[u_+(t) + u_-(t)] \\ &\quad = u(t) - \tfrac{1}{2}(D_+ D_- + D_- D_+)X(t)(2\Delta t)\,. \end{aligned} \qquad (31)$$

Given $u(x, t)$, $v(x, t)$ we can consider:

$$\begin{aligned} &u_\Delta = u + (\mathscr{D}_a^2 X)(t)\Delta t\,, \quad v_\Delta = v + (\mathscr{D}_s^2 X)(t)\Delta t\,, \\ &b_\Delta = b + (\mathscr{D}_s^2 + \mathscr{D}_a^2)X(t)\Delta t \\ &\quad = b + \tfrac{1}{2}(D_+^2 + D_-^2)X(t)\Delta t, \\ &b_{*\Delta} = b_* + (\mathscr{D}_s^2 - \mathscr{D}_a^2)X(t)\Delta t \\ &\quad = b_* + \tfrac{1}{2}(D_+ D_- + D_- D_+)X(t)\Delta t\,. \end{aligned} \qquad (32)$$

In the case of non-zero $b_{*\Delta} - b_*$ we would deal with the attraction (or repulsion) of the incoming flow to a given point. Let us say that the *diffusion process remains in a stochastic equilibrium if the incoming flow is not accelerated at any spatial point for all times: then neither point is particularly distinguished by the process.* It means that

$$\tfrac{1}{2}(D_+ D_- + D_- D_+)X(t) = (\mathscr{D}_s^2 - \mathscr{D}_a^2)X(t) = 0 \qquad (33)$$

for all times, and accordingly

$$\begin{aligned} &b_{*\Delta}(x,t) = b_*(x,t)\,, \quad v_+(t) - v_-(t) = 0\,, \\ &\tfrac{1}{2}[u_+(t) + u_-(t)] = u(x,t)\,. \end{aligned} \qquad (34)$$

The process still allows for non-zero accelerations of outgoing flows, but they are an intrinsic feature of fluctuation phenomena. We shall choose a specific way to destroy the stochastic equilibrium in the above by applying force fields external to the process. If they are conservative, we can produce the field of accelerations for the stochastic process by setting

$$\tfrac{1}{2}(D_+ D_- + D_- D_+)X(t) = -\frac{1}{M}\nabla V(X(t), t)\,, \qquad (35)$$

where M is an arbitrary constant with the dimensions of mass.

Remark. If the diffusion process pertains to identical particles of mass m each, then M should be identified with m. If we do not attribute any concrete mass to diffusing particles then by setting $m_* = \hbar/2\nu$ where $\hbar$ is the Planck constant, we can associate a certain stochastic (effective) mass m_* with the diffusing particles. For future applications let us introduce a constant with dimensions of the action, $\hbar_* = 2m\nu$, in the case of particles of mass m. (Since ν need not be a constant but may in principle slowly vary with x and t, eq. (35) would apply nevertheless for not too large space–time regions).

The acceleration formulas (33), (35) are of profound importance, since as indicated in ref. [8] *the problem of solving the joint system* (13), (35) *can be uniquely replaced by that of solving the partial differential equation*

$$i(2m\nu)\partial_t\psi(x,t) = [-\tfrac{1}{2}m\nu^2\triangle + V(x,t)]\psi(x,t) \tag{36}$$

in the time interval $[0, T]$, provided we confine $\rho_0(x)$ to a contractible spatial area where the density has no zeroes, and

$$\psi(x,t) = \exp[(R+iS)(x,t)],$$

$$\langle S\rangle_0 = \int dx\, S(x,0)\rho_0(x) = 0,$$

$$2\nu\nabla R(x,t) = u(x,t), \quad 2\nu\nabla S(x,t) = v(x,t). \tag{37}$$

In the case of particles of mass m undergoing the diffusion, by setting

$$\nu = \hbar_*/2m \tag{38}$$

we transform (36) into the familiar Schrödinger type equation. In particular we can set $\hbar_* = \hbar$ where $\hbar$ is the Planck constant, which replaces the corresponding diffusion process by the quantum mechanical Schrödinger problem.

