ALMOST FERMI, BOSE DISTRIBUTIONS OR SPIN 1/2 APPROXIMATION OF BOSE MODES IN QUANTUM THEORY

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A metamorphosis of bosons into fermions is characterized in terms of finite temperature statistical distributions.

1. A statistical average of a quantity A over the grand canonical ensemble at temperature T is given by:

$$\langle A\rangle = Z^{-1}(\beta)\,\mathrm{Tr}\,(A\exp(-\beta\mathcal{H}))\,,\qquad \mathcal{H} = H - \mu N\,,\quad Z(\beta) = \mathrm{Tr}\,(\exp(-\beta\mathcal{H}))\,, \tag{1}$$

where H is a total hamiltonian of a single elementary quantum system while μ is the chemical potential. Let us suppose that $\langle A\rangle$ can be given in the form of the ground state expectation value:

$$\langle A\rangle = (0(\beta)|A|0(\beta)) = Z^{-1}(\beta)\sum_n (n|A|n)\exp(-\beta E_n)\,, \tag{2}$$

where $\mathcal{H}|n) = E_n|n)$, $(n|m) = \delta_{nm}$, $|n) = (1/\sqrt{n!})a^{*n}e_0$ and $|0(\beta))$ arises as a "temperature dependent ground state" [1], in the form:

$$|0(\beta)) = Z^{-1/2}(\beta)\sum_n \mathrm{e}^{-\beta E_n/2}|n,\tilde{n})\,, \tag{3}$$

$$(\tilde{n},n|A|n',\tilde{m}') = (n|A|n')\delta_{n'm'}\,,\qquad |n,\tilde{n}) = |n)\,|\tilde{n})\,.$$

The tilde system playing the role of the reservoir appears according to:

$$\tilde{\mathcal{H}}|\tilde{n}) = E_n|\tilde{n})\,,\qquad (\tilde{n}|\tilde{m}) = \delta_{nm}\,,\qquad |\tilde{n}) = (1/\sqrt{n!})\tilde{a}^{*n}\tilde{e}_0\,,\quad \tilde{a}\tilde{e}_0 = 0\,,\quad [\tilde{a},\tilde{a}^*]_- = \mathbf{1}_\mathrm{B}\,. \tag{4}$$

The use of temperature dependent expectation values for a spin 1/2 system implies:

$$H = \omega b^*b\,,\qquad [b,b^*]_+ = \mathbf{1}_\mathrm{F}\,,$$
$$|0(\beta))_\mathrm{F} = [1+\exp(-\beta\omega)]^{-1/2} \times \{|0) + \mathrm{e}^{-\beta\omega/2}b^*\tilde{b}^*|0)\} = \mathrm{e}^{-\mathrm{i}G}|0)\,, \tag{5}$$
$$|0) = e_0\otimes\tilde{e}_0\,,\qquad G = -\mathrm{i}\Theta(\beta)\{\tilde{b}b - b^*\tilde{b}^*\}\,,$$
$$\cos\Theta(\beta) = (1+\mathrm{e}^{-\beta\omega})^{-1/2}\,,$$

so that: ${}_\mathrm{F}(0(\beta)|b^*b|0(\beta))_\mathrm{F} = \mathrm{e}^{-\beta\omega}/(1+\mathrm{e}^{-\beta\omega})$, while for a Bose system $H = \omega a^*a$, $[a,a^*]_- = \mathbf{1}_\mathrm{B}$ it gives:

$$|0(\beta))_\mathrm{B} = [1-\exp(-\beta\omega)]^{-1/2}\{\exp(a^*\tilde{a}^*\,\mathrm{e}^{-\beta\omega/2})\}|0) = \mathrm{e}^{-\mathrm{i}G}|0)\,,\qquad G = -\mathrm{i}\Theta(\beta)(\tilde{a}a - a^*\tilde{a}^*)\,, \tag{6}$$
$$\cosh\Theta(\beta) = (1-\mathrm{e}^{-\omega\beta})^{-1/2}\,,$$

hence: ${}_\mathrm{B}(0(\beta)|a^*a|0(\beta))_\mathrm{B} = \mathrm{e}^{-\beta\omega}/(1-\mathrm{e}^{-\beta\omega})$.

