NONGRASSMANN QUANTIZATION OF THE DIRAC SYSTEM

Piotr GARBACZEWSKI
Institute of Theoretical Physics, University of Wrocław, 50-205 Wrocław, Poland

Received 3 July 1979

We develop a path integral formalism which allows understanding of the Dirac equation in terms of the conventional canonical (phase space) variables: the internal, which are constrained and the external.

1. We denote $\mathcal{F}$ an 8-dimensional euclidean manifold parametrized by the canonical coordinates $\{\rho_\mu, \pi_\mu\}_{\mu=1,2,3,4}$ with μ being an euclidean label. The following Poisson bracket structure is imposed on $\mathcal{F}$:

$$\{\rho_\mu, \pi_\nu\} = \delta_{\mu\nu}, \quad \{\rho_\mu, \rho_\nu\} = 0 = \{\pi_\mu, \pi_\nu\}, \tag{1}$$

so that the antisymmetric second rank tensor

$$F_{\mu\nu} = \rho_\mu \pi_\nu - \rho_\nu \pi_\mu, \tag{2}$$

can be used to define the two three-vectors: $A_i = F_{i4} = \rho_i \pi_4 - \rho_4 \pi_i$, $L_i = \frac{1}{2}\epsilon_{ijk} F_{jk}$, $i, j, k = 1, 2, 3$ which satisfy the O(4) group Lie algebra commutation relations on $\mathcal{F}$:

$$\{L_i, L_j\} = \epsilon_{ijk} L_k, \quad \{L_i, A_j\} = \epsilon_{ijk} A_k,$$
$$\{A_i, A_j\} = \epsilon_{ijk} L_k, \tag{3}$$

and set on $\mathcal{F}$ in a linear way, according to

$$-\{F_{\mu\nu}, \rho_\lambda\} = \rho_\mu \delta_{\nu\lambda} - \rho_\nu \delta_{\mu\lambda},$$
$$-\{F_{\mu\nu}, \pi_\lambda\} = \pi_\mu \delta_{\nu\lambda} - \pi_\nu \delta_{\lambda\mu}. \tag{4}$$

2. From now on, we will only use the natural system of units $\hbar = c = 1$. Let us make a conventional canonical quantization step, by introducing a Schrödinger representation of the canonical commutation relations

$$[\rho_\mu, \pi_\nu]_- = i\delta_{\mu\nu}, \quad [\rho_\mu, \rho_\nu]_- = 0 = [\pi_\mu, \pi_\nu]_-,$$
$$[a_\mu, a_\nu^*]_- = \delta_{\mu\nu}, \quad [a_\mu, a_\nu]_- = 0 = [a_\mu^*, a_\nu^*]_-,$$
$$a_\mu^* = (1/\sqrt{2})(\rho_\mu - i\pi_\mu), \quad a_\mu = (1/\sqrt{2})(\rho_\mu + i\pi_\mu). \tag{5}$$

In terms of a_μ^*, a_μ an operator $F_{\mu\nu}$ appears in the form

$$F_{\mu\nu} = i(a_\mu^* a_\nu - a_\nu^* a_\mu), \tag{6}$$

so that $[F_{\mu\nu}, N]_- = 0$ with $N = \Sigma_\mu a_\mu^* a_\mu$ and

$$A_k = i(a_k^* a_4 - a_4^* a_k), \quad L_k = i\epsilon_{ijk} a_j^* a_k. \tag{7}$$

As a consequence the O(4) Lie algebra commutation relations are immediately satisfied.

