BROWNIAN MOTION AND ITS CONDITIONAL DESCENDANTS

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Contribution in honor of the 60th birthday of John R. Klauder

It happened before [1] that I have concluded my publication with a special dedication to John R. Klauder. Then, the reason was John's PhD thesis [2] and the questions (perhaps outdated in the eyes of the band-wagon jumpers, albeit still retaining their full vitality [3]): *(i) What are the uses of the classical (c-number,non-Grassmann) spinor fields, especially nonlinear ones, what are they for at all ? (ii) What are, if any, the classical partners for Fermi models and fields in particular ?* The present dedication, even if not as conspicuously motivated as the previous one by John's research, nevertheless pertains to investigations pursued by John through the years and devoted to the analysis of random noise. Sometimes, re-reading old papers and re-analysing old, frequently forgotten ideas might prove more rewarding than racing the fashions. Following this attitude, let us take as the departure point Schrödinger's original suggestion [4] of the existence of a special class of random processes, which have their origin in the Einstein-Smoluchowski theory of the Brownian motion and its Wiener's codification. The original analysis due to Schrödinger of the probabilistic significance of the heat equation and of its time adjoint in parallel, remained unnoticed by the physics community, and since then forgotten. It reappeared however in the mathematical literature as an inspiration to generalise the concept of Markovian diffusions to the case of Bernstein stochastic processes. But, it stayed without consequences for a deeper understanding of the possible physical phenomena which might underly the corresponding abstract formalism. Schrödinger's objective was to initiate investigations of possible links between quantum theory and the theory of Brownian motion, an attempt which culminated later in the so-called Nelson's stochastic mechanics [8] and its encompassing formalism [7] in which the issue of the Brownian implementation of quantum dynamics is placed in the framework of Markov-Bernstein diffusions. Schrödinger's discussion of the analogy between wave mechanics and random phenomena of classical statistical physics starts with recalling an obscurity present in the notion of probability (Born's postulate) adopted in quantum theory. For the purposes of the probabilistic interpretation, it seems that one should decide in advance, whether one is considering a probablity *after* one knows what has happened, or rather a probability of what *is to* happen. To conform with the classical notion of probability (event, i.e. sample space is needed to have the axiomatic definition of the probability space) the most natural way is to look at a classical probabilistic system, which structurally is as close as possible to the wave (Schrödinger) equation of quantum mechanics. In case of the free (V=0) propagation, the heat equation with its time adjoint well fit the purpose:

$$\begin{array}{lcl} i\partial_t\psi = -D\triangle\psi & & \partial_t\theta_* = \ \ D\triangle\theta_* \\ i\partial_t\overline{\psi} = \ \ D\triangle\overline{\psi} & t\rightarrow it & \partial_t\theta \ = -D\triangle\theta \end{array} \qquad (1)$$

where the familiar imaginary time transformation is indicated as a pedestrian recipe to pass from quantum theory to statistical physics. Here $\psi, \overline{\psi}$ are complex, while θ_*, θ are real functions, and the diffusion constant D is left unspecified ($D = \hbar/2m$ gives rise to the Schrödinger equation in its standard form). Let us consider the transition probablity density (heat kernel, the Brownian law of random displacements) $h(x,s,y,t)$ for the Brownian motion $(Y_t, t_1 \leq t \leq t_2)$ on the interval $[t_1, t_2]$ i.e.

$$h(x,s,y,t) = [4\pi D(t-s)]^{-1/2} \exp{[-\frac{(y-x)^2}{4D(t-s)}]} \qquad (2)$$

where Y_t takes values in R^1 (Brownian motion in one spatial dimension). If we prescribe the initial particle distribution $\rho_1(x)$ for the random variable Y_t , then all intermediate distributions of $(Y_t, t_1 \leq t \leq t_2)$ are determined in terms of $\rho_1(x)$ and $h(x,s,y,t)$, including the terminal one as well. We have indeed:

$$\rho(x,t) = \int h(z,t_1,x,t)\rho_1(z)\,dz \qquad (3)$$

where $t_1 \leq t$. Starting from the classical Brownian law of random displacements we can ask the following question: *Assuming that a test particle originates from x_1 at t_1 , and terminates its route in (not far from) x_2 at t_2 , what is the probability to find it between x and $x + \triangle x$ at the intermediate time t with $t_1 \leq t \leq t_2$?* The pertinent intermediate probability distribution is given by the conditional transition probability density formula :

$$\rho(x,t) = P(x_1,t_1;x,t;x_2,t_2) = \frac{h(x_1,t_1,x,t)\,h(x,t,x_2,t_2)}{h(x_1,t_1,x_2,t_2)} \qquad (4)$$

