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Dynamics of Coherent States as the Diffusion Process:Schrödinger's system,Bernstein processes and quantum evolution

Piotr Garbaczewski[1]

Institute of Theoretical Physics,University of Wroclaw, pl.M.Borna 9,PL-50 204 Wroclaw,Poland

Abstract By following the probabilistic analysis due to Schrödinger,Nelson and Zambrini we derive the stochastic diffusion processes uniquely defining the *wave packet* evolution in the nonrelativistic quantum theory. We make a detailed sample path analysis of configuration space Markovian diffusions to arrive at the most general dynamical restriction on the forward drift response to the action of external (extrinsic to the random motion proper) force fields.Under the assumption that the Brownian noise is the only generator of randomness ,we identify the microscopic conservation principles ,which are responsible for the drift dynamics.The primary concept in this formalism is the *microscopic level* notion of the sample path of the underlying stochastic process ,and the Schrödinger equation plays the role of the *macroscopic* (through conditional averages) dynamical rule,valid exclusively for particle ensembles.The Nelson-Newton laws are the necessary but insufficient conditions for a unique specification of the diffusion involved .The induced dynamical semigroups allow to set uniqueness criterions,at the same time embedding the Schrödinger wave mechanics in the theory of Bernstein stochastic processes.As a by-product of the discussion we demonstrate that the Brownian motion in a field of force, in the Smoluchowski approximation is an example of the Bernstein diffusion as well.

1. Introduction

In the recent discussion[1] of the Brownian diffusion admitting environmental recoil effects, one arrives at a reformulation of Nelson's stochastic mechanics[2–8]. The very existence of the random environment and the required strict validity of the momentum conservation law on all scales adopted for the investigation of the individual (random) particle scattering by the medium,implies that the collective dynamics of the statistical ensemble is described by the Schrödinger equation (the diffusion constant D takes place

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of $\hbar/2m$ in the stochastic derivation).It demonstrates convincingly that the intrinsically statistical theory might drive into the domain exclusively (according to folk lore) reserved for the study of quantum phenomena. However ,to the author's knowledge,the discussion of why the ensemble dynamics is a crucial factor here,and how to reconcile the individual and collective (ensemble)features of the diffusive propagation[2-8] ,for the first time was given in Ref.1.Individual particles do follow a diffusion process independently (in the repetitive series of the single particle trials) .The collective dynamics allows to identify them as members of the very concrete particle ensembles,the relevant information being encoded in the initial data (the familiar quantum mechanical state preparation procedure might be justifiably invoked at this point). The pertinent ensemble dynamics is uniquely determined once the Cauchy problem is solved for the coupled system of nonlinear equations.It is composed of the probability conservation law

$$\partial_t \rho = -\nabla(\rho v) \tag{1.1}$$

and one of the momentum balance equations (the kinetic theory lore is quite appropriate here)

$$(\partial_t + v\nabla)v = \frac{1}{m}\nabla(V - Q) \quad (Zambrini) \tag{1.2}$$

$$(\partial_t + v\nabla)v = \frac{1}{m}\nabla(Q - V) \quad (Nelson) \tag{1.3}$$

where (we consider diffusions in one spatial dimension for simplicity of the arguments)

$$Q = 2mD^2\frac{\Delta\rho^{1/2}}{\rho^{1/2}} \tag{1.4}$$

and the initial data $\rho_0(x), v_0(x)$ are given. In general we consider evolutions taking place in the finite time interval $[t_0, T]$ with unspecified endpoints,eventually if possible extending to both infinities.In the above m denotes the mass of each individual particle in random motion,D is a diffusion constant,V=V(x) is a Rellich class potential (continuity and boundedness from below are sufficient for our purposes) of a conservative (this condition might be relaxed) force field. Furthermore,$\rho = \rho(x,t)$ stands for the ensemble density (same as a probability distribution of the random variable X(t) undergoing the diffusion process) and:

$$b(x,t) = v(x,t) + D\frac{\nabla\rho}{\rho} \tag{1.5}$$

is the mean local velocity field (forward drift) of the diffusion. It is instructive to notice that the momentum balance equations (1.2),(1.3) are the equivalent expressions for Nelson's acceleration-in-the-mean formulas[7,8] ,the only time reversal invariant analogs of the second Newton law in case of the diffusion processes,whose sample paths are continuous but not differentiable.

Assume to have a priori given $b(x,t)$ for all times of interest.The individual particle dynamics which might underlie either (1.1),(1.2) or (1.1),(1.3) is then governed by the stochastic differential equation[4,8,9]

$$dX(t) = b(X(t),t)dt + \sqrt{2D}\,dW(t) \tag{1.6}$$

$$X(t_0) = x_0$$

The random displacements are generated by the Wiener noise W(t),which superimposes probabilistic fluctuations upon the deterministic contribution $b\Delta t$.The latter is a typical path ensemble input,since it is the mean velocity evaluated over all sample paths originating from x at time t, in the repeatable series of single particle trials.Accordingly, $b(x,t)$ encodes the *mean tendency* of motion,which is basically unidentifiable unless sufficiently many sample flight data are accumulated. By means of the stochastic Ito calculus[9] ,the problem (1.6) gives rise to the statistical transport recipe (the microscopic law of random displacements) $p(y,s,x,t), s \le t$ with the properties

$$\rho(x,t) = \int \rho(y,s)\,p(y,s,x,t)dy \quad s \le t$$

$$\partial_t p = D\triangle_x p - \nabla_x(bp) \quad p = p(y,s,x,t) \tag{1.7}$$

$$b(x,t) = lim_{\Delta t\to 0}\frac{1}{\Delta t}\int (y-x)p(x,t,y,t+\Delta t)dy = v(x,t) + (D\nabla\rho/\rho)(x,t)$$

where the dynamics of $v(x,t)$ and hence $b(x,t)$ is specified by the coupled problems (1.1),(1.2) or (1.1),(1.3). It is a peculiarity of Markov diffusions that the backward propagation rule allowing to reproduce the past statistical data of the process (1.6),(1.7), can be deduced

$$\rho(y,s) = \int p_*(y,s,x,t)\rho(x,t)dx \quad s \le t$$

$$p_*(y,s,x,t)\rho(x,t) = \rho(y,s)p(y,s,x,t) \tag{1.8}$$

$$\partial_t p_* = -D\triangle_y p_* - \nabla_y(b_* p_*)$$

$$b_*(y,s) = lim_{\Delta s\to 0}\frac{1}{\Delta s}\int (y-x)p_*(x,s-\Delta s,y,s)dx = v(x,t) - (D\nabla\rho/\rho)(x,t)$$

To avoid unnecessary complications (see however Ref.6) we assume that a diffusion proceeds in a simply connected area where the density does not vanish except possibly at the boundaries. Let us notice that a straightforward consequence of the assumption (1.5) which basically makes the Wiener (Brownian) noise responsible for whatever is going on,is that

$$\partial_t \rho = -\nabla(\rho v) = D\triangle\rho - \nabla(b\rho) = -D\triangle\rho - \nabla(b_*\rho) \tag{1.9}$$

where $v = \frac{1}{2}(b + b_*)$, so that the forward and backward Fokker-Planck equations for the probability density are naturally built into the formalism.

Example 1: *Free Brownian dynamics*

Let us consider the initial probability distribution of the random variable $X(0)$ of the Wiener (Brownian in the high friction regime) process in the form

$$\rho_0(x) = (\pi\alpha^2)^{-1/2} exp[-\frac{x^2}{\alpha^2}] \tag{1.10}$$

Then its statistical evolution is given by the familiar heat kernel

$$p(y,s,x,t) = [4\pi D(t-s)]^{-1/2} exp[-\frac{(x-y)^2}{4D(t-s)}] \tag{1.11}$$

$$\rho(x,t) = [\pi(\alpha^2+4Dt)]^{-1/2} exp[-\frac{x^2}{\alpha^2+4Dt}]$$

where $s \le t$.

