THE QUANTUM PENDULUM AS SPIN 1/2

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We derive the conditions under which the quantum pendulum becomes equivalent to the elementary spin 1/2. The relation of the quantum and classical angular momenta is considered.

1. A potential $V = mgl(1-\cos\phi)$ implies the motion of a plane pendulum of mass m, length l in a plane in which an acceleration $g = (g,0,0)$ is applied. The first quantization of the problem arises in the following form:

$$\{(-\hbar^2/2ml^2)\,\mathrm{d}^2/\mathrm{d}\phi^2 + mgl(1-\cos\phi)\}\,\psi = E\psi\,,$$

which after the substitutions

$$2z = \phi,\ q = 4m^2l^3g/\hbar^2,\ a = 8ml^2(E-mgl)/\hbar^2\,,$$

goes over to the well-known Mathieu equation

$$\{(\mathrm{d}^2/\mathrm{d}z^2) + (a - 2q\cos 2z)\}\,\psi = 0\,.$$

This last equation is solvable [1], and provides us with a complete description of the Hilbert space $h = \mathcal{L}^2(0,4\pi)$, $\phi\in[0,4\pi]$. The spectrum of the quantum pendulum for $g\neq 0$ is nondegenerate and both eigenfunctions and eigenvalues exhibit a manifest q-dependence. The Mathieu functions:

$$\mathrm{ce}_{2n}(z\pm\pi) = \mathrm{ce}_{2n}(z)\,,$$

$$\mathrm{se}_{2n+2}(z\pm\pi) = \mathrm{se}_{2n+2}(z)\,,$$

$$\mathrm{ce}_{2n+1}(z\pm\pi) = -\mathrm{ce}_{2n+1}(z)\,,$$

$$\mathrm{se}_{2n+1}(z\pm\pi) = -\mathrm{se}_{2n+1}(z)\,,\qquad n = 0,1,2,\ldots\,,$$

$$\frac{1}{\pi}\int_0^{2\pi}\mathrm{ce}_k(z)\,\mathrm{ce}_l(z)\,\mathrm{d}z = \delta_{kl} = \frac{1}{\pi}\int_0^{2\pi}\mathrm{se}_k(z)\mathrm{se}_l(z)\,\mathrm{d}z\,,$$

$$\int_0^{2\pi}\mathrm{ce}_k(z)\,\mathrm{se}_l(z)\,\mathrm{d}z = 0\,,$$

form a complete orthonormal system (basis) in $h = \mathcal{L}^2(0,4\pi)$, where $h = h^{\mathrm{ce}}\oplus h^{\mathrm{se}}$.

In the limit $g\to 0$ the following properties of eigenvalues arise [1]: $a_0(q)\to 0$, $a_k^{\mathrm{ce}}(q)\to a_k^{\mathrm{se}}(q)\to a_k(0)\neq 0$, $a_k(0) < a_{k+1}(0)$, while in the limit $m\to\infty$, g/l fixed, the spectrum of the quantum pendulum goes over to that of the doubly degenerate harmonic oscillator:

$$E_{2n}^{\mathrm{ce}}\to E_{2n+1}^{\mathrm{ce}}\to(2n+1/2)\hbar(g/l)^{1/2}\,,$$

$$E_{2n+1}^{\mathrm{se}}\to E_{2n+2}^{\mathrm{se}}\to\{(2n+1)+1/2\}\hbar(g/l)^{1/2}\,.$$

Moreover, under the reflection $q\to -q$, we have $a_{2n+1}^{\mathrm{ce}}(\pm q) = a_{2n+1}^{\mathrm{se}}(\mp q)$, while an even part of the spectrum remains unchanged. Let us add that conventionally [3,4] an odd part of the spectrum is omitted as "unphysical", but its presence is crucial in below.

2. Let us now introduce the following indexation of the Mathieu functions: $\mathrm{ce}_{2n}(z) = e_{4n}(z)$, $\mathrm{ce}_{2n+1}(z) = e_{4n+1}(z)$, $\mathrm{se}_{2n+2}(z) = e_{4n+2}(z)$, $\mathrm{se}_{2n+1}(z) = e_{4n+3}(z)$, $n = 0,1,\ldots$. Notice that with respect to their magnitude the eigenvalues a_{4n+2}, a_{4n+3} appear in the inverse order.

