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A Canonical Description of the Solitary Quantum Decay (*).

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Summary. — A quantum picture of solitary motion is presented. Time evolution of a quantum N-soliton is described as a transition through infinitely many unitarily inequivalent representations of the canonical commutation relations. The statistical nature and the irreversibility of the process naturally emerge. Their origin is found in the nonunitary character of the transformation among the unitarily inequivalent representations.

1. – Introduction: classical solitary motion.

Let us consider a soliton sector of the sine-Gordon system in two space-time dimensions:

$$L=\frac{1}{2}\left(\left(\frac{\partial\varphi(x,t)}{\partial x}\right)^2-\left(\frac{\partial\varphi(x,t)}{\partial t}\right)^2+2m(1-\cos\varphi(x,t))\right), \tag{1.1}$$

whose general constituents are introduced in the form [1]

$$\begin{cases}\cos\varphi(x,t)=1-\dfrac{2}{m^2}\left(\dfrac{\partial^2}{\partial x^2}-\dfrac{\partial^2}{\partial t^2}\right)\ln f(x,t)\,,\\ f(x,t)=\det|M|;\end{cases} \tag{1.2a}$$

(*) To speed up publication, the authors of this paper have agreed to not receive the proofs for correction.

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(1) P. J. CAUDREY, J. C. EILBECK and J. D. GIBBON: *Nuovo Cimento*, **25** B, 497 (1975); V. E. ZAKHAROV, L. A. TAKHTADIAN and L. D. FADDEEV: *Dokl. Akad. Nauk. USSR*, **219**, 1334 (1974).

$|M| = \{M_{ij}\}$ is the $N\times N$ matrix of elements

$$(1.2b)\qquad \begin{cases} M_{ij} = \dfrac{2}{a_i + a_j}\cosh\dfrac{\theta_i + \theta_j}{2}\,, \\ \theta_i = \pm m\gamma_i(x - v_i t) + \delta\,, \qquad \gamma_i^2 = (1 - v_i^2)^{-1}\,, \\ a_i^2 = \dfrac{1 - v_i}{1 + v_i}\,, \end{cases}$$

the solution $\varphi(x, t) \equiv \varphi_N(x, t)$ is called the N-soliton solution. The solution is parametrized by the number N of parameters $\{a_i\}$ which are in general complex. We shall restrict our considerations to real parameters only. The major point of interest then becomes an asymptotic structure of the solution (1.2) under the assumption that constants a_i are real:

$$(1.3)\qquad \begin{cases} \lim\limits_{t\to\pm T, |T|\gg 0} \varphi(x, t) = 4\sum\limits_{i=1}^{N} \mathrm{tg}^{-1}\exp[\theta_i + \eta_i^{\pm}] = 4\sum\limits_{i}\varphi_i(x, t)\,, \\ \eta_i^{+} = \pm\dfrac{1}{2}\sum\limits_{j=1}^{i-1}\ln a_{ji} \mp \dfrac{1}{2}\sum\limits_{j=i+1}^{N}\ln a_{ji}\,, \\ a_{ij} = \left(\dfrac{a_i - a_j}{a_i + a_j}\right)^2. \end{cases}$$

The asymptotic fields φ_i are called 1-solitons. The corresponding general solution $\varphi(x, t) \equiv \varphi_N(x, t)$ describes the scattering process of the number N of 1-(anti)solitons, where the number of particles is preserved.

Let us reduce our interest to a positive part only of the time evolution scale, and start to consider a solitary motion at the point $t = 0$ towards $t = +\infty$. Now, at the point $t = 0$ we begin from a nonlinear structure which represents a certain initial interacting-N-particle function, where the term N particles indicates that asymptotically N free solitons will appear. After a sufficiently large time period the asymptotic separation appears. Thus we deal in fact with *a classical decay model*, where at $t = 0$ the unstable solution is given which further decomposes into stable particles, 1-solitons, which for all times preserve their 1-soliton identity.

The well-known property of soliton solutions is that they represent the *localized* energy and momentum distributions

$$(1.4)\qquad \begin{cases} E = \displaystyle\int \mathscr{H}(\varphi(x, t))\,\mathrm{d}x = \sum_{i=1}^{N} E_i\,, \\ P = \displaystyle\int \partial_x\varphi(x, t)\,\partial_t\varphi(x, t)\,\mathrm{d}x = \sum_{i=1}^{N} P_i\,, \end{cases}$$

where E_i and P_i are energies and momenta, respectively, of the asymptotic components; $\mathscr{H}(\varphi)$ is the energy density corresponding to the solution $\varphi(x, t)$. Moreover, these global quantities E and P are strictly preserved, despite of the time chosen. Hence, our classical decay is a purely elastic process. In general, any N-soliton solution, depending on its complex parametrization, can be asymptotically decomposed into a number n of 1-solitons and m bound-state excitations, which are called bions. Then $N = n + 2m$, and again the global energy and momentum have the form

$$E = \sum_{i=1}^{n+m} E_i , \qquad P = \sum_{i=1}^{n+m} P_i ,$$

which follows from the asymptotic decomposition.