Let us notice that since the density distribution is now defined for all times $t > s$ we can introduce a convenient device allowing to reproduce a statistical past of the (irreversible on physical grounds,but admitting this specific inversion mathematically)

$$p_*(y,s,x,t) = p(y,s,x,t)\frac{\rho(y,s)}{\rho(x,t)} \tag{1.12}$$

with the properties (set $s = t - \triangle t$)

$$\int p_*(y,s,x,t)\rho(x,t)dx = \rho(y,s) \quad s \le t \tag{1.13}$$

$$\int y\, p_*(y,s,x,t)dy = x\,\frac{\alpha^2+4Ds}{\alpha^2+4Dt} = x - \frac{4Dx}{\alpha^2+4Dt}\triangle t = x - b_*(x,t)\triangle t$$

where $b_*(x,t) = -2D\nabla\rho(x,t)/\rho(x,t)$ and quite trivially $b(x,t) = 0$.Notice that by defining $v(x,t) = \frac{1}{2}b_*(x,t)$,because of the heat equation we have satisfied (1.1),(1.2) with $V = 0$,and

$$(\rho v)(x,t) = \int p(y,s,x,t)\rho_0(y)v_0(y)dy \tag{1.14}$$

Example 2: *Free quantum evolution as the diffusion process*

By defining

$$p(y,0,x,t) = (4\pi Dt)^{-1/2} exp[-\frac{(x-y+2Dty/\alpha^2)^2}{4Dt}] \tag{1.15}$$

we realise that

$$\int p(y,0,x,t)(\pi\alpha^2)^{-1/2} exp(-y^2/\alpha^2)dy =$$

$$\frac{\alpha}{[\pi(\alpha^4+4D^2t^2)]^{1/2}} exp[-\frac{x^2\alpha^2}{\alpha^4+4D^2t^2}] = \rho(x,t) \tag{1.16}$$

and

$$\int p(y,0,x,t)[\frac{2Dy}{\alpha^2}(\pi\alpha^2)^{-1/2}]dy = \frac{2D(\alpha^2-2Dt)x}{\alpha^4+4D^2t^2} = b(x,t)\rho(x,t) \tag{1.17}$$

where evidently

$$v(x,t) = b(x,t) - D\nabla\rho(x,t)/\rho(x,t) \tag{1.18}$$

solves equations (1.1),(1.3) with $V = 0$ and via the familiar Madelung transcription of the free Schrödinger dynamics $i\partial_t\psi(x,t) = -D\triangle\psi(x,t)$ with $\psi = exp(R+iS), \rho = exp(2R), v = 2D\nabla S$ the link between the Brownian type diffusion and the quantum mechanical evolution is established.

Example 3: *Uses of the imaginary time transformation*

For V continuous and bounded from below ,the generator $H = -2mD^2\triangle + V$ is essentially selfadjoint,and then the kernel $h(x,s,y,t) = ker[exp[-(t-s)H]]$ of the related dynamical semigroup is strictly positive.On the other hand it is quite traditional to relate this dynamical semigroup evolution to the quantum mechanical unitary evolution operator $exp(iHt)$ by the imaginary time substitution $t \rightarrow it$. In the most pedestrian and naive interpretation of this fact,one might be tempted to invent the concept of "diffusion process in the imaginary time".Actually nothing like that is here allowed,and if taken seriously, becomes self-contradictory. The routine illustration for the imaginary time transformation is provided by considering the force-free propagation, where apparently (see.e.g.the Ref.15) the formal recipe gives rise to (one should be aware that to execute a mapping for concrete solutions,the proper adjustment of the time interval boundaries is indispensable):

$$i\partial_t\psi = -D\triangle\psi \longrightarrow \partial_t\theta_* = D\triangle\theta_*$$

$$i\partial_t\overline{\psi} = D\triangle\overline{\psi} \longrightarrow \partial_t\overline{\theta} = -D\triangle\overline{\theta} \tag{1.19}$$

with $it \rightarrow t$. Then

$$\psi(x,t) = [\rho^{1/2} exp(iS)](x,t) = \int dx' G(x-x',t)\psi(x',0)$$

$$G(x-x',t) = (4\pi iDt)^{-1/2} exp[-\frac{(x-x')^2}{4iDt}] \tag{1.20}$$

$$\overline{\theta_*}(x,t) = \int dx' h(x-x',t)\overline{\theta_*}(x',0)$$

$$h(x-x',t) = (4\pi Dt)^{1/2} exp[-\frac{(x-x')^2}{4Dt}]$$

where the imaginary time substitution recipe

$$h(x-x',it) = G(x-x',t)\ ,\ h(x-x',t) = G(x-x',-it) \tag{1.21}$$

seems to persuasively suggest the previously mentioned "evolution in imaginary time" notion, except that one *must decide in advance* ,which of the two considered evolutions:the heat or Schrödinger transport,would deserve the status of the "real time diffusion".

At this point let us recall that given the initial data

$$\psi(x,0) = (\pi\alpha^2)^{-1/4} exp(-\frac{x^2}{2\alpha^2}) \tag{1.22}$$

the free Schrödinger evolution $\partial_t\psi = -D\triangle\psi$ implies

$$\psi(x,t) = (\frac{\alpha^2}{\pi})(\alpha^2 + 2iDt)^{-1/2} exp[-\frac{x^2}{2(\alpha^2+2iDt)}] \tag{1.23}$$

On the other hand , we have

$$\psi(x,-it) = \overline{\theta}_*(x,t) = (\frac{\alpha^2}{\pi})^{1/4}(\alpha^2+2Dt)^{-1/2} exp[-\frac{x^2}{2(\alpha^2+2Dt)}] \tag{1.24}$$

Let us confine t to the time interval $[-T/2, T/2]$ with $DT < \alpha^2$.Then we arrive at

$$\partial_t\overline{\theta}_* = D\triangle\overline{\theta}_* \quad , \quad \partial_t\overline{\theta} = -D\triangle\overline{\theta} \quad , \quad -\frac{T}{2} \leq t \leq \frac{T}{2} \tag{1.25}$$

$$\overline{\theta} = (\frac{\alpha^2}{\pi})^{1/4}(\alpha^2 - 2Dt)^{-1/2} exp[-\frac{x^2}{2(\alpha^2-2Dt)}]$$

where

$$\overline{\rho}(x,t) = (\overline{\theta}\overline{\theta}_*)(x,t) = [\frac{\alpha^2}{\pi(\alpha^4 - 4D^2t^2)}]^{1/2} exp[-\frac{\alpha^2x^2}{\alpha^4 - 4D^2t^2}] \tag{1.26}$$

with the interesting, and certainly unpredictable if to follow the traditional Brownian intuitions, outcome: $\overline{\rho}(x,-T/2) = \overline{\rho}(x,T/2)$ However strange this probabilistic evolution would seem, it simply refers to a conditional Brownian motion (in fact the Brownian bridge with smooth ends),and clearly nothing like the"imaginary time diffusion" is here involved. We have rather executed a mapping from one real time diffusion to another, with the *incompatible* dynamical principles at work.

2. Finite difference discussion of Markovian diffusions,local conservation principles and the drift dynamics

In the general theory of Markov diffusions the forward drift $b(x,t)$ of the process is frequently viewed as the stochastic control field,whose properties (smooth, externally driven field) can be quite divorced from the random noise proper.Indeed, if we assume $b(x,t)$ to be a priori given from the beginning,for all times of interest ($t \in [t_0, T]$),then the forward Fokker-Planck equation

$$\partial_t\rho = D\triangle\rho - \nabla(b\rho) \quad , \quad \rho(x,t_0) = \rho_0(x) \tag{2.1}$$

gives rise to the well defined Cauchy problem for $\rho(x,t)$.Apparently,the sole knowledge of $b(x,t)$ (given $\rho_0(x)$) determines both the evolution of $\rho(x,t)$ and the fundamental law of random displacements $p(y,s,x,t), s \leq t$ for the case (2.1). Indeed,by general arguments (Ref.9,chap.4) the forward transition probability density for short times can be deduced from (2.1):

$$p(y,t,x,t+\triangle t) \simeq [4\pi D\triangle t]^{-1/2} exp[-\frac{[x-y-b(y,t)\triangle t]^2}{4D\triangle t}] \tag{2.2}$$

under the assumption $0 < \triangle t \ll t$ and via the chain rule (with the Chapman-Kolmogorov equation consecutively utilized) gives rise to the standard[9,10] path integral expression for the transition density:

$$p(y,s,x,t) = \tag{2.3}$$

$$lim_{\triangle t\to 0}\int dz_1 ... \int dz_{n-1}(4\pi D\triangle t)^{-n/2} exp[-\frac{1}{4D\triangle t}\sum_{k=0}^{n-1}[z_{k+1} - z_k - b(z_k,t_k)\triangle t]^2]$$

where $\triangle t = (t-s)/n$, $z_0 = y$, $z_n = x$,and the $\triangle t$ partitioning of the interval $[s,t]$ becomes finer with the growth of n:$n \to \infty \Rightarrow \triangle t \to 0$.