By defining $L_i = s_i - \zeta_i$, $A_i = s_i + \zeta_i$ we find furthermore:

$$s_1 = \tfrac{1}{2}i(a_2^* a_3 - a_3^* a_2 + a_1^* a_4 - a_4^* a_1),$$
$$s_2 = \tfrac{1}{2}i(a_3^* a_1 - a_1^* a_3 + a_2^* a_4 - a_4^* a_2),$$
$$s_3 = \tfrac{1}{2}i(a_3^* a_4 - a_4^* a_3 + a_1^* a_2 - a_2^* a_1),$$
$$\zeta_1 = \tfrac{1}{2}i(a_1^* a_4 - a_4^* a_1 - a_2^* a_3 + a_3^* a_2),$$
$$\zeta_2 = \tfrac{1}{2}i(a_2^* a_4 - a_4^* a_2 - a_3^* a_1 + a_1^* a_3),$$
$$\zeta_3 = \tfrac{1}{2}i(a_3^* a_4 - a_4^* a_3 - a_1^* a_2 + a_2^* a_1), \tag{8}$$

where

$$[s_i, N]_- = 0 = [\zeta_i, N]_- = 0 = [N, \mathbf{s}^2]_-, \quad \mathbf{s}^2 = \boldsymbol{\xi}^2,$$

and

$$[s_i, \zeta_j]_- = 0, \quad [s_i, s_j]_- = i\epsilon_{ijk} s_k,$$
$$[\zeta_i, \zeta_j]_- = i\epsilon_{ijk} \zeta_k.$$

3. By virtue of the above commutation relations, we can represent the O(4) group Lie algebra in the $N = 1$

sector of the carrier Hilbert space $\otimes_{\mu=1}^{4} h_\mu = h$ for $\{a_\mu^*, a_\mu\}_{\mu=1,2,3,4}$. For that purpose, we shall use a weaker than $N = 1$ constraint and demand state vectors of interest to belong to a subspace $\mathbf{1}_F \otimes_{\mu=1}^{4} h_\mu = h_F$ of h, where

$$\mathbf{1}_F = \prod_\mu \mathbf{1}_{F\mu}$$

$$= \prod_\mu \{:\exp(-a_\mu^* a_\mu): + a_\mu^* :\exp(-a_\mu^* a_\mu): a_\mu\} . \quad (9)$$

Then by taking advantage of the projection theorem (theorem 4) of ref. [1], we get the following identities *on* h_F:

$$\mathbf{1}_F s(a^*, a) \mathbf{1}_F = s(a^* \to b^*, a \to b) = s_F ,$$
$$\mathbf{1}_F \boldsymbol{\zeta}(a^*, a) \mathbf{1}_F = \boldsymbol{\zeta}(a^* \to b^*, a \to b) = \boldsymbol{\zeta}_F , \quad (10)$$

where the mere replacement of the Bose generators a_μ^*, a_μ by Fermi generators b_μ^*, b_μ in eqs. (8) is enough to produce a representation of the O(4) Lie algebra, which is irreducible in the $N = 1$ subspace $h_{1/2}$ of h_F. Here, on $h_{1/2}$ we have: $s^2 = s(s+1) = 3/4$ and $\dim h_{1/2} = 4$. The explicit formulas for the Bose constructed generators b_μ^*, b_μ can be found in refs. [1,2] or quite easily deduced by noting that $\mathbf{1}_F a_\mu^* \mathbf{1}_F = \sigma_\mu^+$, $\mathbf{1}_F a_\mu \mathbf{1}_F = \sigma_\mu^-$, where $\sigma_\mu^+, \sigma_\mu^-$ are the familiar Pauli operators, and then using the Jordan–Wigner transformation [3,2].

4. It was shown by Dahl [4], in his study of the spinning relativized quantum top, that by using the O(4) group Lie algebra generators $s_F, \boldsymbol{\zeta}_F$ and the external space–momentum variables $\boldsymbol{p}, \boldsymbol{x}$, the nonmatrix form of the Dirac hamiltonian arises in the form

$$H_F \phi = \mathrm{i}\partial\phi/\partial t , \quad H_F = 2m\zeta_{F3} + 4\zeta_{F1}(s_F \boldsymbol{p}) , \quad (11)$$

where $p_k = -\mathrm{i}\partial/\partial x_k$, $(\boldsymbol{x}, \mathrm{i}ct) = x$ and throughout the paper $\hbar = c = 1$. Then