Let us turn back to the N-soliton (1.2), which decomposes into a number N of 1-(anti)solitons. Any $\varphi(x, t)$ can be parametrized by the number N of quite arbitrary velocity parameters

$$\varphi_N(x, t) = f_N(v_1, \dots, v_N)(x, t) . \tag{1.5}$$

The only restriction here is $0 < |v_i| < 1 = c$, for any i, with c being the light velocity and that, if any pair of velocities coincides, then the solution vanishes [2,3]: $v_i = v_j$, for any i, j, implies $f_N = 0$. From (1.2) one sees that each θ_k depends on v_k only; v_k then characterizes in the asymptotic limit the k-th 1-(anti)soliton, as one can see from (1.3). Since 1-solitons are energy distributions, $f_N(v_1, \dots, v_N)(x, t)$ can be thought of as a function which describes the motions of the N energy centres of the underlying distributions and the motions relative to them. We admit here the interaction between energy centres. Let us now consider the solitary evolution in the lattice approximation. Namely, we shall assume to have the linear lattice so that the line $\mathbf{R}^1$ is countably covered by the set $\{\Delta\}$ of nonintersecting open intervals $\Delta_s \cap \Delta_t = 0$, for $s \neq t$. The lattice constant is assumed to be given: $\mu(\Delta_s) = \lambda$ for all $s = 0, \pm 1, \pm 2, \dots$. Let us assume that $\varphi_N(x, t)$ at the time $t = 0$ represents the number N of interacting–single-particle energy distributions, whose energy centres occupy the positions $y_1, \dots, y_N$, each belonging to a respective interval $\Delta \in \mathbf{R}^1$. Then the point x can be completely identified (in the lattice sense) when we write

$$\varphi_N(x, 0) = f_N(v_1, \dots, v_N; y_1 + n_1\lambda, y_2 + n_2\lambda, \dots, y_N + n_N\lambda) = \\ = \varphi^{n_1, n_2, \dots, n_N}(v_1, \dots, v_N; y_1, \dots, y_N) . \tag{1.6}$$

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(2) S. ORFANIDIS: *Phys. Rev. D*, **14**, 472 (1976).

(3) P. GARBACZEWSKI: *Self-quantization of the sine-Gordon system in the soliton sector*, University of Salerno preprint (1977); *Boson expansion methods in quantum theory*, to appear in *Phys. Rep. C*.

Any single parameter n_k establishes that the site to which x belongs is n_k-th relative to the y_k position of the energy centre: $n_k = 0, \pm 1, \pm 2, \ldots$. Next, let $v = |v_k|$ be the smallest velocity in the set $(v_1, \ldots, v_N)$; we introduce then the identification $v_k = r_k v$ with $r_k = \pm 1, \pm 2, \ldots$ (we have thus a discrete velocity scale).

The time evolution of (1.6) reads (³)

$$(1.7) \qquad \varphi_N\left(x, t = \frac{\nu\lambda}{v}\right) = f_N\left(v_1, \ldots, v_N, y_1 + n_1\lambda - \nu\lambda r_1 + \int_0^{t=\nu\lambda r_1/v_1} \dot{y}_1(t)\,\mathrm{d}t, \ldots \right.$$
$$\left. \ldots, y_N + m_N\lambda - \nu\lambda r_N + \int_0^{t=\nu\lambda r_N/v_N} \dot{y}_N(t)\,\mathrm{d}t\right),$$

where terms of the form $\nu\lambda r_k$ (ν is an integer) represent a uniform motion of the k-th energy centre (asymptotic free motion; compare what happens if 1-soliton is considered in the place of φ_N) and $\int_0^{t=\nu\lambda r_k/v_k} \dot{y}_k(t)\,\mathrm{d}t$ represent the result of accelerated motion suffered by the k-th energy centre in the interval $[0, t = \nu\lambda/v]$ (interactions with other energy distributions). For sufficiently large ν_s's ($t \gg 0$) we can, in fact, decompose φ_N into a linear sum of single 1-(anti)-soliton distributions, where each single term is of the form

$$(1.8) \qquad 4\,\mathrm{tg}^{-1} \exp\left[\pm m\gamma_k\left(y_k + n_k\lambda - \nu\lambda r_k + \int_0^{+\infty} \dot{y}_k(t)\,\mathrm{d}t\right)\right],$$

where $y_k + n_k\lambda = x_k$ is the initial position of interest (varying y_k, we get the information about space properties of the 1-soliton solution) and $\int_0^{+\infty} \dot{y}_k(t)\,\mathrm{d}t = \eta_k^{\pm}$ is the k-th phase shift of the solution, see eq. (1.2).