The set $\{e_k\}_{k=0,1,\ldots}$ can be used to define densely in h the pair of operators:

$$a^* = \sum_{k=0}^{\infty}\sqrt{k+1}\;e_{k+1}\otimes\bar{e}_k\,,\quad a = \sum_{k=1}^{\infty}\sqrt{k}\;e_{k-1}\otimes\bar{e}_k\,,$$

which as generating in h a Fock representation of the CCR (canonical commutation relations) algebra, give a complete characterization of the quantum pendulum problem, $\{a^*, a, |0) = \mathrm{ce}_0\}$. On the other hand, if one denotes:

$$|n,1) = [(2n+1)!]^{-1/2}(a^*)^{2n+1}|0)\,,$$

$$|n,0) = [(2n)!]^{-1/2}(a^*)^{2n}|0)\,,$$

and notices that:

$$\sum_{n=0}^{\infty} |n,0)(0,n| = \cos^2(\pi N/2)\,, \quad N = a^*a\,, \tag{1}$$

one finds [5] that the operators:

$$b = \frac{\cos^2(\pi N/2)}{(N+1)^{1/2}}\,a\,, \quad b^* = a^*\,\frac{\cos^2(\pi N/2)}{(N+1)^{1/2}}\,, \tag{2}$$

generate in $h = \mathcal{L}^2(0, 4\pi)$ a reducible representation of the CAR (canonical anticommutation relations) algebra, which becomes reduced on each two-dimensional subspace $h_n \subset h$; an even basis vector is the vacuum state for this irreducibility sector,

$$be_{2n} = 0\,, \quad b^*e_{2n} = e_{2n+1}\,, \quad \forall n = 0, 1, \dots .$$

Consequently, with respect to b, b^* we get the following splitting of h:

$$h = \bigoplus_{n=0}^{\infty} (h_n^{\mathrm{ce}} \oplus h_n^{\mathrm{se}})\,,$$

where h_n^{ce} is a linear span of $\{\mathrm{ce}_{2n}, \mathrm{ce}_{2n+1}\}$, and h_n^{se} of $\{\mathrm{se}_{2n+2}, \mathrm{se}_{2n+1}\}$. Notice that b^* creates an energy quantum in h_n^{ce}, while annihilating an energy quantum in h_n^{se}. The converse holds true for b.

Because $\dim h = 2$, each h_n carries an elementary spin 1/2. Hence the quantum pendulum is equivalent to the reducible spin 1/2 equipped with an additional degree of freedom I indicating whether the creation of the spin-up state creates or annihilates an energy quantum: $h^{\mathrm{ce}} = h^+, h^{\mathrm{se}} = h^-$.

3. Now we shall introduce the notion of temperature-dependent ground states $|\Theta(\beta))$ according to thermo-field dynamics [6,7]. They give account of the coupling between the quantum system of interest (quantum pendulum here) and the environment (reservoir), under which the ground state expectation value of $N = a^*a$ reads:

$$(\Theta(\beta)|a^*a|\Theta(\beta)) = \sinh^2\Theta(\beta) = [\exp(\beta\omega) - 1]^{-1}\,. \tag{3}$$

As shown in refs. [8–10], it is possible to construct the one-parameter family $\{\Theta_\lambda(\beta)\}_{\lambda\in[0,\infty)}$ of the above couplings, so that:

$$\lim_{\lambda\to\infty} [\exp(\beta\omega_\lambda) - 1]^{-1} \to [\exp(\beta\omega_{\mathrm{F}}) + 1]^{-1}\,,$$

which expresses a nonthermal procedure of the λ-enforcing of the spin 1/2 approximation of the considered system:

$$\lim_{\lambda\to\infty} (\lambda|{:}F^{\mathrm{c}}(a^*, a){:}|\lambda) = (\infty|{:}F(b^*, b){:}|\infty)\,,$$

$$|\lambda) \in h\,, \quad |\infty) \in h_{\mathrm{F}} = 1_{\mathrm{F}}h\,.$$

Here $[b, b^*]_+ = 1_{\mathrm{F}}$ and $\{b^*, b, |0)\}$ is a Fock representation of the CAR (canonical anticommutation relations) algebra. If applied to the quantum pendulum problem, such a choice of the temperature-dependent ground state, mathematically represents a selection of an irreducible spin 1/2 component from the mixture, and says that the system–reservoir coupling prefers spin 1/2 based on $h_0^{\mathrm{ce}} = h_0^+$.