$$\phi = \phi(x, t) = \sum_{\mu=1}^{4} \psi_\mu(x, t)\theta_\mu ,$$

with θ_μ's being the eigenvectors of $\boldsymbol{s}^2 = \boldsymbol{\zeta}^2 = 3/4$ in $h_{1/2}$. The orthonormality conditions $(\theta_\mu, \theta_\nu) = \delta_{\mu\nu}$ allow us to introduce a conventional matrix realization of the Dirac equation

$$(m\beta + \boldsymbol{\alpha p})\psi = \mathrm{i}\partial\psi/\partial t ,$$

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} , \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} , \quad k = 1, 2, 3, \quad (12)$$

where ψ is a column consisting of the expansion coefficients $\psi_\mu(x, t)$ of ϕ. Our basic assumption here is to use the spin 1/2 representation of the O(4) group Lie algebra as given before i.e. the Bose constructed one.

5. The Dirac hamiltonian H_F is densely defined in the tensor product Hilbert space $h_{1/2} \otimes \mathcal{H}$ where $\mathcal{H} = \otimes_{i=1}^{3} h_i$ carries a Schrödinger representation of the canonical space–momentum variables $\{x_i, p_i\}_{i=1,2,3}$. If $P_{1/2}$ denotes a projection onto the $N = 1$ subspace $h_{1/2}$ of $h = \otimes_{\mu=1}^{4} h_\mu$ we find that $H_F = P_{1/2} H P_{1/2}$, where the operator H is densely defined in $h \otimes \mathcal{H}$ and is *completely determined in terms of the quantized phase space variables* for the system

$$H = H(\rho, \pi, p) = H(a^*, a, A^*, A) ,$$

with

$$A_k^* = (1/\sqrt{2})(x_k - \mathrm{i}p_k) , \quad A_k = (1/\sqrt{2})(x_k + \mathrm{i}p_k) .$$

By inspection, one easily finds that:

$$H = :H(\rho, \pi, p):$$
$$+ p_1(a_1^* a_1 + a_4^* a_4 - a_2^* a_2 - a_3^* a_3)$$
$$+ p_2(a_1^* a_2 + a_2^* a_1 - a_4^* a_3 - a_3^* a_4)$$
$$+ p_3(a_1^* a_3 + a_4^* a_2 + a_2^* a_4 + a_3^* a_1) , \quad (13)$$

where $:H:$ stands for a normal ordered form of H: recall that under the sign of the normal ordering all ρ, π variables commute to 0.

We have thus proved that the quantized Dirac system can be represented as a conventional hamiltonian system with a total number of 7 (4 internal and 3 external) quantum degrees of freedom, among which the 4 internal ones are subject to the constraint $(N - 1)\phi = 0$, with $N = \Sigma_\mu a_\mu^* a_\mu$.

6. The transition amplitude for any quantum hamiltonian system with constraints can be expressed in the path integral representation (as a sum over all phase space trajectories), according to the general formalism of Faddeev [5], see e.g. ref. [6].

The quantum constraint $(\Sigma_\mu a_\mu^* a_\mu - 1)\phi = 0$ is an image of the following classical one:

$$\mathcal{N} = \sum_\mu \bar{\alpha}_\mu \alpha_\mu = (1/2)(\rho^2 + \pi^2) = 1 \ .$$

In the Faddeev formalism a classical constraint must be accompanied by the "gauge" condition $\varphi(\bar{\alpha}, \alpha) = 0$, which is restricted by the demand that the Poisson bracket (in the ρ, π variables) $\{\varphi(\bar{\alpha}, \alpha), \mathcal{N}\}$ does not vanish. A convenient choice is

$$\varphi(\bar{\alpha}, \alpha) = \sum_\mu \rho_\mu \pi_\mu = (\rho, \pi) = 0$$

$$\Rightarrow \det(\{\varphi, \mathcal{N}\}) = \pi^2 - \rho^2 \ . \qquad (14)$$

Then a physical phase space for the "classical" Dirac system is 12 dimensional (with 6 internal and 6 external dimensions). The "classical" hamiltonian for the Dirac system is