3. Weights for random trajectories and respective path integrals

For small flight times the transition probability density of the diffusion process (13) is given by [1,8]

$$p(y,t,x,t+\Delta t) = (4\pi\nu\Delta t)^{-3/2} \times \exp\left(-\frac{[x-y-b(y,t)\Delta t]^2}{4\nu\Delta t}\right). \tag{39}$$

Since it is necessary that (35) is satisfied, we utilize $b(x,t) = 2\nu\nabla(R+S)(x,t)$ and the system which is equivalent to the Schrödinger equation (38):

$$\partial_t S - \nu[(\nabla R)^2 - (\nabla S)^2 + \triangle R] + V/\hbar = 0,$$

$$\partial_t R + \nu[\triangle S + 2(\nabla S)(\nabla R)] = 0, \quad \langle S\rangle_0 = 0, \tag{40}$$

to evaluate contributions to (39) from the gradient (i.e. b), by means of the Ito formula [5,9] (applied to R and S as functions of the random variable $X(t)$ in the finite difference scheme),

$$\begin{aligned} &f(X(t+\Delta t), t+\Delta t) - f(X(t), t), \\ &\quad t = t_i, \quad t+\Delta t = t_{i+1}, \\ &\quad X(t+\Delta t) = x_{i+1}, \quad X(t) = x_i \\ &\quad \to f(x_{i+1}, t_{i+1}) - f(x_i, t_i) \\ &\quad = (\partial_t f)(x_i, t_i)\Delta t + (\nabla f)(x_i, t_i)\Delta x \\ &\quad + \nu(\triangle f)(x_i, t_i)\Delta t, \\ &\quad \Delta t = t_{i+1} - t_i, \quad \Delta x = x_{i+1} - x_i, \end{aligned} \tag{41}$$

with $\nu = \hbar/2m$. The transition probability density between two fixed points x_0 and x in the time interval $[0, T]$ can be *formally* represented by the random path summation formula [9,10] (see also ref. [8]),

$$\begin{aligned} &p(x_0, 0, x, T) \\ &\quad = \lim_{n\to\infty} \int \prod_{i=0}^{n} p(x_i, t_i, x_{i+1}, t_{i+1}) \prod_{k=1}^{n-1} dx_k \\ &\quad = \frac{\exp[(R+S)(x,T)]}{\exp[(R+S)(x_0,0)]} \\ &\quad \times \int_{X(0)=x_0}^{X(T)=x} [DX] \exp\left[-\frac{1}{\hbar}\int_{t_0}^{t}\left(\tfrac{1}{2}m\dot{q}^2 - V + \frac{\hbar^2}{2m}\frac{\Delta\rho^{1/2}}{\rho^{1/2}}\right)\right], \\ &\quad x_{n+1} = x, \quad t_{n+1} = T, \quad t_0 = 0. \end{aligned} \tag{42}$$

It collects contributions from probabilistic weights, associated with random paths of the stochastic process connecting points x_0 and x during the flight time $T<0$. In the case of $\hbar_* \neq \hbar$, $\hbar_*$ should be used instead of $\hbar$ above.

A formal link of the above path summation procedure with the Feynman path integral was established in ref. [9] and was claimed to provide a stochastic mechanics derivation of the Feynman propagator formula. However, the arguments of refs. [9,10] remain incomplete: one cannot be satisfied with the replacement of the diffusion constant ν process by the "process" with the imaginary diffusion constant $\mathrm{i}\nu$. What is necessary, is to demonstrate that Feynman's path integral can be represented as the summation of phase contributions associated with random trajectories of the same stochastic process as the one underlying (42).