The function (1.7) can be easily transformed into the form exhibiting the hopping motion from site to site, which is modified by the time-dependent phase shift:

$$(1.9) \qquad \varphi_N\left(x, t = \frac{\nu\lambda}{v}\right) = \varphi^{n_1-\nu r_1, \ldots, n_N - \nu r_N}\left(v_1, \ldots, v_N; y_1 + \eta_1(t), \ldots, y_N + \eta_N(t)\right).$$

Now the discrete time $t = \nu\lambda/v$ labels the instants of our classical decay process. Let us add that, provided the velocities are given, a global momentum of the N-soliton is established, so it can be introduced as an additional parameter P labelling momentum properties of the functions (1.7)-(1.9). In the next section we will introduce a quantum picture for the solitary time evolution. The quantum decay process will be modelled by the classical solitary decay motion introduced in the present section. In sect. **3** a description of the solitary quantum decay process is presented, which is based on the existence of infinitely many unitarily inequivalent representations of the canonical commutation relations. The statistical nature and the irreversibility of the solitary decay process naturally emerges from the nonunitarity of the transformation among the inequivalent representations.

2. – Solitary quantum decay.

Let us start from the classical N-soliton solution (1.6) *at time* $t = 0$. Let us assume to have a family $\{f_k\}$ of complex test functions with the properties

$$\begin{cases} f_k = f_k(v_1, \dots, v_N)\,, \\ \int \mathrm{d}\boldsymbol{v}_N \boldsymbol{f}_k(\boldsymbol{v}_N) \boldsymbol{f}_l(\boldsymbol{v}_N) = \delta_{kl}\,, \\ \sum_k f_k(\boldsymbol{v}_N) f_k(\boldsymbol{v}'_N) = \delta_{\boldsymbol{v}_N \boldsymbol{v}'_N}\,, \\ f_k(\boldsymbol{v}_N) = 0 \text{ for } k \neq \sum_{i=1}^{N} k_i\,, \end{cases} \tag{2.1}$$

where $\boldsymbol{v}_N \equiv (v_1, \dots, v_N)$ and each of the k's is a 1-soliton asymptotic momentum defined by the parameter v_i. In addition, let us assume that the initial positions of energy centres belong to a finite interval Δ_{in} with a characteristic function

$$\chi_{\text{in}}(y_1, \dots, y_N) = \begin{cases} 1 & y_1, \dots, y_N \in \Delta_{\text{in}}\,, \\ 0 & \text{if any of the } y\text{'s} \notin \Delta_{\text{in}}. \end{cases}$$

We define

$$\varphi^n(\boldsymbol{v}_n) = \int \mathrm{d}y_1 \dots \mathrm{d}y_N \varphi^n(\boldsymbol{v}_N, y_1, \dots, y_N)\chi_{\text{in}}(y_1, \dots, y_N) \tag{2.2}$$

and further

$$\begin{cases} \varphi_k^{+n} = \int \mathrm{d}\boldsymbol{v}_N \varphi^n(\boldsymbol{v}_N) \boldsymbol{f}_k(\boldsymbol{v}_N)\,, \\ \varphi_k^{-n} = \int \mathrm{d}\boldsymbol{v}_N \varphi^n(\boldsymbol{v}_N) f_k(\boldsymbol{v}_N)\,, \end{cases} \tag{2.3}$$

where $\boldsymbol{n} \equiv (n_1, \dots, n_N)$ and $\mathrm{d}\boldsymbol{v}_N \equiv \mathrm{d}v_1 \dots \mathrm{d}v_N$. We restrict considerations to a finite number of k's only, and to a finite number of real parameters s, which we need to define a classical lattice field

$$\varphi_s^n = \frac{1}{\sqrt{V}} \sum_k \left\{ \varphi_k^{+n} \exp\left[\frac{iks\pi}{V}\right] + \varphi_k^{-n} \exp\left[-\frac{iks\pi}{V}\right] \right\}. \tag{2.4}$$