It is then obvious that the formula for $\rho(x,t)$ can always be rewritten as a product of solutions $\theta(x,t), \theta_*(x,t)$ of the heat equation and its time adjoint: $\rho(x,t) = (\theta\theta_*)(x,t)$ provided $t_1 \leq t \leq t_2$. Let us now define $\rho(x,t_0) = \rho_0(x)$ and $\rho(x,T) = \rho_T(x)$ to be the initial and final probability distributions *determined* by the Bernstein transition density P, with $t_1 < t_0 < T < t_2$. Although we know the general Brownian transition mechanism (the law of random displacements), h , the conditioning present in the Bernstein (in fact Brownian, since V=0) bridge construction allows us to formulate a new probabilistic problem. We are now revisiting the original question due to Schrödinger [5,6]: *What is the most likely way for the particles to evolve as t goes from t_0 to T, once we have prescribed in advance both the initial $\rho_0(x)$ and final (terminal) $\rho_T(x)$ probability densities for the process, given the prior transition mechanism ?* The answer is given by deriving from the original (prior) process $(Y_t, t_1 \leq t \leq t_2)$, the new one $(X_t, t_1 < t_0 \leq t \leq T < t_2)$ which is now known as the Markov-Bernstein process [7]. The above discussion can be rephrased in more phenomenological terms. Suppose that the observer is measuring a coordinate x of the event (particle entering the observation area A, the measurement accuracy does not matter) at time t_0 viewed as the initial time instant in the repeatable series of one particle x experiments. Accumulating the data one arrives at the empirical distribution which asymptotically is found to approximate a probability distribution $\rho_0(x)$. It is then taken to characterise the "state of the system" at time t_0. Assume also that the observer is collecting the coordinate data of these repeatable events (entering particles) in the detection area B at a later time T and let them approximate the terminal probability distribution $\rho_T(x)$. If $\rho_T(x)$ is far from what it should be according to the law of large numbers (i.e. when $\rho_T(x)$ is much different from $\int h(x, t_0, y, T)\rho_0(y)dy$, with h given before), then we arrive at the *core* of the original Schrödinger's discussion: ***What are the intermediate probability distributions $\rho(x,t)$ and what is the particular transition mechanism responsible for the probabilistic evolution from $\rho_0(x)$ to $\rho_T(x)$, if no external forces are affecting the particle, except for Brownian agitation (Brownian noise)?*** Let us proceed more generally [6] than the previous discussion would suggest, and assume to have given a pair of diffusion equations in duality for real functions θ, θ_* :

$$\begin{aligned} \partial_t\theta_* &= \ \ D\triangle\theta_* - V\theta_*/2mD \\ \partial_t\theta \ &= -D\triangle\theta \ + V\theta \ /2mD \end{aligned} \tag{5}$$

D is a diffusion constant, m the mass of a particle subject to diffusion. The potential V is assumed to be continuous and bounded from below, which implies the existence of the strictly positive semigroup kernel generated by the operator $H = -2mD^2\triangle + V$. Let $h = h(x,s,y,t)$, $s \leq t$ be the fundamental solution of the first diffusion equation. Then, the initially chosen function $\theta_*(x, -T/2)$, $0 \leq T$ is propagated forward $\theta_*(x,t) = \int \theta_*(z, -T/2)h(z, -T/2, x, t)dz$, $-T/2 \leq t$ while the terminal choice of $\theta(x, T/2)$, $0 \leq T$ allows one to reproduce the past data $\theta(x,t)$, $t \leq T/2$ through the backward propagation $\theta(x,t) = \int h(x,t,y,T/2)\theta(y,T/2)dy$. By virtue of the semigroup property of the kernel h we have also:

$$\begin{aligned} \theta_*(x,t) &= \int \theta_*(z,s)h(z,s,x,t)\,dz \\ \theta(x,s) &= \int h(x,s,z,t)\theta(z,t)\,dz \end{aligned} \tag{6}$$

where $(-T/2 \leq s < t \leq T/2)$; hence a solution of the dual system with the prescribed boundary data at $\pm T/2$ might be given, such that $\int \theta_*(x,t)\theta(x,t)\,dx = 1$ holds true for all times t in $[-T/2, T/2]$. We have here determined the Markov-Bernstein process [7], which allows one to propagate (hence both predict the future and reproduce the past, given the present) the probability distribution:

$$\rho(x,t) = \theta(x,t)\theta_*(x,t) \tag{7}$$

respectively forward and backward in time. Statistical predictions about the future can be accomplished by means of the forward transition probability density:

$$p(x,s,y,t) = h(x,s,y,t)\,\frac{\theta(y,t)}{\theta(x,s)} \tag{8}$$

while the past can be reproduced statistically by means of the backward density

$$p_*(x,s,y,t) = h(x,s,y,t)\,\frac{\theta_*(x,s)}{\theta_*(y,t)} \tag{9}$$

for the diffusion with fixed boundary probability distributions $\rho(x,-T/2)$ and $\rho(x,T/2)$.