At this place we shall generalize the spin 1/2 approximation concept of refs. [8–10] to allow the selection of other than h_0^+ spins 1/2. We take: $f = \Sigma_{k=0}^{\infty} f_k e_k \in h$ and define:

$$\sinh^2\Theta_\lambda^{2s}(\beta) = \frac{\Sigma_{k=1}^{\infty} |k-2s\,\|f_k|^2 \exp(-\beta E_k^\lambda)}{\Sigma_k |f_k|^2 \exp(-\beta E_k^\lambda)}\,, \tag{4}$$

where the λ-scaling of the quantum pendulum spectrum is chosen by demanding: $\lim_{\lambda\to\infty} E_k^\lambda = \infty$ for all $k < 2s, k > 2s+1$ while $\lim_{\lambda\to\infty} E_{2s}^\lambda = E_{2s}$, $\lim_{\lambda\to\infty} E_{2s+1}^\lambda = E_{2s+1}$ (for an explicit construction of such scaling in the limit $m \to \infty$, which corresponds to ϕ_1^4, see refs. [2,10]). Now, we have:

$$\lim_{\lambda\to\infty} \sinh^2\Theta_\lambda^{2s}(\beta) = [\exp\{\beta(\omega_{\mathrm{F}} + \epsilon_{2s})\} + 1]^{-1}\,, \tag{5}$$

where $\omega_{\mathrm{F}} = (2/\beta)\ln|f_{2s}/f_{2s+1}|$ and $\epsilon_{2s} > 0$ for $n = 0, 2, 4, \dots, \epsilon_{2s} < 0$ for $n = 1, 3, 5, \dots$.

By repeating the arguments of refs. [8,9], one easily finds that each 2sth coupling of the quantum pendulum with the environment, in the limit $\lambda \to \infty$ selects a single irreducible spin 1/2 component of the quantum pendulum, which is based on h_n^+ for $n = s/2, s = 0, 2, 4, \dots$

and on h_n^- for $n = (s-1)/2, s = 1,3,5, ...$, respectively.

4. The quantum pendulum understood as a triple $\{a^*, a, \mathrm{ce}_0 = |0)\}$ defines in $h = \mathcal{L}^2(0,4\pi)$ a symmetry group E(2) of the euclidean plane:

$$T^+ = a^* , \quad T^- = a , \quad J_z = -\tfrac{1}{2} + a^* a ,$$

$$[J_z, T^\pm]_- = \pm T^\pm , \quad [T^-, T^+]_- = 1_B ,$$

so that $T_\alpha = \exp(\alpha a^* - \bar{\alpha} a)$ translates the vacuum $T_\alpha|0) = |\alpha)$ into a coherent state, while J_z realizes rotations in the complex α-plane:

$$a \to a \exp(-i\phi) = a_\phi \Rightarrow (\alpha|a_\phi|\alpha) = \alpha \exp(-i\phi) .$$

Let us now consider the temperature-dependent expectation value of the generator J_z in the limit $\lambda \to \infty$:

$$\lim_{\lambda\to\infty} (\Theta_\lambda^{2s}(\beta)|J_z|\Theta_\lambda^{2s}(\beta)) = -\tfrac{1}{2} + [\exp\{\beta(\omega_F + \epsilon_{2s})\} + 1]^{-1} \in [-\tfrac{1}{2}, +\tfrac{1}{2}] . \tag{6}$$

We have here realized the generalized spin 1/2 approximation of the quantum pendulum. Namely, we find in h vectors $|\lambda, 2s)$ so that:

$$\lim_{\lambda\to\infty} (\lambda, 2s|{:}F_N^c(a^*, a){:}|\lambda, 2s) = (2s|{:}F(b^*, b){:}|2s) ,$$

where $|2s) = \lim_{\lambda\to\infty}|\lambda, 2s) \in h_n$ (i.e. h_n^+ for $2n = s$, h_n^- for $2n+1 = s$) and the index N at ${:}F_N^c(a^*, a){:}$ means that in the explicit normal ordered operator expansion for ${:}F^c(a^*, a){:}$ all a^*, a should be replaced by $a^* = (1/(N+1)^{1/2})a^*$, $a = (1/(N+1)^{1/2})a$, respectively, cf. e.g. ref. [2]. But it also means that

$$\lim_{\lambda\to\infty} (\lambda, 2s|a^*/(2s+1)^{1/2}|\lambda, 2s) = (2s|b^*|2s) ,$$

$$\lim_{\lambda\to\infty} (\lambda, 2s|a/(2s+1)^{1/2}|\lambda, 2s) = (2s|b|2s) ,$$

i.e. the whole SU(2) Lie algebra for spin 1/2 emerges in the place of $(T^\pm, J_z)$: $S^+ = \sigma^+, S^- = \sigma^-, S_z = -1/2 + \sigma^+\sigma^-$.