A generalization onto the level of quantum field theory is immediate, thus leading to a countable collection of identities:

$$ {}_\mathrm{B}(0(\beta)|a_s^*a_s|0(\beta))_\mathrm{B} = \mathrm{e}^{-\beta\omega_s}/(1-\mathrm{e}^{-\beta\omega_s})\,,\qquad {}_\mathrm{F}(0(\beta)|b_s^*b_s|0(\beta))_\mathrm{F} = \mathrm{e}^{-\beta\omega_s}/(1+\mathrm{e}^{-\beta\omega_s})\,, \tag{7}$$

with the new (infinite tensor product structure) ground states:

$$|0(\beta))_\mathrm{B}\,,\quad |0(\beta))_\mathrm{F}\,,\quad |0(\beta)) = \Pi_s\otimes|0(\beta))_s\,.$$

Assume now a lower bound $\omega_0 \leqslant \omega_s$ for the admissible frequency spectrum, and let $\beta \gg 0$ (low temperature limit). Then:

$${}_\mathrm{B}(0(\beta)|a_s^*a_s|0(\beta))_\mathrm{B} \approx \exp(-\beta\omega_s) \approx {}_\mathrm{F}(0(\beta)|b_s^*b_s|0(\beta))_\mathrm{F}\,,$$

thus establishing an *almost equivalence of boson and spin 1/2 (fermion) modes* with coinciding frequencies. The distinction between boson and spin 1/2 modes becomes meaningful with the temperature increase. At this place our aim is to construct the mechanism which is able to compensate the difference between Fermi and Bose distributions so that both cases become indistinguishable within experimental accuracy limits for all finite temperature values.

2. Assume to have fixed a countable sequence $\{f_s\}_{s=1,2,\ldots}$ of state vectors in a Hilbert space h, and $|f) = \Pi_s \otimes (f \otimes \tilde{e}_0)_s$. Define furthermore (notice a non-unitarity of U_λ in h):

$$\prod_s \otimes (h \otimes \tilde{h})_s \ni |f, \lambda) = \prod_s \otimes (f^\lambda \otimes \tilde{e}_0)_s ,$$

$$f_s^\lambda = U_\lambda f_s / \| U_\lambda f_s \| , \tag{8}$$

$$\| U_\lambda f_s \|^2 = \sum_k |f_s^k|^2 \left(\frac{1}{1+\lambda}\right)^{\Sigma_{j=1}^k (j-1)} .$$

Let further the product vector $|f, \lambda)$ in $\Pi_s \otimes (h \otimes \tilde{h})_s$ be constructed in the following way:

$$f_s^0 \in \mathbf{R}, \quad f_s^0 > 0 \forall_s , \quad |f, \lambda) = \prod_s \otimes (f^\lambda \otimes \tilde{e}_0)_s ,$$

where:

$$\sum_s \ln \left\{ \frac{\|f_s\|}{f_s^0} \left[1 + \sum_{k=1}^\infty \frac{k|f_s^k|}{\|f_s\|^2} \right]^{1/2} \right\} < \infty .$$

Then for each fixed value of $\lambda \in (0, \infty)$ there exists in the incomplete direct product space IDPS ($|f, \lambda)$) an associated vector $|\Theta(\lambda))$ satisfying:

$$(\Theta(\lambda)| a_s^* a_s |\Theta(\lambda)) = \sinh^2 \Theta_s(\lambda) = (f, \lambda | a_s^* a_s | f, \lambda) , \tag{9}$$

for all s. Hence by using the basis $\{e_k\}_{k=0,1,\ldots}$ in h, we can write:

$$h = h_s \ni f_s^\lambda = \sum_{k=0}^\infty f_s^k e_k^\lambda \cdot (1/\| U_\lambda f_s \|) ,$$

$$(f, \lambda | a_s^* a_s | f, \lambda) = \sinh^2 \Theta_s(\lambda) = \sum_{k=1}^\infty \frac{k|f_s^k|^2}{\|U_\lambda f_s\|} \left(\frac{1}{1+\lambda}\right)^{\Sigma_{j=1}^k (j-1)} . \tag{10}$$

More details can be found in ref. [2]. By defining:

$$\Omega_s(\lambda) = -\ln [\sinh^2 \Theta_s(\lambda) / (1 + \sinh^2 \Theta_s(\lambda))] , \tag{11}$$

where:

$$\sinh^2 \Theta_s(\lambda) = \sum_{k=1}^\infty \frac{k|f_s^k|^2}{\|U_\lambda f_s\|^2} \left(\frac{1}{1+\lambda}\right)^{\Sigma_{j=1}^k (j-1)} , \tag{12}$$

we get:

$$\sinh^2 \Theta_s(\lambda) = 1/(\exp \Omega_s(\lambda) - 1) , \tag{13}$$

where

$$\lim_{\lambda \to \infty} \Omega_s(\lambda) = \ln \left\{ \left(1 + \frac{|f_s^1|^2}{|f_s^1|^2 + |f_s^0|^2}\right) \bigg/ \frac{|f_s^1|^2}{|f_s^0|^2 + |f_s^1|^2} \right\} ,$$

so that (14)

$$\lim_{\lambda \to \infty} \frac{1}{e^{\Omega_s(\lambda)} - 1} = \frac{1}{1 + |f_s^0/f_s^1|^2} := \frac{1}{1 + e^{\Omega_s^F}} . \tag{15}$$

