Rotational diffusions as seen by relativistic observers

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The major unsolved problem in the framework of Nelson's stochastic mechanics is addressed and an attempt is made to provide a description of relativistic spin-$\frac{1}{2}$ particles in terms of Markovian diffusions on S_3. Random rotations are here labeled by the proper time of a particle in relativistic motion and are continuously distributed along a space-time trajectory followed by the particle in Minkowski space.

I. INTRODUCTION: THE PROBLEM OF RANDOM ROTATIONS IN SPECIAL RELATIVITY

The description of spin-$\frac{1}{2}$ in the framework of Nelson's stochastic mechanics[1,2] involves a harmonic analysis on the group $\mathscr{G}$ of rotations in R^3. For functions on the group the Hilbert space structure is induced by the invariant Haar measure on S_3. The scalar product reads[3,4] as

$$(f_1,f_2)=\int dg\, f_1(g)\overline{f}_2(g)$$
$$=\frac{1}{8\pi^2}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta$$
$$\times\int_0^{2\pi}d\psi\sin\theta f_1(g)\overline{f}_2(g),$$
$$S\ni g=(\theta,\phi,\psi), \tag{1.1}$$

where θ, ϕ, and ψ are the Euler angles locally parametrizing the group manifold in a given Cartesian frame of reference. The transformations $U_{g_1}f(g) = f(gg_1)$ are known to form an infinite unitary representation of the group of rotations in the Hilbert space $L^2(S_3)$, whose resolution into irreducible components gives rise to a four-dimensional space $H_{1/2}$ characterizing spin-$\frac{1}{2}$. The familiar SU(2) harmonics form an orthonormal basis system in it.

Let $G(t)$ be a random variable taking values in S_3, which undergoes a nondissipative (rotational) Markovian diffusion.[1,2] The ith SU(2) harmonic describes the ith state of stationary diffusion, $i=1,2,3,4$. For each state we can introduce a vector-valued function of this random variable $\mathbf{L}=\mathbf{L}(G(t))$, which following Refs. 1 and 2 may be attributed the role of an angular momentum induced by the rotation $G(t)$ in a given state of rotational diffusion. Let $e_i(g)$ denote the SU(2) harmonic for the spin-$\frac{1}{2}$ case. Then $|e_i(g)|^2$ stands for the probability distribution of $G(t)$ in the state $e_i(g)$. We can evaluate the expectation values:

$$\langle\mathbf{L}\rangle_i=\int\mathbf{L}(g)|e_i(g)|^2\,dg=\epsilon_i\frac{\hbar}{2}\mathbf{s}$$
$$\epsilon_i=\begin{cases}+1, & i=1,2,\\ -1, & i=3,4,\end{cases} \tag{1.2}$$

where $\mathbf{s}$ is the unit (spin polarization) vector in R^3, identifying the direction of the quantization axis in space. In the standard quantum mechanical lore it is the direction on which spin projections are equal to $\pm\hbar/2$.

In Dankel's paper,[1] the case of $\mathbf{s}=\mathbf{k}$ (the z direction in the Cartesian frame) was investigated. In virtue of Ref. 2 the mere change of Euler parametrization allows us to consider quite arbitrary $\mathbf{s}$, eventually allowing for a smooth time dependence $\mathbf{s}=\mathbf{s}(t)$ characteristic for the spin precession.

According to Ref. 2 rotational diffusions characterizing the spin-$\frac{1}{2}$ particle at rest involve the Cartesian system frame, whose orientation relative to the laboratory frame is given by the Euler angles $(\overline{\theta},\overline{\phi},\overline{\psi})=\overline{g}$. They determine the quantization axis direction,

$$R(\overline{g})\mathbf{k}=\mathbf{s}=(\sin\overline{\psi}\sin\overline{\theta},\cos\overline{\psi}\sin\overline{\theta},\cos\overline{\theta}). \tag{1.3}$$

All random fluctuations (e.g., rotations) are intrinsic to the system frame, and described in terms of another set of Euler angles $(\theta,\phi,\psi)=g$ referring to intrinsic rotation axes $\mathbf{e}_\theta$, $\mathbf{e}_\phi$, $\mathbf{e}_\psi$ in the system frame.

The discussion[1,2] of random rotations implementing spin-$\frac{1}{2}$ is purely nonrelativistic and effectively confined to the system rest frame (inhomogeneous magnetic fields alter this picture[2]).

We denote by K' the inertial rest frame, in which the spin-$\frac{1}{2}$ rotational diffusion takes place. We admit, furthermore, that K' moves uniformly with the velocity $\boldsymbol{v}$, $|\boldsymbol{v}|<c$ relative to another inertial frame, and address the following relativistic problem.

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How does the K observer perceive the Markovian diffusion taking place in K′? Is it a stochastic diffusion process again?