With p and p_* in hand, we can straightforwardly evaluate the conditional expectation values, which are necessary to establish the mean forward and backward derivatives in time for functions of the random variable $X(t)$. The backward $(D_-X)(t) = b_*(x,t)$ and forward $(D_+X)(t) = b(x,t)$ drifts of the Markovian diffusions in the above read thus $b(x,t) = 2D\nabla\theta/\theta$ and $b_*(x,t) = -2D\nabla\theta_*/\theta_*$ so that the continuity equation follows:

$$\partial_t\rho = -\nabla(\rho v) = D\triangle\rho - \mathrm{div}(\rho b) = -D\triangle\rho - \mathrm{div}(\rho b_*) \quad \text{with} \quad v = (b+b_*)/2 \tag{10}$$

If we define $\theta = \exp(R+S)$, $\theta_* = \exp(R-S)$ with R and S being real functions, then there holds:

$$v = 2D\nabla S \qquad u = (b-b_*)/2 = 2D\nabla R \tag{11}$$

and the continuity equation can be rewritten as follows (with its gradient form, due to Nelson [8], included) :

$$(1/2D)\partial_t R = (-1/2)\triangle S - (\nabla R)(\nabla S) \;\rightarrow\; \partial_t u = -D\triangle v - \nabla(uv) \tag{12}$$

If the continuity equation holds true, then the necessary consequence of the dual system of diffusion equations is the (generalised) Hamilton-Jacobi equation:

$$V = 2mD\{\partial_t S + D(\nabla S)^2 + D[(\nabla R)^2 + \triangle R]\} \tag{13}$$

which can furthermore be rewritten as :

$$2mD^2[(\nabla R)^2 + \triangle R] = 2mD^2\,\frac{(\triangle\rho^{[1/2]})}{\rho^{1/2}} = Q \tag{14}$$

Except for the sign inversion, Q has the familiar functional form of the de Broglie-Bohm-Vigier "quantum potential" [5,6,9,10]. We can here argue in reverse, and recover the dual system of diffusion equations, given the continuity and the generalised Hamilton-Jacobi equations. By evaluating the forward and backward time derivatives of $b(x,t)$ and $b_*(x,t)$ we can verify that the gradient form of the Hamilton-Jacobi equation reads:

$$\partial_t v = 2D\triangle u + (1/2)\nabla u^2 + (1/2)\nabla v^2 + (1/m)\nabla V \tag{15}$$

which in turn implies the validity of the so-called Nelson-Newton law [6] with the sign inverted potential (conventionally interpreted as the signal that we are in the Euclidean framework, see however [9,10]):

$$(m/2)(D_+D_+ + D_-D_-)X(t) = \nabla V \tag{16}$$

This stochastic acceleration (in the conditional mean) formula was primarily rejected by Nelson [8] as the physically relevant characteristic of the Markovian diffusion. Notice that the above equation can be written in the form :

$$(\partial_t + v\nabla)v = (-1/m)\nabla(Q - V) = (1/m)\nabla(V - Q) \tag{17}$$

reminiscent of the momentum balance equation in the kinetic theory of gases and liquids, which should in principle apply to all conceivable osmotic diffusions.

To illustrate the previous discussion, let us come back to the previous ($V = 0$) considerations. We can ask for a probabilistic interpolation between the coinciding boundary distributions:

$$\rho(x, t_0) = \rho(x, T) = [\frac{\alpha}{2\pi D(\alpha^2 - \beta^2)}]^{1/2} \exp[-\frac{\alpha x^2}{2D(\alpha^2 - \beta^2)}] \tag{18}$$

in the time interval $[t_0, T]$. No physicist would expect such an evolution while having a traditional picture of the Brownian motion in memory. However it immediately comes out from the Bernstein (actually Brownian) bridge construction. Indeed, let us set $t_0 = -\beta = -T$, $t_1 = -\alpha = -t_2$, $0 < \beta < \alpha$ and choose $x_1 = 0$ as a source of particles. If we confine attention to these Brownian particles only, which after time 2α from their emission are bound to be back at (or at least not far away from) the initial location $x_1 = 0 = x_2$ then the Bernstein transition density P reduces to :