Let us notice that $\Omega_s(\lambda)$, Ω_s^F arise as dimensionless quantities thus providing us with a continuous spectrum of allowed dimensional frequencies for each fixed value $\Omega_s(\lambda) \in \mathbf{R}^+ : \Omega_s(\lambda) = \beta\omega_s(\lambda, \beta)$. An index β in $\omega_s(\lambda, \beta)$ means that we have established a frequency $\omega_s(\lambda)$ relative to the β unit. By fixing $\beta = \bar{\beta}$ we have selected a single frequency curve $\omega_s(\lambda, \bar{\beta}) = \omega_s(\lambda) = \Omega_s(\lambda)/\bar{\beta}$ which characterizes the λ-dependence at the thermal $\bar{\beta}$-equilibrium. Obviously states $|\Theta(\lambda))$ cannot exhibit any particular temperature dependence as giving account of the λ-constraint for all possible temperature values jointly.

3. Let us now extract from $|f, \lambda)$ a finite tensor product vector $|f, \lambda)_J = \Pi_{s \in J} \otimes f_s^\lambda$. We define a vector $|\lambda)$ associated with $|f, \lambda)_J$ by putting:

$$(\lambda | a_s^* a_s | \lambda) = \sinh^2 \Theta_s(\lambda) \cdot \prod_{s \in J} \| U_\lambda f_s \|^2 ,$$

$$|\lambda) = \prod_{s \in J} \otimes U_\lambda f_s = \left(\prod_{s \in J} U_\lambda^s \right) \cdot \prod_{s \in J} \otimes f_s := U_\lambda |0) , \tag{16}$$

$$|0) = \lim_{\lambda \to 0} |\lambda) .$$

The redundant $\tilde{e}_0 \otimes \ldots \otimes \tilde{e}_0$ terms in the tensor product were for simplicity omitted. Observe now [2–7], that $\mathbf{1}_F^s = :\exp(-a_s^* a_s): + a_s^* :\exp(-a_s^* a_s): a_s$ projects onto $\lim_{\lambda \to \infty} f_s^\lambda = f_s^\infty = \Sigma_{k=0,1} f_s^k e_k$ and is a projection in h onto a linear span of $\{e_0, e_1\}$. Consequently: $\mathbf{1}_F^s f_s^\lambda = f_s^\infty = \mathbf{1}_F^s f_s$. Take further $\Pi_{s \in J} \otimes f_s$. One can easily check by a straightforward calculation that [2–7]:

$$\prod_{s\in J} 1_F^s \prod_{s\in J} \otimes f_s = \prod_{s\in J} \otimes f_s^\infty = 1_F \prod_{s\in J} \otimes f_s^\lambda = 1_F \prod_{s\in J} \otimes f_s . \tag{17}$$

On the other hand it is easy to check that:

$$\lim_{\lambda\to\infty} (\lambda| a_s^* |\lambda) = (0| 1_F^s a_s^* 1_F^s |0)$$
$$= (0| a_s^* : \exp(-a_s^* a_s) : |0) = (0| \sigma_s^+ |0) ,$$

$$\lim_{\lambda\to\infty} (\lambda| a_s |\lambda) = (0| 1_F^s a_s 1_F^s |0)$$
$$= (0| : \exp(-a_s^* a_s) : a_s |0) = (0| \sigma_s^- |0) , \tag{18}$$

$$\lim_{\lambda\to\infty} (\lambda| (-1/2) + a_s^* a_s |\lambda) = (0| \{(-1/2) 1_F^s + \sigma_s^+ \sigma_s^- |0) ,$$

which forms a weak metamorphosis of the E(2) group Lie algebra $T_s^+ = a_s^* \to S_s^+$, $T_s^- = a_s \to S_s^-$, $J_s^3 = (-1/2) + a_s^* a_s \to S_s^3$ into that of SU(2) in the case of spin 1/2. By comparing with refs. [2–7], we find that the $\lambda \to \infty$ limit gives us the *spin 1/2 approximation of a given Bose field theory: each single boson degree of freedom can be effectively replaced by an associated spin 1/2 degree.* Due to eqs. (11)–(15) the λ-enforcing of the spin 1/2 approximation can be understood as a transition through a one-parameter family $\omega_s(\lambda, \bar{\beta})$ of frequencies of a quantum Bose system at a fixed thermal equilibrium $\beta = \bar{\beta}$. The λ-mechanism is obviously temperature independent.

Remark. A study of relations between Bose and Fermi systems can be found in refs. [6–10]. Let us however add that some (historically first) aspects of the sine-Gordon–Thirring model equivalences were studied in refs. [8,9].

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