The finite time increment analysis of the diffusion process is a standard[3,12,13] and quite efficient tool.Let us choose an arbitrary point $x \in R^1$ and investigate what happens in the course of the Markovian diffusion in its vicinity,at different time instants. We shall consider sample paths originating from x at time t (in terms of particles it pertains to the repeatable sampling series of the single particle flights).An average over all points,reached by them time $\triangle t$ later,is given by the conditional mean value of the random variable $X(t+\triangle t)$:

$$< X > (x,t+\triangle t) = \int p(x,t,y,t+\triangle t)\, y dy \simeq x + b_*(x,t)\triangle t \tag{2.4}$$

with $X(t+\triangle t) = y$. Now let us consider sample paths converging to a common endpoint x at time t,and evaluate the conditional expectation value over all points of origin for these paths at time $t - \triangle t$:

$$< X > (x,t-\triangle t) = \int y\, p_*(y,t-\triangle t,x,t)\, dy \simeq x - b_*(x,t)\triangle t \tag{2.5}$$

$$p_*(y,t-\triangle t,x,t)\,\rho(x,t) = \rho(y,t-\triangle t)\, p(y,t-\triangle t,x,t)$$

with $X(t-\triangle t) = y$. We shall use the standard stochastic notations of Refs.4-8 in below.Let X(t) be our random variable.Then ,in terms of left and right (mean) time derivatives for the diffusion, we can write

$$b(x,t) = (D_+X)(t) \quad , \quad b_*(x,t) = (D_-X)(t) \quad , \quad X(t) = x \tag{2.6}$$

Given the local drift $b(x,t)$.Any random displacement of a particle from the point x comprises the same for each sample path purely deterministic contribution $b(x,t)\triangle t$, and the essentially unrestricted noise component $\sqrt{2D}\triangle W(t)$.In virtue of the Itô calculus, the forward Fokker-Planck equation defines the time developement of $\rho(x,t)$ once the initial probability density $\rho_0(x)$ of the random variable $X(t_0) = x$ is specified. Let us consider an ensemble of random paths,all of which are to leave x at time t before arriving at their points of destination $X(t+\triangle t)$ time $\triangle t$ later.The mean local velocity (forward drift) in x at time t equals $b(x,t)$,and all outgoing paths have an identical deterministic contribution .However we know that after time $\triangle t$ the field of local drifts is no longer $b(y,t)$ but $b(y,t+\triangle t)$.Consequently,if we would allow our sample paths to continue the random propagation in the subsequent time $\triangle t$ interval,then each of the sample paths segments would display the new deterministic contribution, now depending on the new point of origin.If we know the field of drifts $b(y,t+\triangle t)$,then we can evaluate how the mean velocity (drift) of the diverging from x path bundle does change in time:

$$b(x,t) \to b(X(t+\triangle t), t+\triangle t) \Rightarrow b(x,t) \to \langle b \rangle (x,t+\triangle t) \tag{2.7}$$

$$\langle b \rangle (x,t+\triangle t) = \int p(x,t,y,t+\triangle t)\, b(y,t+\triangle t)\, dy \simeq b(x,t) + (D_+b)(x,t)\triangle t =$$

$$= b(x,t) + (D_+^2X)(t)\triangle t$$

with $X(t) = x$.

Let us proceed analogously with an ensemble of random (sample) paths $X(t-\triangle t) \to x\, at\, time\, t$ which originate from randomly distributed sources but after time $\triangle t$ (at t) all of them are bound to converge to the same point x.For each particular sample path the deterministic contribution depends on the point of origin,while for the continued (from x at t, in the next time interval of duration $\triangle t$) motion ,the outgoing deterministic contribution is to be the same ($b(x,t)$) for all paths. Consequently,the forward drift change in time can be evaluated along the considered (sample) path ensemble:

$$b(X(t-\triangle t), t-\triangle t) \to b(x,t) \Rightarrow \langle b \rangle (x,t-\triangle t) \to b(x,t) \tag{2.8}$$

$$\langle b \rangle (x,t-\triangle t) = \int b(y,t-\triangle t)\, p_*(y,t-\triangle t,x,t)\, dy \simeq b(x,t) - (D_-D_+X)(t)\triangle t$$

The change in time of the backward drift along the diverging from x at t trajectories comes out as well:

$$b_*(x,t) \to b_*(X(t+\triangle t), t+\triangle t) \to \langle b_* \rangle (x,t+\triangle t) \Rightarrow$$

$$\langle b_* \rangle (x,t+\triangle t) \simeq b_*(x,t) + (D_+D_-X)(t)\triangle t \tag{2.9}$$

while for paths converging to x at t we have:

$$b_*(X(t-\triangle t), t-\triangle t) \to b_*(x,t) \Rightarrow \langle b_* \rangle (x,t-\triangle t) \simeq b_*(x,t) - (D_-^2X)(t)\triangle t \tag{2.10}$$

compare e.g. also arguments of Refs.3,13. At this point it is not useless to mention that for a smooth function $f(X(t),t)$ of the random variable $X(t)$, the left (backward) and right (forward) time derivatives in the conditional mean read:

$$(D_+f)(X(t),t) = (\partial_t + b\nabla + D\triangle)f(X(t),t)$$

$$(D_-f)(X(t),t) = (\partial_t + b_*\nabla - D\triangle)f(X(t),t) \tag{2.11}$$

$$X(t) = x\,,\; b = b(X(t),t)\,,\; b_* = b_*(X(t),t)$$

Let us,quite formally ,instead of b and b_* introduce the new local velocity fields u and v according to

$$b = D_+X = v + u \quad , \quad b_* = D_-X = v - u \tag{2.12}$$

Then,the mere consequence of (2.11) is that the mean acceleration terms can be represented as follows

$$D_+^2X = D_+(u+v) = \partial_t v + (u+v)\nabla v + D\triangle v + \partial_t u + (v+u)\nabla u + D\triangle u \tag{2.13}$$

$$D_-D_+X = D_-(v+u) = \partial_t v + (v-u)\nabla v - D\triangle v + \partial_t u + (v-u)\nabla u - D\triangle u$$

$$D_+D_-X = D_+(v-u) = \partial_t + (v+u)\nabla v + D\triangle v - \partial_t u - (v+u)\nabla u - D\triangle u$$

$$D_-^2X = D_-(v-u) = \partial_t v + (v-u)\nabla v - D\triangle v - \partial_t u - (v-u)\nabla u + D\triangle u$$

We shall restrict further considerations to these Markovian diffusions only ,for which one of the velocity fields introduced above,$u = \frac{1}{2}(b - b_*)$ takes Nelson's *osmotic* velocity form[7,8]

$$u = u(x,t) = D\,\frac{\nabla\rho(x,t)}{\rho(x,t)} \tag{2.14}$$

and accordingly $v = v(x,t)$ deserves its (explicitly ensemble by origin) name of the *current* velocity. The assumed from the very beginning Fokker -Planck equation for the diffusion

$$\partial_t\rho = D\triangle\rho - \nabla(b\rho) = -\nabla(v\rho) \to \partial_t u = -D\triangle v - \nabla(uv) \tag{2.15}$$

allows to rewrite formulas (2.13) in the surprisingly simple and familiar (kinetic theory associations should be born immediately) form:

$$D_+^2X = \partial_t v + v\nabla v + \frac{1}{m}\nabla Q = D_-^2X \tag{2.16}$$

while

$$D_-D_+X = (\partial_t v + v\nabla v - \frac{1}{m}\nabla Q) - 2(D\triangle v + u\nabla v) \tag{2.17}$$

$$D_+D_-X = (\partial_t v + v\nabla v - \frac{1}{m}\nabla Q) + 2(D\triangle v + u\nabla v)$$

Apparently all four acceleration expressions in the above do refer to the same diffusion process,and describe its different mean dynamical features.