At this point we shall introduce a quantum lattice field

$$\hat{\varphi}_s^n = \frac{1}{\sqrt{V}} \sum_k \left\{ V_k^{*n} \exp\left[i\frac{ks\pi}{V}\right] + V_k^n \exp\left[-i\frac{ks\pi}{V}\right]\right\}, \tag{2.5}$$

where

$$\begin{cases} [V_k^n, V_{k'}^{*n'}] = \delta_{kk'}\delta_{nn'}, \\ [V_k^{n*}, V_{k'}^{n'*}] = [V_k^n, V_{k'}^{n'}] = 0, \qquad \text{for all } k\text{'s}, \\ V_k \Omega_B = 0, \end{cases} \tag{2.6}$$

so that the corresponding Fock space is given. The notation here is $\boldsymbol{n} = \boldsymbol{n}' \Rightarrow \Rightarrow (n_1, \ldots, n_N) = (n_1', \ldots, n_N')$.

We introduce then

$$V_k = \sum_{\{n\}} \boldsymbol{\alpha}^n V_k^n, \qquad \sum_n |\alpha^n|^2 = 1, \tag{2.7}$$

so that

$$[V_k, V_{k'}^*] = \delta_{kk'}, \qquad [V_k^*, V_{k'}^*] = [V_k, V_{k'}] = 0. \tag{2.8}$$

To compare this quantum construction with the previously introduced one, let us write explicitly

$$V_k^{*n} = \int \mathrm{d}\boldsymbol{v}_N \hat{\varphi}_k^n(\boldsymbol{v}_N) \boldsymbol{f}_k(\boldsymbol{v}_N) \tag{2.9}$$

with $\hat{\varphi}_k^n$ being a Hermitian field.

We construct the coherent-state domain for the field $\hat{\varphi}_s^n$, so that its coherent-state expectation value will coincide with φ_s^n, see, *e.g.*, (2.5) and (2.4). The coherent-state domain is given by

$$|B\rangle = \exp\left[-\tfrac{1}{2}\sum_k \sum_{\{n\}} \varphi_k^{+n}\varphi_k^{-n}\right] \exp\left[\sum_k \sum_{\{n\}} V_k^{*n}\varphi_k^{-n}\right]\Omega_B, \tag{2.10}$$

so that

$$\langle B|\hat{\varphi}_s^n|B\rangle = \varphi_s^n. \tag{2.11}$$

The correspondence relation (2.11) between the classical and quantum levels, provided we allow time to flow, establishes the relation between the classical solitary dynamics and the corresponding quantum solitary dynamics. Namely we expect to have time dependence of the operator $\hat{\varphi}_s^n$ once $\varphi_s^n(t)$ is given. By assuming that for each time t one has a representation of the canonical algebra, then, through the GNS construction, we always have a corresponding carrier Hilbert space H_t. We expect that in H_t there exist state vectors $|B, t\rangle$ such that

$$\langle B, t|\hat{\varphi}_s^n(t)|B, t\rangle = \varphi_s^n(t) \tag{2.12}$$

holds. Let W_t be a time evolution operator such that

$$|B, t\rangle = W_t|B\rangle$$

and hence

$$\langle B|W_t^* \hat{\varphi}_s^n(t)\, W_t|B\rangle = \varphi_s^n(t)\,. \tag{2.13}$$

In general

$$W_t \neq W_t^{-1}\,.$$

Suppose

$$\langle B\,|\hat{\varphi}_s^n(t)|B\rangle = \varphi_s^n(t) \tag{2.14}$$

for some $\hat{\varphi}_s^n(t)$, so that the influence of the general evolution rule (2.12) can be studied by the use of pure Fock techniques. In consequence of (2.14) there exists a unitary motion operator U_t such that

$$\hat{\varphi}_s^n(t) = U_t^{-1}\hat{\varphi}_s^n(0)\, U_t\,, \tag{2.15}$$

in such a way that (2.14) holds. The form of U_t is predicted by the classical solitary-motion law (1.9), which shows that energy centres become shifted by suitable space intervals when time flows. These shifts imply the corresponding form of $\varphi_s^n(t)$. Let us namely restrict to all the energy centre initial positions and velocities such that the values

$$\int_0^t \dot{y}(t)\,\mathrm{d}t + y = y(t) \tag{2.16}$$

for all times t still belong to the initial interval Δ_{in}. As a consequence, any asymptotic phase shifts completely loose their importance: with or without them, the $\varphi_s^n(t)$ can be satisfactorily defined by taking into account the free-motion shifts which have their origin in the uniform terms vt. Thus, for $t = \nu\lambda/v$,