5. With a moving plane pendulum (classical) we can associate an angular momentum vector, averaged over the period τ, oriented in the z-direction and with length $L_0 = \tau^{-1}\int_0^\tau ml^2\dot{\phi}\,dt$. In the case of rotating motion we have $\pm 2\pi ml^2/\tau = L_0$, where the sign depends on whether sign $\dot{\phi} = \pm$, respectively, while for an oscillation: $L_0 = 0$.

On the other hand, the quantal result in the spin 1/2 approximation would be 1/2 and no apparent relation between the classical and quantum angular momenta is seen.

For simplicity (but with no loss of generality) we restrict our considerations to the case $s = 0$ and demand:

$$(\Theta_\lambda(\beta)| \pm J_z|\Theta_\lambda(\beta)) = \pm(-\tfrac{1}{2} + \sinh^2\Theta_\lambda(\beta)) = \pm 2\pi ml^2/\tau_\lambda\hbar . \tag{7}$$

In the limit $\lambda \to \infty$ we get then:

$$\pm 1/\tau_\infty = (\hbar/2\pi ml^2)(-\tfrac{1}{2} + [\exp(\omega_F + \epsilon) + 1]^{-1}) , \tag{8}$$

where: $\pm 1/\tau_\infty \in [-\hbar/4\pi ml^2, \hbar/4\pi ml^2]$ and consequently a lower bound for the classically admitted rotation period arises: $\tau_\infty \geqslant 4\pi l^2 m/\hbar$. In addition, because the spin 1/2 approximation defines an irreducible spin 1/2 in h^+, we can introduce a rotation through an angle θ about an axis $(\sin\varphi, -\cos\varphi, 0)$ by the use of S [11,12],

$$R_{\theta,\varphi} = \exp(\xi S^+ - \bar{\xi} S^-) ,$$

$$\xi = \tfrac{1}{2}\theta \exp(-i\varphi) , \quad |\theta,\varphi) = R_{\theta,\varphi}|0) ,$$

$$S_{\theta,\varphi} = R^*_{\theta,\varphi} S R_{\theta,\varphi} \Rightarrow (0|S_{\theta,\varphi}|0) = (\theta,\varphi|S|\theta,\varphi) ,$$

$$(\theta,\varphi|S_z|\theta,\varphi) = \tfrac{1}{2}\cos\theta , \quad (\theta,\varphi|S_x|\theta,\varphi) = \tfrac{1}{2}\sin\theta\cos\varphi ,$$

$$(\theta,\varphi|S_y|\theta,\varphi) = \tfrac{1}{2}\sin\theta\sin\varphi , \quad (\theta,\varphi|S^2|\theta,\varphi) = 3/4 , \tag{9}$$

$|0)$ being the spin-down (vacuum) state in a spin 1/2 sector. The rotated vacuum state $|\theta,\varphi)$ is known as the Bloch state and provides us with a classical-like image of S.

The above formulas suggest to define the deflection angle by demanding:

$$\cos\theta = 4\pi l^2 m/\tau_\infty\hbar \quad \text{if } \theta \in [0, \pi/2) ,$$

$$= -4\pi l^2 m/\tau_\infty\hbar \quad \text{if } \theta \in [\pi/2, \pi] ,$$

which establishes the link between eqs. (7) and (9): the classical plane (m, l) pendulum rotating with the period $\tau_\infty = 4\pi l^2 m/\hbar$ around the z-axis, can be considered as a classical relative (descendant or ancestor) of the irreducible spin 1/2 component of the quantum pendulum, here $\theta = 0$. Taking $\tau_\infty > 4\pi l^2 m/\hbar$ induces a corresponding θ-rotation of eq. (9) for all possible φ.

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