We shall loosely adopt the kinetic (gas) theory lore to discuss local (in the mean) properties of the ensemble of sample paths of the diffusion process. Let us think in terms of a "swarm" of sample points (particle locations in the repetitive series of single particle trials) which at time $t-\triangle t$ are distributed with the (conditional) density $p_*(y,t-\triangle t,x,t)$ about the mean position $x-b_*(x,t)\triangle t$.At time $t-\triangle t$ the diffusion involves a field of backward drifts $b_*(x,t-\triangle t)$,so that at each randomly selected location $y=X(t-\triangle t)$ we know the *mean velocity of incoming to y particles* ,hence the way particles are delivered to y at time $t-\triangle t$ from all surrounding points of origin (left at time $t-2\triangle t$).So, $< b_* > (x,t-\triangle t)$ is the mean velocity of incoming particles evaluated over the whole ensemble of random locations $X(t-\triangle t)$ from which random paths are next bound to converge to x at time t. In the same way we can proceed with $b(x,t)$ i.e.*the mean velocity of outgoing from a given point particles*.Then $< b > (x,t+\triangle t)$ is the mean velocity of outgoing particles evaluated over the "swarm" of random points $X(t+\triangle t)$,hence referring to the time interval $[t+\triangle t, t+2\triangle t]$ of the path bundle evolution.We have:

$$t-2\triangle t \to t-\triangle t \to t \to t+\triangle t \to t+2\triangle t \tag{2.18}$$

$$b_*(x,t) - < b_* > (x,t-\triangle t) = (D_-^2X)(t)\triangle t = (D_+^2X)(t)\triangle t = \\ < b > (x,t+\triangle t) - b(x,t) \tag{2.19}$$

Let us recall that in the course of the previous discussion ,no explicit information was utilised about the particular dynamics of the forward drift $b(x,t)$.As mentioned before ,the drift might in principle be an arbitrary smooth function (the control field for the Markovian diffusion) of x and t.We realise however that the above purely stochastic processing is insufficient to generate $b(x,t+\triangle t)$, given the data at time t.For this purpose we must know $\partial_t b = \partial_t u + \partial_t v$. Because of the assumption (2.14), $\partial_t u$ comes out from the Fokker-Planck equation for the density ρ

$$\partial_t u = -D\triangle v - \nabla(uv) \tag{2.20}$$

and the only freedom left in the developed stochastic formalism,pertains to $\partial_t v(x,t)$.Whatever is the dynamics expected (demanded) from $b(x,t)$,the way it can be incorporated is through the formulas:

$$(D_+^2X)(t) = (D_-^2X)(t) = \partial_t v + v\nabla v + \frac{1}{m}\nabla Q = F(x,t) \tag{2.21}$$

$$\frac{1}{2}(D_+D_- + D_-D_+)X(t) = \partial_t v + v\nabla v - \frac{1}{m}\nabla Q = G(x,t)$$

$$G(x,t) - F(x,t) = -\frac{2}{m}\nabla Q(x,t)$$

where $F(x,t), G(x,t)$ are quite arbitrary, but *not independent* smooth functions,responsible for all acceleration-looking phenomena admitted by the Markovian diffusion (compare e.g. also Refs.3,13). The formulas (2.21) provide us with a sound basis for Nelson's stochastic acceleration *postulates* (the Nelson-Newton laws,whose equivalent version is given in terms of (1.2),(1.3)).

Would we have imposed the dynamical restriction (the local drift conservation law respected by the diffusion):

$$(D_-^2X)(t) = 0 = (D_+^2X)(t) \tag{2.22}$$

$$\partial_t v + v\nabla v = -\frac{1}{m}\nabla Q \tag{2.23}$$

then ,in the finite time difference regime,we have a working procedure to establish how the current velocity $v(x,t)$ and hence $b(x,t)$ changes in time between t and $t+\triangle t$:

$$v(x,t+\triangle t) \simeq v(x,t) - (v\nabla v + \frac{1}{m}\nabla Q)\triangle t \tag{2.24}$$

given the data $v(x,t), \rho(x,t)$ at the earlier moment.Apparently:

$$F(x,t) = 0 \Leftrightarrow G(x,t) = -\frac{2}{m}Q(x,t) \tag{2.25}$$

and the admitted class of diffusions is much broader than the Brownian motion proper,see e.g.Ref.6 and Ref.15 in particular for a discussion of the conditional Brownian motions. A more general discussion of diffusions *consistent* with the dynamical constraint:

$$F(x,t) = \frac{1}{m}\nabla V(x) \Leftrightarrow G(x,t) = \frac{1}{m}\nabla[V(x) - 2Q(x,t)] \tag{2.26}$$

can be found in Refs.5,6,16,17 under the name of the "Euclidean quantum mechanics".In the above V(x) is considered to give rise to the traditional force field $-\frac{1}{m}\nabla V = F$ hence it is worth emphasizing that it appears with the "wrong"(Euclidean) sign :on the purely physical grounds it generally corresponds to the replacement of the problem with attracting forces by this with the repulsive ones.

If we decide (following Nelson) to investigate the conservative forces merely,then in principle we can introduce the following dynamical constraints:

$$D_+^2X = D_-^2X = \pm\frac{1}{m}\nabla V \Leftrightarrow \frac{1}{2}(D_+D_- + D_-D_+)X = \pm\frac{1}{m}\nabla(V \mp 2Q) \tag{2.27}$$

$$\frac{1}{2}(D_+D_- + D_-D_+)X = \pm\frac{1}{m}\nabla V \Leftrightarrow D_+^2X = D_-^2X = \mp\frac{1}{m}\nabla(2Q \pm V) \tag{2.28}$$

However,by general arguments (see e.g.Ref.6) we know that for continuous and bounded from below potentials,the Markovian diffusion process is followed by the individual particle if either:

$$D_+^2X = D_-^2X = \frac{1}{m}\nabla V \Leftrightarrow \frac{1}{2}(D_+D_- + D_-D_+)X = \frac{1}{m}\nabla(V - 2Q)$$

$$D_+^2 X = D_-^2 X = \frac{1}{m}\nabla(2Q - V) \Leftrightarrow \frac{1}{2}(D_+D_- + D_-D_+)X = -\frac{1}{m}\nabla V \quad (2.29)$$

holds true as the dynamical restriction (in fact the Nelson-Newton law) on the time developement of the forward drift $b = u + v$. Irrespective of whether we adopt Nelson's dynamical restrictions or proceed more generally with relatively arbitrary $F(x,t), G(x,t)$,we can devise a recipe for the forward drift at the future time instant,given the presenta data.In particular with the choice (2.22) we arrive at:

$$b(x, t + \Delta t) \simeq b(x,t) - (v\nabla v + \frac{1}{m}\nabla Q)(x,t)\Delta t - [D\Delta v + \nabla(uv)](x,t)\Delta t \quad (2.30)$$

and apparently we have $b(x, t+\Delta t) \simeq b(x,t) + (\partial_t v)(x,t)\Delta t + (\partial_t u)(x,t)\Delta t$, as should be. We realize that the dynamical restriction (local drift conservation law) (2.22) does indeed specify the time developement of , otherwise arbitrary,forward drift $b(x,t)$. However , as observed before in more general context, a straightforward consequence of (2.26) is that another momentum (velocity) balance formula is respected by the diffusion process as well

$$\frac{1}{2}(D_+D_- + D_-D_+)X(t) = -\frac{2}{m}\nabla Q \quad (2.31)$$

What is it about ?