$$\varphi_s^n(t) = \varphi_s^{n_1-\nu r_1,\dots,n_N-\nu r_N}\,, \tag{2.17}$$

and this is the motion rule which we must generate at the quantum level. To derive a quantum motion which will imply (2.17) through (2.11), we need the appropriate quantum transformation of $\hat{\varphi}_s^n$ while time flows:

$$\exp[-\,iMt]\,\hat{\varphi}_s^n \exp[iMt] = \hat{\varphi}_s^{n_1-\nu r_1,\dots,n_N-\nu r_N} \tag{2.18}$$

with $t = \nu\lambda/v$, so that the Hermitian operator M is just the generator of quantum solitary motion. Spectral properties of M are well defined due to (2.6) and (2.8). Notice that immediately

$$\langle B|\hat{\varphi}_s^{n_1-\nu r_1,\dots,n_N-\nu r_N}|B\rangle = \varphi_s^{n_1-\nu r_1,\dots,n_N-\nu r_N}\,. \tag{2.19}$$

Obviously, the right-hand side of (2.19) gives account of the classical solitary decay exhibiting the most important feature (*i.e.* shifting in space) of solitary motion in the lattice approximation. Thus, to the classical decay there corresponds through (2.19) an associate quantum decay implied by the classical time evolution. In this sense, we speak of *solitary quantum decay.*

3. – Statistical description of solitary quantum decay.

In this section the statistical properties of the solitary quantum decay process are of main interest to us: we want to study the probability distribution informing about the fraction of still undecayed particles in the given fraction of particles with momentum k.

We consider the time scale divided into a sequence of time intervals Δt much smaller than the characteristic lifetime of our decay process. Let $t = 0$ be the initial time; a generic time interval is denoted by Δ_τ, $\tau=0, 1, 2, \dots, +\infty$. We recall that each of the operators V_k^*, V_k was defined by (2.7), (2.8) for N-soliton excitation at $t = 0$; we shall put a subscript N to the state vectors representing states of N-soliton particles, *e.g.* $\Omega_B \equiv |O\rangle_N$ so that (cfr. (2.6))

$$V_k|O\rangle_N = 0\,, \qquad {}_N\langle O|O\rangle_N = 1\,. \tag{3.1}$$

Our task is the construction of a state $|O(\tau)\rangle_N$ such that the expectation value of the number operator in this state is a time-dependent function which describes the quantum decay of the quantum N-soliton excitation. We expect namely that the value

$$ {}_N\langle B|\exp[-iM\tau]\,V_k^*\,V_k\exp[iM\tau]\,|B\rangle_N = \Gamma_{k,\tau}\,, \qquad \text{for } t\in\Delta_\tau\,, \tag{3.2}$$

will be the expectation number of quantum N-soliton excitations of momentum k at the time $t\in\Delta_\tau$. Recall that M is the generator of the solitary time evolution (cf. eqs. (2.18) and (2.19)). The time-dependent function $\Gamma_{k,\tau}$ describes the decay law whose explicit form is irrelevant to our description. For simplicity, we shall assume

$$\begin{cases} \Gamma_{k,\tau} = \begin{cases} 1 & \text{at } \tau = 0\,, \\ 0 & \text{at } \tau = +\infty\,, \end{cases} \\ 0 < \Gamma_{k,\tau} < 1 & \text{for } \tau\in(0, +\infty)\,. \end{cases} \tag{3.3}$$

To construct the state $|O(\tau)\rangle_N$ such that

$$ {}_N\langle O(\tau)|V_k^* V_k|O(\tau)\rangle_N = \Gamma_{k,\tau}\,, \tag{3.4} $$

we follow a standard procedure ([4]) which has already been used in the description of unstable particles in quantum field theory ([5]).

We introduce the operators $\tilde{V}_k^*$ and $\tilde{V}_k$ at $t\in\Delta_0$ with the commutation relations

$$ [\tilde{V}_k, \tilde{V}_{k'}^*] = \delta_{kk'}\,, \qquad [\tilde{V}_k, \tilde{V}_{k'}] = [\tilde{V}_k^*, \tilde{V}_{k'}^*] = [V_k, \tilde{V}_{k'}^*] = 0\,. \tag{3.5} $$

The « tilde » vacuum for $t\in\Delta_0$ is denoted by $|\tilde{O}\rangle_N$:

$$ \tilde{V}_k|\tilde{O}\rangle_N = 0\,, \qquad {}_N\langle\tilde{O}|\tilde{O}\rangle_N = 1\,. \tag{3.6} $$