This goal can be indeed achieved along the same lines as (39)–(42), provided that instead of the probability density we shall utilize the proper complex expression for the phase accumulation between points x and y in the small time interval Δt. Let us observe that

$$K(y,t,x,t+\Delta t)=(4\pi\mathrm{i}\nu\Delta t)^{-3/2} \times \exp\left(\frac{\mathrm{i}}{4\nu}\frac{[x-y-v(y,t)\Delta t]^2}{\Delta t}\right) \times \exp\{-\mathrm{i}[\partial_t S+\nu(\nabla S)^2+\nu\Delta S-V/\hbar](y,t)\Delta t\}. \tag{43}$$

After suitable reordering of terms and next utilizing the Ito formula (41) gives rise to

$$K(x_i,t_i,x_{i+1},t_{i+1}) = [4\pi\mathrm{i}\nu(t_{i+1}-t_i)]^{-3/2}\exp\left(\frac{\mathrm{i}}{4\nu}\frac{(x_{i+1}-x_i)^2}{t_{i+1}-t_i} -\mathrm{i}[\partial_t S\Delta t+\nabla S\Delta x+\nu\Delta S\Delta t+V\Delta t/\hbar](x_i,t_i)\right)$$
$$= [4\pi\mathrm{i}\nu(t_{i+1}-t_i)]^{-3/2}\exp\left(\frac{\mathrm{i}}{4\nu}\frac{(x_{i+1}-x_i)^2}{t_{i+1}-t_i} -\mathrm{i}[S(x_{i+1},t_{i+1})-S(x_i,t_i) +V(x_i,t_i)(t_{i+1}-t_i)/\hbar]\right)$$
$$\doteq \exp\{\mathrm{i}[S(x,t_i)-S(x_{i+1},t_{i+1})]\} \times k(x_i,t_i,x_{i+1},t_{i+1}),$$
$$\Delta x=x_{i+1}-x_i,\quad \Delta t=t_{i+1}-t_i. \tag{44}$$

This enables us to write down the phase accumulation formula as a *formal* path integral over random trajectories of the stochastic process (13), (35),

$$K(x_0,0,x,T) = \exp\{\mathrm{i}[S(x_0,0)-S(x,T)]\}k(x_0,0,x,T)$$
$$k(x_0,0,x,T) = \int[\mathrm{D}X]\exp\left(\frac{\mathrm{i}}{\hbar}\int_0^T(\tfrac{1}{2}m\dot{q}^2-V)\,\mathrm{d}t\right). \tag{45}$$

Hence Feynman's expression for the quantum mechanical propagator comes out, and *Feynman paths are essentially random paths of the well defined diffusion process (13) affected by the conservative force field (35).*

Remark 1. The path integral measure for (45) is known not to exist in the literal sense, see ref. [11].

Remark 2. The positive probability weights were attributed to Feynman trajectories in refs. [12,13] (see also references therein). As emphasized in ref. [13] it appears that a common stochastic basis unifies the three viewpoints of quantum particle behaviours: the Bohm–Vigier causal approach, Fenyes–Nelson–Guerra stochastic mechanics and the Feynman path description of quantum mechanics.

References

[1] H. Risken, The Fokker–Planck equation (Springer, Berlin, 1989).

[2] C.W. Gardiner, Handbook of stochastic methods (Springer, Berlin, 1983).

[3] E. Nelson, Quantum fluctuations (Princeton Univ. Press, Princeton, 1985).

[4] Ph. Blanchard, Ph. Combe and W. Zheng, in: Lecture notes in physics, Vol. 281 (Springer, Berlin, 1987).

[5] F. Guerra, Phys. Rep. 77 (1981) 263.

[6] J.C. Zambrini, J. Math. Phys. 27 (1986) 2307.

[7] T. Misawa and K. Yasue, J. Math. Phys. 28 (1987) 2569.

[8] P. Garbaczewski, Phys. Lett. A 143 (1990) 85;
E. Carlen and M. Loffredo, Phys. Lett. A 141 (1989) 9.

[9] M.S. Wang, Phys. Rev. A 38 (1988) 5401.

[10] M.S. Wang, Phys. Lett. A 137 (1989) 437.

[11] E. Nelson, J. Math. Phys. 5 (1964) 332.

[12] N. Cufaro-Petroni, Phys. Lett. A 141 (1989) 370.

[13] J.P. Vigier, Real physical paths in quantum mechanics. Equivalence of Einstein–de Broglie and Feynman points of view on quantum particle behaviour, to appear in: Proc. 3rd Int. Symp. on Foundations of quantum mechanics, Tokyo (1989).