Let us recall that $< b > (x, t - \Delta t)$ averages the outgoing velocities at all random locations $X(t - \Delta t)$ of origin, for random paths which are bound to converge to x at t.Hence $< b > (x, t - \Delta t)$ stands for a valid *tendency* of the uniform rectilinear motion attributed to the particle "swarm" in the time interval $[t - \Delta t, t]$,while $b(x,t)$ is to be the one in the subsequent time interval $[t, t + \Delta t]$.We have

$$t - \Delta t \to t \to t + \Delta t \Rightarrow < b > (x, t - \Delta t) \to b(x,t) \quad (2.32)$$

while for the backward drift we find

$$t - \Delta t \to t \to t + \Delta t \Rightarrow b_*(x,t) \to < b_* > (t + \Delta t) \quad (2.33)$$

The *causal sequences* (2.32),(2.33) indicate that in the time interval $[t - \Delta t, t]$, in the *vicinity* of the chosen (reference) point x, we have coexistent:

(i)the flow of sample paths directed towards x and destined to reach (cross) x at time t,its mean velocity (of the incoming particles) equals $b_*(x,t)$,and the flow originates at time $t - \Delta t$ from randomly distributed points $X(t - \Delta t)$

(ii)a multitude of outgoing flows ,which send particles away in random directions from sample points $X(t - \Delta t)$ surrounding x,their "swarm" average equals $< b > (x, t - \Delta t)$. Following intuitions of Ref.13,let us denote

$$v_-(x,t) = \frac{1}{2}[< b > (x, t - \Delta t) + b_*(x,t)] \quad (2.34)$$

the average (inward and outward flows combined together) velocity of sample motions occurring in the vicinity of x in the time interval $[t - \Delta t, t]$. The analogous quantity for the subsequent time interval is

$$v_+(x,t) = \frac{1}{2}[b(x,t) + < b_* > (x, t + \Delta t)] \quad (2.35)$$

so that the local velocity (momentum) balance identity reads:

$$v_+(x,t) - v_-(x,t) = -\frac{2}{m}\nabla Q(x,t)\Delta t \quad (2.35)$$

which connects the average flows about x in time intervals $[t - \Delta t, t]$ and $[t, t + \Delta t]$,respectively. The $\nabla Q(x,t)$ rate of change per Δt is here a consequence of the assumed local conservation law (2.22).

The strict drift conservation law (2.22) can be written as

$$b_*(x,t) - < b_* > (x, t - \Delta t) = 0 = < b > (x, t + \Delta t) - b(x,t) \quad (2.36)$$

while an alternative to (2.22) strict local conservation law:

$$v_+(x,t) = v_-(x,t) \Leftrightarrow \frac{1}{2}(D_+D_- + D_-D_+)X(t) = 0 = \partial_t v + v\nabla v - \frac{1}{m}\nabla Q \quad (2.37)$$

has a microscopic version

$$b(x,t) - < b > (x, t - \Delta t) = -[< b_* > (x, t + \Delta t) - b_*(x,t)] \quad (2.38)$$

While (2.36) tells us that particles are delivered to x with a conserved mean (over the ensemble of converging paths) velocity, and the same conservation law is shared by the outgoing flow,the microscopic conservation law (2.38) tells us something strikingly different.A very specific dynamical equilibrium is locally maintained in the course of the diffusion:if the outgoing drift drops down between $t - \Delta t$ and t, then the incoming drift will show up the growth between t and $t + \Delta t$ in the very same rate,and in reverse.

Consequently (2.38) embodies the definite *action-reaction phenomena* in the causal sequence $t - \Delta t \to t \Rightarrow t \to t + \Delta t$. Let us invoke the *local Brownian recoil principle* of Ref.1 at this point: *"If Brownian fluctuations due to the medium produce an average field of local particle flows $\overline{v}(x,t)$,then an average field of local drifts $-\overline{v}(x,t)$ is induced in the medium itself.The $-\overline{v}$ local drag of particles does compensate the $m\overline{v}$ local momentum associated with (transferred to) the ensemble of Brownian scattered particles"* We denote

$$< b > (x, t - \Delta t) = b(x,t) + \Delta\overline{v}(x,t) \Rightarrow < b > (x, t - \Delta t) \to < b > -\Delta\overline{v} = b \quad (2.39)$$

According to the Brownian recoil principle, the mean momentum information cannot simply *evaporate* .We demand[1] the validity of the momentum conservation law on all conceivable scales adopted for the investigation of the individual particle scattering on the medium constituents.Hence (2.38) tells us that the outgoing particle flow $\Delta\overline{v}$ is being built in the interval $[t-\Delta t,t]$.Since the flow $\Delta\overline{v}$ is transporting particles away from the area surrounding x, it cannot affect $b_*(x,t)$,albeit it *must* contribute to $< b_* >(x,t+\Delta t)$:

$$b_*(x,t) \to b_* + \Delta\overline{v} = < b_* > (x,t+\Delta t) \tag{2.40}$$

The formula (2.38) is a microscopic statement that the local Brownian recoil principle governs the diffusion.It gives account of the detailed structure (strictly observed momentum conservation law) of the particle-random medium interaction.Hence we deal here with a universal space-time independent property of the medium.Therefore it can be taken as a defining feature of the environment which is capable of generating a specific class of Markovian diffusions.Quite alike the Brownian law of random displacements, which is another way to tell that the Brownian medium remains statistically homogeneous. The arguments of Ref.1 imply here that the individually negligible phenomena *on the ensemble average*,give rise to the highly non-trivial turbulent medium structure, which is eventually responsible for the quantum mechanical looking evolution of particle ensembles (wave order from the corpuscular chaos).

3. Drift dynamics in terms of dynamical semigroups

In case of conservative force fields, the dynamical constraints defining the appropriate Markovian diffusion read:

$$D_+^2 X = \frac{1}{m}\nabla V = D_-^2 X \quad (Zambrini) \tag{3.1}$$

$$D_+^2 X = \frac{1}{m}\nabla(2Q - V) = D_-^2 X \quad (Nelson) \tag{3.2}$$

Let us once more emphasize that they express the very same dynamical content as the more familiar[7,8] Nelson-Newton laws:

$$\frac{1}{2}(D_+^2 + D_-^2)X = \frac{1}{m}\nabla V \ (Zambrini) \tag{3.3}$$

$$\frac{1}{2}(D_+D_- + D_-D_+)X = -\frac{1}{m}\nabla V \ (Nelson) \tag{3.4}$$

The important observation is that the potentials explicitly present in (3.1),(3.2) are precisely the ones defining the semigroup dynamics,whose behind-the-scene presence is indispensable (Ref.6) for a rigorous investigation of the Markov-Bernstein diffusions.Provided we take $V(x)$ to be continuous and bounded from below,and require the same property from $2Q - V$ .The latter demand is highly non-trivial and seemingly impossible to control because of the explicit $\rho(x,t)$ dependence mediated by $Q(x,t)$.However a careful examination shows[6] that our demand amounts to selecting a class of diffusions which obey the finite energy restriction:

$$\int \rho(x,t)[\frac{1}{2}(u^2 + v^2) + V]dx < \infty \tag{3.5}$$

Assume to have given the Wiener measure (in fact, the heat kernel characterising the free Brownian diffusion).Let $\Re(X(t),t)$ be an arbitrary continuous and bounded from below function of the random variable $X(t)$ of the Wiener process.Let y at time t be the source point for random paths of the process, while x the terminal one for all of them,at time $t > s$.We shall follow the finite time difference procedure in the spirit of (1.7),but with $b = 0$) and define:

$$\Delta t = (t-s)/n\, ,\, t_j = j\Delta t\, ,\, j = 0,1,...,n\, ,\, t_0 = s\, ,\, t_n = t\, ,\, X(t_j) = x_j\, ,\, x_0 = y\, ,\, x_n = x \tag{3.6}$$

Let us define the following integral (with respect to the conditional Wiener measure,see Refs.18-20):

$$K(\Re;y,s,x,t) = lim_{n\to\infty}\frac{1}{(4\pi D\Delta t)^{n/2}}\int_{-\infty}^{+\infty} dx_1 ... \int_{-\infty}^{+\infty} dx_{n-1}$$

$$exp(-\frac{x_1^2}{4D\Delta t} - \sum_{j=1}^{n-1}[\frac{1}{2mD}\Re(y+x_j,t_j)\Delta t + \frac{(x_{j+1}-x_j)^2}{4D\Delta t}]) \tag{3.7}$$

It is well known that this (Feynman -Kac) formula defines the semigroup kernel in the sense that ($D = \hbar/2m$ might be set for comparison)

$$\partial_t K = D\triangle_x K - \frac{1}{2mD}\Re(x,t)K \quad , \quad lim_{t\to s}K(\Re;y,s,x,t) = \delta(x-y) \tag{3.8}$$

holds true.We are here in the situation more general than this originally[6] set for the investigation of the Markov-Bernstein diffusions.One should really keep in mind that time dependent potentials $\Re(x,t)$ are allowed, in conformity with the general case (2.25).