We will use the notation $|O\rangle_N \equiv |O, \tilde{O}\rangle_N$ for the direct product of the V-vacuum state and the $\tilde{V}$-vacuum state. We introduce the operators $V_k(\tau)$ and $\tilde{V}_k(\tau)$ by a Bogoliubov transformation as follows:

$$ \begin{cases} V_k(\tau) = V_k\cosh\theta - \tilde{V}_k^*\sinh\theta\,, \\ \tilde{V}_k(\tau) = \tilde{V}_k\cosh\theta - V_k^*\sinh\theta \end{cases} \tag{3.7} $$

and their Hermitian conjugates; here θ is a function of τ. The generator of (3.7) is

$$ G_N(\tau) = i\theta(\tau)(V_k^*\tilde{V}_k^* - \tilde{V}_k V_k)\,, \tag{3.8} $$

$$ V_k(\tau) = \exp[-iG_N]\,V_k\exp[iG_N]\,, \qquad \tilde{V}_k(\tau) = \exp[-iG_N]\,\tilde{V}_k\exp[iG_N]\,. \tag{3.9} $$

The state $|O(\tau)\rangle_N$ is given by

$$ |O(\tau)\rangle_N = \exp[-iG_N(\tau)]\,|O\rangle_N\,, \tag{3.10} $$

i.e.

$$ |O(\tau)\rangle_N = \frac{1}{\cosh\theta}\exp[\mathrm{tgh}\,V_k^*\tilde{V}_k^*]|0\rangle_N\,. \tag{3.11} $$

The state $|O(\tau)\rangle_N$ is the vacuum state for $V_k(\tau)$ and $\tilde{V}_k(\tau)$

$$ V_k(\tau)|O(\tau)\rangle_N = \tilde{V}(\tau)|O(\tau)\rangle_N = 0 \tag{3.12} $$

([4]) Y. Takahashi and H. Umezawa: *Collective Phenomena*, **2**, 55 (1975).
([5]) S. De Filippo and G. Vitiello: *Lett. Nuovo Cimento*, **19**, 92 (1977).

and

$$ {}_N\langle O(\tau)|O(\tau)\rangle_N = 1\,, \qquad \forall\, \tau. \tag{3.13} $$

By cyclic operation of $V_k^*(\tau)$ and $\tilde{V}_k^*(\tau)$ on $|O(\tau)\rangle_N$ we can generate a Hilbert space whose vacuum state is $|O(\tau)\rangle_N$ (our construction is thus the analogue of the GNS construction). Note that transformation (3.7) preserves the canonical commutation relations (2.8) and (3.5). We have now

$$ {}_N\langle O(\tau)|V_k^* V_k|O(\tau)\rangle_N = \sinh^2\theta = \Gamma_{k,\tau} \qquad \text{for } t\in\Lambda_\tau, \tag{3.14} $$

due to our requirement of eq. (3.4). Equation (3.14) fixes θ as a function of τ (cf. also eq. (3.3)).

Due to eqs. (3.3) and (3.14) (cf. also eq. (3.11)) we see that

$$ |O(0)\rangle_N = \frac{1}{\sqrt{2}}\exp\left[\frac{1}{\sqrt{2}}\, V_k^* \tilde{V}_k^*\right] |O\rangle_N \tag{3.15} $$

represents the V_k (N-solitary) excitation state at $t\in\Lambda_0$; while

$$ |O(+\infty)\rangle_N = |O\rangle_N \tag{3.16} $$

is the no-V_k-state at $t\in\Lambda_{+\infty}$. Note that the time evolution (decay) of the N-solitary excitation is thus described by the transition through the $|O(\tau)\rangle_N$ states. Note also that in the $|O(\tau)\rangle_N$ state there is an equal number of V_k excitations and of $\tilde{V}_k$ excitations; furthermore,

$$ \begin{cases} V_k^*(\tau)|O(\tau)\rangle_N = \dfrac{1}{\cosh\theta}\, V_k^*|O(\tau)\rangle_N = \dfrac{1}{\sinh\theta}\, \tilde{V}_k|O(\tau)\rangle_N\,, \\ \tilde{V}_k^*(\tau)|O(\tau)\rangle_N = \dfrac{1}{\cosh\theta}\, \tilde{V}_k^*|O(\tau)\rangle_N = \dfrac{1}{\sinh\theta}\, V_k|O(\tau)\rangle_N\,, \end{cases} \tag{3.17} $$

which suggest to interpret the tilde excitations as the holes of the V_k excitations, since the addition of one V_k excitation to $|O(\tau)\rangle_N$ is equivalent to the annihilation of one $\tilde{V}_k$. One could interpret the tilde system as a « reservoir ».