$$\begin{aligned}\mathcal{H}_0 &= H(a^* \to \bar{\alpha}, a \to \alpha, A^* \to \bar{\mathcal{A}}, A \to \mathcal{A}) \\ &= H_{\rm cl}(\rho, \pi, \boldsymbol{p}) + (p_1/2) \\ &\times (\rho_1^2 + \pi_1^2 + \rho_4^2 + \pi_4^2 - \rho_2^2 - \pi_2^2 - \rho_3^2 - \pi_3^2) \\ &+ p_2(\rho_1\rho_2 + \pi_1\pi_2 - \rho_3\rho_4 - \pi_3\pi_4) \\ &+ p_3(\rho_1\rho_3 + \pi_1\pi_3 + \rho_2\rho_4 + \pi_2\pi_4) \ , \end{aligned} \qquad (15)$$

and obviously has nothing in common with any conventional $\hbar \to 0$ limit, as in our case $\hbar = c = 1$ both on the classical and quantum levels. Our "classical" level can be interpreted as to give account of the true classical (i.e. continuous) motions but then on a microscopic scale. Any $\hbar \to 0$ limit means approaching a macroscale on which all finite volume contributions of microclassical fluctuations to macrotrajectories are negligible.

The path integral representation of the transition amplitude reads

$$Z = \int \prod_\mu \mathcal{D}\rho_\mu \mathcal{D}\pi_\mu \prod_i \mathcal{D}p_i \mathcal{D}x_i \prod_t \delta(\rho^2 + \pi^2 - 2)$$

$$\times \delta((\rho, \pi))(\pi^2 - \rho^2) \exp\left[i \int L \, dt \right] , \qquad (16)$$

where:

$$L = \sum_\mu \pi_\mu \dot{\rho}_\mu + \sum_i p_i \dot{x}_i - \mathcal{H}_0(\rho, \pi, \boldsymbol{p}) \qquad (17)$$

and $\mathcal{D}$ indicates an appropriate measure.

7. Let us make a short comment on the role of the gauge condition $(\rho, \pi) = 0$ which divides the internal phase space into two orthogonal subspaces. One can easily check that the quantity

$$G = (1/2)(L^2 + A^2) = (1/2)\rho^2\pi^2 - (\rho, \pi)^2 \ ,$$

is an invariant of the O(4) group on the classical manifold, and that by fixing the value of G, we choose in the physical internal phase space of the Dirac system a five-dimensional submanifold, which is an orbit with respect to the action of O(4). The constraint

$$\rho^2 + \pi^2 - 2 = 0 = 2G/\pi^2 + \pi^2 - 2 \ ,$$

fixes the variability range of G to the interval [0, 1/2] in which $\pi_\pm^2 = 1 \pm (1 - 2G)^{1/2}$, $\rho_\pm^2 = 1 \mp (1 - 2G)^{1/2}$, are the allowed radii of the mutually orthogonal hyperspheres.

Remark. For an example of the Grassmann quantization of the Dirac system, see e.g. ref. [8]. Some references to the earlier nonGrassmann quantization approaches can also be found there, see also refs. [6,4].

This work was partially supported by the National Science and Engineering Research Council of Canada and the Faculty of Science of the University of Alberta. I would like to thank Prof. H. Umezawa for making this financial assistance available.

References

[1] P. Garbaczewski, J. Math. Phys. 19 (1978) 642.
[2] P. Garbaczewski, Phys. Rep. 36C (1978) 65.
[3] P. Garbaczewski, Bull. Acad. Pol. Sci. ser. astr. math. phys. 25 (1977) 711.
[4] J.P. Dahl, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. 39 (1977) No. 12.
[5] L.D. Faddeev, Teor. Mat. Fiz. 1 (1969) 3.
[6] A. Jevicki and N. Papanicolau, Semiclassical spectrum of the Heisenberg chain, IAS Princeton preprint (1978).
[7] V.B. Serebrennikov, A.E. Shabad and Y.A. Smorodinskiy, Nuovo Cimento 47A (1978) 333.
[8] F.A. Berezin and M.S. Marinov, Ann. Phys. (NY) 104 (1977) 336.