As a dynamical semigroup kernel,in addition to (3.8), $K(x,t)$ solves:

$$\partial_t K = -D\triangle_x K + \frac{1}{2mD}\Re(x,t)K \tag{3.9}$$

as well (we assume $\Re(x,t) = \Re(x,-t)$ here to secure the time reversal invariance). From now on we shall use the notation:

$$K(V;y,s,x,t) = h(y,s,x,t)\ ,\ K(2Q-V;y,s,x,t) = k(y,s,x,t) \tag{3.10}$$

Assume to have given a pair of diffusion equations in duality (i.e. mutually time adjoint) for real functions $\overline{\theta}, \overline{\theta}_*$:

$$\partial_t\overline{\theta}_* = D\triangle\overline{\theta}_* - \frac{V}{2mD}\overline{\theta}_* \quad , \quad \partial_t\overline{\theta} = -D\triangle\overline{\theta} + \frac{V}{2mD}\overline{\theta} \tag{3.11}$$

D is the diffusion constant, m mass of a particle in the course of the diffusive transport.By our assumptions about V we have granted the existence of the strictly positive semigroup kernel generated by the operator $H = -2mD^2\triangle + V$. Let $h = h(y,s,x,t), s \leq t$ be the fundamental solution (dynamical semigroup kernel) of the diffusion equations (3.11). Then, the initially chosen function $\overline{\theta_*}(x,-T/2), T \geq 0$ is propagated forward $\overline{\theta_*}(x,t) = \int \overline{\theta_*}(z,-T/2)h(z,-T/2,x,t)dz$, $t \geq -T/2$ while the terminal choice of $\overline{\theta}(x,T/2)$, $T \geq 0$ allows to reproduce the past data $\overline{\theta}(x,t)$, $t \leq T/2$ through the backward propagation $\overline{\theta}(x,t) = \int h(x,t,y,T/2)\overline{\theta}(y,T/2)dy$. In virtue of the semigroup property of the kernel h we have also :

$$\overline{\theta_*}(x,t) = \int \overline{\theta_*}(z,s)h(z,s,x,t)dz \quad , \quad \overline{\theta}(x,s) = \int h(x,s,z,t)\overline{\theta}(z,t)dz \qquad (3.12)$$

where $-T/2 \leq s < t \leq T/2$, hence a solution of (3.11) with the prescribed boundary data at $\pm T/2$ might be given.

By Ref.6 , we have here determined the Markov -Bernstein stochastic process, which allows to propagate (hence predict the future and reproduce the past ,given the present) the probability distribution

$$\overline{\rho}(x,t) = \overline{\theta}(x,t)\overline{\theta_*}(x,t) \qquad (3.13)$$

respectively forward and backward in time .The statistical predictions about the future can be accomplished by means of the forward transition probability density

$$\overline{p}(x,s,y,t) = h(x,s,y,t)\frac{\overline{\theta}(y,t)}{\overline{\theta}(x,s)} \qquad (3.14)$$

while the past can be statistically reproduced by means of the backward density

$$\overline{p}_*(x,s,y,t) = h(x,s,y,t)\frac{\overline{\theta_*}(x,s)}{\overline{\theta_*}(y,t)} \qquad (3.15)$$

for the diffusion with fixed (!) boundary probability distributions $\overline{\rho}(x,-T/2), \overline{\rho}(x,T/2)$.

With p, p_* in hands ,we can straightforwardly evaluate the conditional expectation values which are necessary to establish the mean forward and backward derivatives in time for functions of the random variable $X(t)$.The drifts read here[6]

$$(D_-\overline{X})(t) = \overline{b_*}(x,t) = -2D\frac{\nabla\overline{\theta_*}}{\overline{\theta_*}} \qquad (3.16)$$

$$(D_+\overline{X})(t) = \overline{b}(x,t) = 2D\frac{\nabla\overline{\theta}}{\overline{\theta}} \qquad (3.17)$$

so that $v = \frac{1}{2}(\overline{b} + \overline{b_*})$ and

$$\partial_t\overline{\rho} = -\nabla(\overline{\rho v}) = D\triangle\overline{\rho} - \nabla(\overline{\rho b}) = -D\triangle\overline{\rho} - \nabla(\overline{\rho b_*}) \qquad (3.18)$$

If we define

$$\overline{\theta} = exp(\overline{R} + \overline{S}) \ , \ \overline{\theta_*} = exp(\overline{R} - \overline{S}) \qquad (3.19)$$

with $\overline{R}, \overline{S}$ being real functions ,then

$$\overline{v} = 2D\nabla\overline{S} \ , \ \overline{u} = \frac{1}{2}(\overline{b} - \overline{b_*}) = 2D\nabla\overline{R} \qquad (3.20)$$

and (3.18) can be rewritten as follows

$$\frac{1}{2D}\partial_t\overline{R} = -\frac{1}{2}\triangle\overline{S} - (\nabla\overline{R})(\nabla\overline{S}) \rightarrow \partial_t\overline{u} = -D\triangle\overline{v} - \nabla(\overline{uv}) \qquad (3.21)$$

If we assume that the the continuity equation is valid,then the necessary consequence of (3.11) is that[5,6,22] the modified (by the presence of the extra term) Hamilton-Jacobi equation holds true

$$V = 2mD(\partial_t\overline{S} + D(\nabla\overline{S})^2 + D[(\nabla\overline{R})^2 + \triangle\overline{R}]) \qquad (3.22)$$

and $\overline{\rho} = \overline{\theta\theta_*}$ implies

$$2mD^2[(\nabla\overline{R})^2 + \triangle\overline{R}] = 2mD^2\frac{\triangle\overline{\rho}^{1/2}}{\overline{\rho}^{1/2}} = \overline{Q} \qquad (3.23)$$

Except for the sign inversion $\overline{Q}$ has the familiar and previously identified form of the de Broglie-Bohm "quantum potential",see e.g. Refs.1,14,15. Apparently we can argue here in reverse (we adopt the stronger version of the Nagasawa argument[22] ;given the continuity equation and the modified Hamilton-Jacobi equation, then the system of diffusion equations in duality (3.11) follows. It is easy to verify that the gradient form of (3.22)is equivalent to the previously discussed Nelson-Newton law

$$\partial_t\overline{v} = 2D\triangle\overline{u} + \frac{1}{2}\nabla\overline{u}^2 + \frac{1}{2}\nabla\overline{v}^2 + \frac{1}{m}\nabla V \Leftrightarrow D_+^2\overline{X} = D_-^2\overline{X} = \frac{1}{m}\nabla V \qquad (3.24)$$

Let us now consider another pair of diffusion equations in duality (actually a non-trivially coupled nonlinear system)

$$\partial_t\theta = D\triangle\theta_* - \frac{1}{2mD}(2Q - V)\theta_* \ , \quad \partial_t\theta = -D\triangle\theta + \frac{1}{2mD}(2Q - V)\theta \qquad (3.25)$$

where V is the same as before , while $Q = 2mD^2(\triangle\rho^{1/2})/\rho^{1/2}, \rho(x,t) = \theta(x,t)\theta_*(x,t)$ differ from the previously utilised objects by the absence of dashes,which is to ditinguish solutions of (3.11) from these of (3.25). In virtue of the semigroup argument applicability (provided we demand $2Q - V$ to be continuous and bounded from below[6]) all previous considerations can be repeated through replacing the diffusion kernel $K(x,s,y,t)$ by the nonlinear diffusion kernel $k(x,s,y,t)$ of (3.25). The transition probability densities (no dash !) p, p_* allow to derive the new drifts b, b_*.The continuity equation in the canonical form $\partial_t\rho = -\nabla(\rho v)$ implies that as a consequence of (3.25) we arrive at the identity (which is a modified version of the Hamilton-Jacobi equation again)

$$2Q - V = 2mD[\partial_t S + D(\nabla S)^2] + Q \rightarrow (\partial_t + v\nabla)v = -\frac{1}{m}\nabla(V - Q) \qquad (3.26)$$

to be compared to the previously considered (dashed case) (3.22),(3.23). Let us here observe that the identity (3.26) implies $2Q = 2V + 4mD[\partial_t S + D(\nabla S)^2]$ hence

$$\frac{1}{2mD}(2Q - V) = \frac{1}{2mD}V + 2\partial_t S + 2D(\nabla S)^2 \quad (3.27)$$

which allows to replace (3.26) by the equivalent identity,where the original potential $\frac{1}{2mD}V$ acquires a correction $2\partial_t S + 2D(\nabla S)^2$.While (3.26) was utilised by Zambrini[6], the system with the corrected potential was utilised by Nagasawa[22],however without notifying that the equivalence is established when diffusions with creation and killing are replaced by the Markov-Bernstein diffusions. It is well known that equations of continuity and the Hamilton-Jacobi problem (3.26) do uniquely (Madelung representation) determine solutions of the Schrödinger equation

$$i(2mD)\partial_t\psi = [-\frac{mD^2}{2}\triangle + V]\psi \quad , \quad \psi(x,t) = exp(R + iS)(x,t) \quad (3.28)$$

but now the solutions of the diffusion system (3.25) do determine R and S in the above

$$R = \frac{1}{2}ln(\theta\theta_*) \,, \; S = \frac{1}{2}ln(\frac{\theta}{\theta_*}) \quad (3.29)$$