To complete our description we need to consider the « decay products »; *i.e.* we must include in our picture the N 1-solitons by which we can represent asymptotically the N-solitary excitations:

$$ V_k \xrightarrow[\tau\to+\infty]{} \sum_{i=1}^{N} V_{k_i} \tag{3.18} $$

with

$$ k = \sum_{i=1}^{N} k_i\,. \tag{3.19} $$

The vacuum for V_{k_i} $(i = 1, \ldots, N)$ at $t \in \Delta_{+\infty}$ is $|O\rangle_{\{i\}}$:

$$V_{k_i}|O\rangle_{\{i\}} = 0\,, \qquad {}_{\{i\}}\langle O|O\rangle_{\{i\}} = 1\,, \qquad \text{for } t \in \Delta_{+\infty}\,. \tag{3.20}$$

The V_{k_i} satisfy the canonical commutation relations

$$[V_{k_i}, V^*_{p_j}] = \delta_{kp}\,\delta_{ij}\,, \qquad i, j = 1, \ldots, N\,, \tag{3.21}$$

etc. As we did in the case of V_k, we introduce the tilde operators $\tilde{V}_{k_i}$ and $\tilde{V}^*_{k_i}$ and construct the operators $\tilde{V}_{k_i}(\tau)$ and $V_{k_i}(\tau)$ by a Bogoliubov transformation similar to the one in (3.7); then construct the state $|O(\tau)\rangle_{\{i\}}$:

$$\begin{aligned}|O(\tau)\rangle_{\{i\}} &= \exp[-iG_{\{i\}}]|O\rangle_{\{i\}} = \\ &= \prod_{i=1}^{N} \frac{1}{\cosh\theta_i} \exp[\operatorname{tgh}\theta_i\, V^*_{k_i}\tilde{V}^*_{k_i}]|O\rangle_{\{i\}}\; {}_{\{i\}}\langle O(\tau)|O(\tau)\rangle_{\{i\}} = 1\end{aligned} \tag{3.22}$$

with

$$G_{\{i\}} = i\sum_{i=1}^{N} \theta_i(\tau)(V^*_{k_i}\tilde{V}^*_{k_i} - \tilde{V}_{k_i}V_{k_i})\,, \tag{3.23}$$

and $\theta_i \equiv \theta_{k_i}(\tau)$ with k_i constrained by (3.19).

We require that

$${}_{\{i\}}\langle O(\tau)|\prod_{i=1}^{N}(V^*_{k_i}V_{k_i})|O(\tau)\rangle_{\{i\}} = \prod_{i=1}^{N}\sinh^2\theta_i = 1 - \Gamma_{k,\tau} \qquad \text{at } t \in \Delta_\tau\,, \tag{3.24}$$

which fixes $\sinh^2\theta_i$ equal to 1 at $\tau = +\infty$ and to zero at $\tau = 0$.

Denote now by $|O\rangle$ the direct product $|O\rangle_N \otimes |O\rangle_{\{i\}}$ and

$$|O(\tau)\rangle = \exp[-iG_{\{i\}}]\exp[-iG_N]\,|O\rangle\,, \qquad \langle O(\tau)|O(\tau)\rangle = 1\,. \tag{3.25}$$

The N-solitary excitation state *at* $t \in \Delta_0$ is

$$|O(0)\rangle = \frac{1}{\sqrt{2}}\exp\left[\frac{1}{\sqrt{2}}V^*_k\tilde{V}^*_k\right]|O\rangle \qquad \text{for } t \in \Delta_0\,. \tag{3.26}$$

As $\tau \to +\infty$, it evolves to

$$|O(+\infty)\rangle = \prod_{i=1}^{N}\frac{1}{\sqrt{2}}\exp\left[\frac{1}{\sqrt{2}}V^*_{k_i}\tilde{V}^*_{k_i}\right]|O\rangle \qquad \text{for } t \in \Delta_{+\infty}\,. \tag{3.27}$$

We observe that one can write $|O(\tau)\rangle$ as [4]

$$|O(\tau)\rangle = \exp[-K/2]\exp[V^*_k\tilde{V}^*_k]\exp\left[\sum_{i=1}^{N}V^*_{k_i}\tilde{V}^*_{k_i}\right]|O\rangle \tag{3.28}$$

with

$$(3.29)\qquad K = -(V_k^* V_k \log \sinh^2 \theta - V_k V_k^* \log \cosh^2 \theta) - \\ - \sum_{i=1}^{N} (V_{k_i}^* V_{k_i} \log \sinh^2 \theta_i - V_{k_i} V_{k_i}^* \log \cosh^2 \theta_i) .$$