$$\begin{aligned} \rho(x, t) &= [\frac{\alpha}{2\pi D(\alpha^2 - t^2)}]^{1/2} \exp[-\frac{\alpha x^2}{2D(\alpha^2 - t^2)}] \\ &= (8\pi D\alpha)^{1/2} h(0, -\alpha, x, t) h(x, t, 0, \alpha) \end{aligned} \tag{19}$$

where the transition densities:

$$\begin{aligned} h(0, -\alpha, x, t) &= [4\pi D(t + \alpha)]^{-1/2} \exp[-\frac{x^2}{4D(t + \alpha)}] \\ h(x, t, 0, \alpha) &= [4\pi D(\alpha - t)]^{-1/2} \exp[-\frac{x^2}{4D(\alpha - t)}] \end{aligned} \tag{20}$$

solve the systems of heat equations in duality ($V = 0$), in the time interval $[-\alpha, \alpha]$. Notice that the pertinent boundary data are induced by the Bernstein (Brownian) bridge itself. By previous arguments we have given here the transition probability densities characterising Markovian diffusions, hence the drifts of the processes can be evaluated as follows:

$$\begin{aligned} b(x, t)\Delta t &\simeq \int p(x, t, y, t + \Delta t) y dy \, - \, x \\ b_*(x, t)\Delta t &\simeq x \, - \, \int p_*(z, t - \Delta t, x, t) z dz \end{aligned} \tag{21}$$

where Δt is a small time increment, b is the forward drift of the diffusion (mean velocity of outgoing –from x at t – particles) while b_* is its backward drift (mean velocity of incoming – to x at t – particles) $t_0 \leq t \leq T$. The drifts stand for substitutes of time derivatives, non-existent in the naive sense for Wiener paths. Here $(D_+X)(t) = b(x,t)$ is the left time derivative in the conditional mean, while $(D_-X)(t) = b_*(x,t)$ is the right one. Up to irrelevant multiplicative constants we can set :

$$\begin{aligned} \theta_*(x,t) &\sim h(0,-\alpha,x,t) \\ \theta(x,t) &\sim h(x,t,0,\alpha) \end{aligned} \qquad (-\beta \leq t \leq \beta < \alpha) \tag{22}$$

and consequently there holds:

$$\begin{aligned} b(x,t) &= 2D\frac{\nabla\theta}{\theta} = -\frac{x}{\alpha - t} = -\frac{x(\alpha+t)}{\alpha^2 - t^2} = u + v \\ b_*(x,t) &= -2D\frac{\nabla\theta_*}{\theta_*} = \frac{x}{\alpha+t} = \frac{x(\alpha-t)}{\alpha^2-t^2} = -u + v \end{aligned} \tag{23}$$

We have here a natural decomposition into two terms, one of which (namely v) is odd while the other (namely u) is even with respect to the time reversal. In addition, we can easily recover the relations:

$$\begin{aligned} u &= 2D\nabla R & v &= 2D\nabla S \\ R &= -\frac{\alpha x^2}{4D(\alpha^2 - t^2)} & S &= -\frac{t x^2}{4D(\alpha^2 - t^2)} \\ \theta_* &\sim \exp(R - S) & \theta &\sim \exp(R + S) \\ \rho(x,t) = (\theta\theta_*)(x,t) &\sim \exp 2R \quad \rightarrow & u &= D\frac{\nabla\rho}{\rho} = 2D\nabla R \end{aligned} \tag{24}$$

Notice the validity of the momentum balance equation (acceleration formula) $(D_+D_+ + D_-D_-)X(t) = 0$.

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References

1. P. Garbaczewski, J.Phys. **A 20** (1987), 1277.
2. J.R. Klauder, Ann.Phys.(NY) **11** (1960), 123.
3. P. Garbaczewski, *Classical and Quantum Field Theory of Exactly Soluble Nonlinear Systems*, World Scientific, Singapore, 1985.
4. E. Schrödinger, Ann. Inst. H.Poincaré **2** (1932), 269.
5. P. Garbaczewski, J.P. Vigier, *Brownian Motion and its Descendants According to Schrödinger*, submitted.
6. P. Garbaczewski, *Physical Significance of the Nelson-Newton Laws*, submitted.

7. J.C. Zambrini, J.Math.Phys. **27** (1986), 2397.
8. E. Nelson, *Quantum Fluctuations*, Princeton Univ.Press, Princeton, 1985.
9. P. Garbaczewski, Phys.Lett. **A 162** (1992), 129.
10. P. Garbaczewski, Phys.Lett. **A 164** (1992), 6.

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