For a discussion of the multiply connected (due to nodal surfaces) configuration space see Ref.6.As before all arguments go in reverse,since the standard Madelung route applies: once we know that the continuity and Hamilton-Jacobi equations holds true, we have the system of diffusion equations (3.25) as the equivalent description of the Schrödinger one.One immediately verifies that the Nelson-Newton law $\frac{1}{2}(D_+D_- + D_-D_+)X(t) = -\frac{1}{m}\nabla V$ comes out as the gradient form of (3.26). The stochastic acceleration formulas, according to our previous discussion of Section 2, play the role of the momentum (velocity) balance equations in the mean for stochastic flows, and by the finite difference arguments give an insight into the random particle transport phenomena.However the knowledge of gradient velocity fields u, v ($\overline{u}, \overline{v}$ respectively) is *insufficient* for a unique reconstruction of the underlying stochastic diffusion theory. The intrinsic ambiguity of the Nelson-Newton laws was bypassed by us in the discussion of Section 2, but not overcome. There , the distinctively alien to each other particle-random medium interaction mechanisms were found to be responsible for the drift dynamics,and we know that the Cauchy problems (1.1),(1.2) and (1.1),(1.3) respectively , do uniquely specify the evolution of ρ and v (and hence b) given the initial data $\rho_0(x), v_0(x)$. On the other hand the potentials R, S ($\overline{R}, \overline{S}$) for gradient velocity fields u, v ($\overline{u}, \overline{v}$) are the primary entities in the diffusion semigroup formalism.Apparently the passage from S to $2D\nabla S = v$ and the gradient operation necessary to derive the stochastic acceleration formulas in this framework,do enforce a certain loss of the differentiation between various quantities provided by the primordial diffusion equations in duality.

Let us consider our initial Cauchy problems (1.1),(1.2) or (1.1),(1.3) to specify the dynamics in a fixed finite time interval $[t_0, T]$.Then ,given ρ_0, v_0, we have the terminal

distribution $\rho_T(x)$ in hands.At this point we shall address the problem originally due to Schrödinger[21],whose refined description can be found in Refs.5,6,15-17:

Given the boundary probability distributions $\rho_0(x)$ and $\rho_T(x)$ .Can we derive the stochastic process interpolating between them ?

The answer can be given in the framework of the Bernstein stochastic processes.If we demand the interpolating process to be Markovian (strictly speaking the Markov-Bernstein) then the interpolating process is specified *uniquely* if the joint probability distribution

$$\theta_*(x,t_0)K(x,t_0,y,T)\theta(y,T) = m(x,y) \quad (3.32)$$

$$\int m(x,y)dy = \rho(x,t_0) \quad , \quad \int m(x,y)dx = \rho(y,T)$$

can be found ,where $K(x,t_0,y,T)$ is a certain strictly positive dynamical semigroup kernel and $\theta_*(x,t_0), \theta(y,T)$ are two real and non-zero functions of the same sign. By specifying the kernel to coincide either with our $k(x,s,y,t)$ or $h(x,s,y,t)$ we have indeed given the evolution of $\theta_*(x,t), \theta(y,t)$ in our prescribed time interval, provided $\theta_*(x,t_0)$ are the initial data for the forward evolution, while $\theta(y,T)$ are the terminal data to be propagated back in time by the backward evolution. These data must be found as a solution of the *Schrödinger system* (3.32), once the kernel K is chosen . In virtue of the uniqueness of the solution, we realize that both $R(x,t)$ and $S(x,t)$ are specified .It however means that any of the evolution scenarios of Section 2 is realized not by a single but by many distinct Markov-Bernstein stochastic processes all of which belong to the same (dynamical) equivalence class.They interpolate between the given pair of boundary densities $\rho_0(x), \rho_T(x)$:each representative being uniquely specified by the two potentials R, S ($\overline{R}, \overline{S}$ respectively).

Remark 1 : It is interesting to notice that $m(x,y)\triangle x\triangle y$ or more generally

$$m(B_1, B_2) = \int_{B_1} dx \int_{B_2} dy\, m(x,y) \quad (3.33)$$

stands for the probability that a particle leaving the area B_1 at time t_0 will reach the area B_2 at time T. This concept is of profound importance from the point of view of experimental procedures where monitoring the particle in the course of its propagation is either not easy ,or impossible without destroying the (otherwise arising) experimental outcomes ,like e.g. in case of the interference experiments. What is usually accessible,is the information about the initial frequency distribution (we approximate it then by an appropriate probability density) due to the chosen statistical "state preparation procedure",and about the terminal frequency distribution (photo-plate data or numer-of-particles data like in most neutron counting experiments) which is again approximated by a certain terminal probability density.

Example 4: *Brownian motion in a field of force as the Bernstein diffusion*

To make the role of dynamical semigroups in the presented formalism more explicit,we shall make an illuminating exercise to demonstrate that the very traditional

Brownian motion in the external force field,if considered in the Smoluchowski approximation,does give rise to a rich class of Markov-Bernstein diffusions. The Fokker-Planck equation governing the time developement of the spatial probability distribution in case of the phase space noise with high friction, in the Smoluchowski form reads

$$\partial_t \rho = D\triangle\rho - \nabla(b\rho) \quad , \quad b(x,t) = \frac{1}{\beta}F(x), \; , \; \rho_0(x) = \rho(x,0) \tag{3.34}$$

where β is the friction constant and the external force we assume to be conservative

$$F(x) = -\nabla\Phi(x) \tag{3.35}$$

It is well known that the substitution

$$\rho(x,t) = \theta_*(x,t)\, exp[-\frac{\Phi(x)}{2D\beta}] \tag{3.36}$$

converts the Fokker-Planck equation into the generalised diffusion equation for $\theta_*(x,t)$

$$\partial_t\theta_* = D\triangle\theta_* - \frac{V(x)}{2mD}\theta_* \tag{3.37}$$

where (the mass m was here introduced per force ,but with a very concrete purpose of embedding our discussion in the formalism of the "Euclidean quantum mechanics")

$$V(x) = \frac{m}{\beta}(\frac{F^2}{2\beta} + D\nabla F) \tag{3.38}$$

Since F^2, D, β are positive, a sufficient condition for the auxiliary potential $V(x)$ to be bounded from below (its continuity is taken for granted) is that the source term $g(x)$ in the familiar Poisson equation

$$\nabla F = -\triangle\Phi = g \tag{3.39}$$

is bounded from below: $g(x) > -c, c > 0, c$ is finite. Under this boundedness condition,we know that the equation(3.37) defines the fundamental *semigroup transition mechanism* underlying the Smoluchowski diffusion.Indeed, we have in hands the well defined semigroup operator $exp[-t(-D\triangle + V/2mD)]$, whose integral kernel is a strictly positive solution of (3.37) with the initial condition $\lim_{t\to 0} h(y,0,x,t) = \delta(y-x)$.