Comparison of (3.28) with (3.4) and (3.2) shows that K plays the same role as $-iM\tau$. By using (3.25) and (3.29) it is possible indeed to show that $-\mathrm{d}K/\mathrm{d}\tau$ generates transition from $|O(\tau)\rangle$ to $|O(\tau')\rangle$ with $\tau \neq \tau'$. The probability of finding a state $|O(\tau)\rangle$ at the time $t \in \Delta_{\tau'}$, $\tau' \neq \tau$, is thus given by $\langle O(\tau)|\exp[-K]|O(\tau)\rangle$: the time evolution of the V_k-N-solitary wave is thus described by the transition through the states $|O(\tau)\rangle$ controlled by the operator K. Note that the operator K gives a measure of the irreversibility of the process. It is indeed called the « entropy » in the Takahashi-Umezawa's formulation of statistical mechanics in QFT ([4]). The expectation value of K on $|O(\tau)\rangle$ is, in fact,

$$(3.30)\qquad \langle O(\tau)|K|O(\tau)\rangle = -\sum_m W_m \log W_m - \sum_{i=1}^{N} \sum_{m_i} W_{m_i} \log W_{m_i} ,$$

as can be seen by writing $|O(\tau)\rangle$ as

$$|O(\tau)\rangle = \sum_m \sqrt{W_m}\, |m, \tilde{m}\rangle_n \otimes \sum_{m_i} \sqrt{W_{m_i}}\, |m_i, \tilde{m}_i\rangle_{\{i\}}$$

with $\sum_m W_m = 1$, $\sum_{m_i} W_{m_i} = 1$, and by using (3.29).

One can verify then that the « entropy »

$$(3.31)\qquad \langle O(0)|K|O(0)\rangle \xrightarrow[\tau \to +\infty]{} +\infty ,$$

i.e. it reaches a maximum as the system evolves towards a stability condition at $t \in \Delta_{+\infty}$.

We observe also that, as we go to the infinite-volume limit (lattice spacing going to zero), the $|O(\tau)\rangle$ states become orthogonal to the $|O\rangle$ state. We have, by using (3.25),

$$(3.32)\qquad \langle O|O(\tau)\rangle = \frac{1}{\cosh\theta} \prod_{i=1}^{N} \frac{1}{\cosh\theta_i} = \exp[-\log\cosh\theta] \exp\left[-\sum_{i=1}^{N} \log\cosh\theta_i\right] ,$$

which is zero in the infinite-volume limit if $\log\cosh\theta$ and/or $\sum_i \log\cosh\theta_i$ go to $+\infty$. In this limit relations as (3.8), (3.23), (3.25) become formal. In the same limit $\langle O(\tau)|O(\tau')\rangle \to 0$, $\tau \neq t'$. The time evolution of the quantum N-soliton excitation is thus naturally described as a transition through the unitarily inequivalent representations $\{|O(\tau)\rangle\}$ of the canonical commutation relations. The unitary inequivalence among the quantum N-soliton states at different values of τ is a remainder of the classical inequivalence, in the statistical sense, among N-solitary waves at different time, *i.e.* with classically different internal structures (notice the strict relation between K and the generator of the solitary motion M introduced in (2.18), as noted after eq. (3.29)). Finally, it is interesting to note that the statistical nature and the irreversibility of the time evolution is strictly connected with the unitary inequivalence: it is indeed the « entropy » K which controls the time evolution as a transition through the inequivalent representations.

● RIASSUNTO (*)

Si dà una descrizione quantistica del moto solitario. Si descrive l'evoluzione nel tempo di un N-solitone quantico come una transizione attraverso infinitamente molte rappresentazioni unitariamente inequivalenti delle relazioni di commutazione canoniche. Emergono naturalmente la natura statistica e l'irreversibilità del processo. Si trova la loro origine nel carattere non unitario della trasformazione tra le rappresentazioni unitariamente inequivalenti.

(*) *Traduzione a cura della Redazione.*

Каноническое описание одиночного квантового распада.

Резюме (*). — Предлагается квантовая картина одиночного движения. Описывается временная эволюция квантового N-солитона. Статистическая природа и неприводимость процесса проявляются непосредственным образом. Обнаружено, что их происхождение связано с неунитарным характером преобразования между унитарно неэквивалентными представлениями.

(*) *Переведено редакцией.*