The kernel is defined by the Feynman-Kac formula (3.7) (in terms of the conditional Wiener measure,which sets an obvious link with the Brownian propagation). It is trivial to check that $h(y,s,x,t)$ propagates $\theta_{*0}(x)$ into a solution of (3.37)

$$\theta_{*0}(x) = \rho_0(x)\, exp[\frac{\Phi(x)}{2D\beta}] \longrightarrow \theta_*(x,t) = \int h(y,0,x,t)\theta_*(y,0)dy \tag{3.40}$$

while,apparently

$$\theta(x,t) = exp[-\frac{\Phi(x)}{2D\beta}] = \int h(x,t,y,T)\theta_T(y)dy = \theta_T(x) \tag{3.41}$$

for all $t \in [0,T]$.Indeed $\theta(x,t)$ solves

$$\partial_t\theta = -D\triangle\theta + \frac{V}{2mD}\theta \tag{3.42}$$

where $\partial_t\theta = 0$ and

$$D\triangle\theta = [\frac{(\nabla\Phi)^2}{4D\beta^2} - \frac{\triangle\Phi}{2\beta}]\theta = \frac{V}{2mD}\theta \tag{3.43}$$

Since the deterministic evolution governed by the Smoluchowski equation gives rise to a definite terminal (in the interval $[0,T]$) outcome $\rho_T(x)$, given $\rho_0(x)$,a straightforward inspection demonstrates that the Schrödinger system is solved by $\theta_{*0}(x)$ and $\theta_T(x)$ with the kernel $h(V;y,s,x,t)$. As a consequence , we have completely specified the unique Markov-Bernstein diffusion interpolating between $\rho_0(x)$ and $\rho_T(x)$,which is identical with the Smoluchowski diffusion itself.We know here the transition probability density (e.g. the law of random displacements modified by the presence of external force fields) $p(y,s,x,t)$ in the form (3.14), which is responsible for the *most likely* particle propagation scenario. We have also automatically satisfied the local conservation laws (1.1),(1.2). In our case,apparently

$$v(x,t) = D\nabla(-\frac{\Phi}{D\beta} - ln\rho) = -\frac{1}{\beta}\nabla\Phi - D\frac{\nabla\rho}{\rho} \longrightarrow \tag{3.44}$$

$$\partial_t\rho = \nabla[\frac{1}{\beta}(\nabla\Phi)\rho] + D\triangle\rho$$

to be compared with the Smoluchowski equation.

Remark 2:The formally oriented reader might benefit from the observation that the whole family of diffusions affiliated[36] to the Smoluchowski process in a force field is embedded in Zambrini's "Euclidean quantum mechanics".A formal recipe of the "imaginary time transformation" allows to map "Euclidean" into Nelson diffusions on the sound mathematical[38] (albeit not-physical[37]) basis.

Example 5:*Coherent state dynamics (harmonic potential)*

To conclude our discussion ,we invoke a notorious harmonic oscillator problem and the related coherent states .Our aim is to consider their Schrödinger evolution from the diffusion process viewpoint (see e.g.Ref.23). However,we shall address the issue from a new perspective[37,39].We depart from the general stochastic differential equation (1.6) and the related Fokker-Planck equation (1.7) for the transition probability density.Following Stratonovich[39] let us transform the transition density (compare e.g.(3.14),(3.15)) by means of the substitution $p(y,s,x,t) = k(y,s,x,t)exp\Phi(y,s)/exp\Phi(x,t)$,which under an assumption that $b(x,t)$ is the gradient field

$$b(x,t) = -2D\nabla\Phi(x,t) \Rightarrow \frac{1}{2}[\frac{b^2}{2D} + \nabla b] = D[(\nabla\Phi)^2 - \triangle\Phi] \tag{3.45}$$

allows to replace the Fokker-Planck equation for p by the generalised diffusion equation (the dynamical semigroups enter the game)

$$\partial_t k = D\triangle_x k - (-\partial_t\Phi - D[\triangle\Phi - (\nabla\Phi)^2])k \tag{3.46}$$

where $lim_{t\to s}k(y,s,x,t) = \delta(x-y)$.Its (to be strictly positive) solution can be represented in terms of the Feynman-Kac formula (3.7),which integrates contributions from the *auxiliary* potential

$$\frac{\Re}{m} = 2D(-\partial_t\Phi - D[\triangle\Phi - (\nabla\Phi)^2]) = -2D\partial_t\Phi + D\nabla b + \frac{1}{2}b^2 \tag{3.47}$$

with respect to the conditional Wiener measure.Hence,with the given Φ and the integral kernel $k(y,s,x,t)$ of the dynamical semigroup operator $exp[-\frac{1}{2mD}\int_s^t(2mD^2\triangle - \Re)du]$,we have the appropriate transition probability density for the diffusion in hands. In the particular case of the harmonic oscillator,the coherent (minimum uncertainty wave packet) solution of the Schrödinger equation $i\partial_t\psi = -D\triangle\psi + \frac{1}{4mD}m\omega^2x^2\psi$ has the canonical form

$$\psi(x,t) = exp(R+iS)(x,t) = \rho^{1/2}(x,t)exp[iS(x,t)] \tag{3.48}$$

$$\rho(x,t) = (2\pi\sigma)^{-1/2}exp[-\frac{1}{4\sigma}(x-q_{cl})^2] \quad , \quad S(x,t) = \frac{1}{2mD}[xp_{cl} - \frac{1}{2}p_{cl}q_{cl} - mD\omega t]$$

In the above $\sigma = D/\omega$ and $D = \hbar/2m$ should be set for comparison with the standard quantum mechanical notation. The classical phase-space variables display the time dependence only: $q_{cl}(t) = q_0 cos\omega t + \frac{p_0}{m\omega}sin\omega t$ and $p_{cl}(t) = p_0 cos\omega t - m\omega q_0 sin\omega t$ .By our previous arguments , we know that the stochastic differential equation (1.6) for the Nelson diffusion associated with (3.48) has the forward drift in the form $b(x,t) = \frac{1}{m}p_{cl}(t) - \omega[x - q_{cl}(t)] = -2D\nabla\Phi(x,t)$ where $\Phi = -[S + \frac{1}{2}ln\rho]$.All the data needed for the Feynman-Kac integration (33.7) are given. Let us add that one could as well begin from the traditional Mehler formula[18], and next utilize the Cameron-Martin theory[40] of linear drift transformations (translations) to arrive at the same goal.

Note:*It is perhaps justified to mention my personal past concerning the coherent states,and their applications to the study of classical relatives of Fermi systems, the identification of Fermi sectors in the state spaces of Bose systems or the classical-quantum relationships in nonlinear field theory models*[31]*. The present research has its roots there.*

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SOME REFLECTIONS ON COHERENCE AND ION TRAPPING†

ROY J. GLAUBER

Lyman Laboratory of Physics, Harvard University

Cambridge, Massachusetts 02138

ABSTRACT

We recall the physical arguments underlying the definition of the coherent states and review some of their related properties. The problem of ion trapping in oscillating quadrupole fields leads us to the consideration of a new class of generalized coherent states that correspond to the motion of periodically pulsating wave packets along classical trajectories.

1. Introduction

In the earliest work on wave mechanics by Schrödinger one mystery abides among its many penetrating insights. The physical meaning of the wave function ψ, which lay somehow at the foundation of the new mechanics, remained completely unclear. Schrödinger turned, seeking clarification, to the analysis of time-dependent states. He evidently felt that the wave function of a particle represents the structure of the particle itself in a much more explicit way than we now do, and so he hoped to find solutions to the wave equation that remained restricted in size and moved along mechanical trajectories akin to those of classical point particles. He found such a solution for the case of the one-dimensional harmonic oscillator.[1] By superposing the normalized wave functions of the n-th excitation states of the oscillator, using the coefficients $\frac{A^n}{\sqrt{2^n n!}}$, where A is a real number much larger than unity, he showed that the resulting wave packet has a Gaussian peak that moves precisely as the classical oscillator does. What he was doing, in effect, was to give the ground state wave function of the oscillator a real-valued initial displacement proportional to A and to observe its subsequent behavior. Since the "width" of the wave function remained that of the ground state no matter how large the amplitude A was made, he could see the classical limit of particle-like behavior quite clearly.

This was surely the first example of what we now call the coherent states, though it dealt with only a narrow subset of those oscillator states (for real rather than complex A). Schrödinger looked forward, a bit optimistically, to the possibility of finding analogous wave packets for other mechanical systems. He foresaw, for example, the construction of a packet that would move about the Kepler ellipses of the hydrogen atom without spreading, but never found one. That problem has come back to life again in experimental terms, and we shall surely hear more about it.

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