The nonrelativistic treatment of Ref. 2 suggests that we should first establish the transformation properties of the polarization axes when passing from one inertial frame to another. The issue has been solved in the context of the Bargmann–Michel–Telegdi equation.[5,6] Specification of the rest frame polarization is known to determine the components of the polarization four-vector in any inertial frame. Indeed, if $\mathbf{s}$ is the rest frame polarization, its K frame image [via the Lorentz transformation Λ^{-1} taking $(c,0,0,0)$ into $(\gamma c,\gamma \mathbf{v})$ with $\gamma=(1-\beta^2)^{-1/2}$, $\boldsymbol{\beta}=\mathbf{v}/c$], is given by[6]

$$(0,\mathbf{s})\rightarrow(S^0,\mathbf{S}),$$

$$S^0=\boldsymbol{\beta}\mathbf{S}=\gamma\boldsymbol{\beta}\mathbf{s}, \tag{1.4}$$

$$\mathbf{S}=\mathbf{s}+\frac{\gamma^2}{1+\gamma}(\boldsymbol{\beta}\mathbf{s})\boldsymbol{\beta},$$

while in reverse we have $\mathbf{s}=\mathbf{S}-[\gamma/(\gamma+1)](\boldsymbol{\beta}\mathbf{S})\boldsymbol{\beta}$.

The normalized vector $\mathbf{s}, |\mathbf{s}|=1$ has the unnormalized spatial image in K, since $(S^0)^2-\mathbf{S}^2=-1$. Nevertheless, $\mathbf{S}$ properly, identifies the polarization (quantization axis) direction as seen by another (e.g., K) inertial observer.

With a fixed Λ^{-1} in hand, we have given a stationary group of the four-velocity vector as a subgroup of Lorentz transformations, which leave this vector in place (i.e., do not take this vector away from a given inertial frame). In K' this stability group coincides with the group of rotations while its isomorphic image in K is given[4] by

$$\mathcal{G}=\mathcal{G}(\mathbf{v})=\Lambda^{-1}\mathcal{G}'\Lambda. \tag{1.5}$$

Unfortunately the action of $\mathcal{G}$ in K is a nontrivial transformation of four-vectors, which modifies the length of their spatial component (unlike $\mathcal{G}'$ in K'). Let us consider the spatial part of the transformation (1.4) as a mapping in R^3. The vectors $\mathbf{S},\mathbf{s},\mathbf{v}$, are coplanar in R^3 and the map is realized merely by the alteration of the longitudinal (along $\mathbf{v}$) component of $\mathbf{s}$,

$$\mathbf{s}\rightarrow\mathbf{S}\Rightarrow\mathbf{s}_{\|}=\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{s})/\beta^2,$$

$$\mathbf{S}_{\|}=\mathbf{s}_{\|}+\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{s})\gamma^2/(\gamma+1). \tag{1.6}$$

We are interested in the relative orientation of the polarization axes along $\mathbf{s}$ and $\mathbf{S}$, respectively. This purely rotational output of the Lorentz transformation Λ^{-1} becomes isolated through.

$$\cos\theta=\mathbf{s}\cdot\hat{\mathbf{S}},\quad \hat{\mathbf{S}}=\mathbf{s}/|\mathbf{s}|,\quad \mathbf{s}\mathbf{S}=1+\frac{\gamma^2}{1+\gamma}(\boldsymbol{\beta}\mathbf{s})^2,$$

$$|\mathbf{S}|=\left\{1+(\boldsymbol{\beta}\mathbf{s})^2\frac{\gamma^2}{1+\gamma}\left[\beta^2\frac{\gamma^2}{1+\gamma}+2\right]\right\}^{1/2}, \tag{1.7}$$

and θ is uniquely defined, given $\mathbf{s}$ and $\mathbf{v}$.

Once in R^3, passing from $\mathbf{s}$ to $\mathbf{S}$ amounts to a rotation by an angle θ about the axis $\mathbf{s}\times\boldsymbol{\beta}$ (the same as about $\mathbf{S}\times\boldsymbol{\beta}$),

$$g_\theta:\mathbf{s}\rightarrow\mathbf{S}=g_\theta\mathbf{s}. \tag{1.8}$$

From now on, the directions $\mathbf{s}$ and $\mathbf{S}$ will be the fixed z-axis directions in the system frames located in K' and K, respectively.

Consider a unit vector $\mathbf{n}$, initially along the z axis of the system frame in K'. Let us execute a rotation $\mathbf{n}\rightarrow g\mathbf{n}$. In view of (2.4), we have here

$$g\mathbf{n}\rightarrow\mathbf{N}_g=g\left[\mathbf{n}+\frac{\gamma^2}{1+\gamma}(\boldsymbol{\beta}'\mathbf{n})\boldsymbol{\beta}'\right]=g\mathbf{N}_{\theta'}, \tag{1.9}$$

where $\boldsymbol{\beta}'=g^{-1}\boldsymbol{\beta}$ induces a rotation of angle v' about the axis $\mathbf{n}\times\boldsymbol{\beta}'$. Accordingly, a unit vector $\hat{\mathbf{N}}_g$ is recovered,

$$\hat{\mathbf{N}}_g=g\hat{\mathbf{N}}_{\theta'}=gg_{\theta'}\mathbf{n}=gg_{\theta'}g_\theta^{-1}\hat{\mathbf{N}}, \tag{1.10}$$

where $\hat{\mathbf{N}}$ was initially along the z axis of the system frame located in K (i.e., parallel to $\mathbf{S}$). If we consider $\hat{\mathbf{N}}_{g1}$ and $\hat{\mathbf{N}}_{g2}$, then

$$\hat{\mathbf{N}}_{g_1}=g_1g_{1\theta'}g_\theta^{-1}\hat{\mathbf{N}}=(g_1g_{1\theta'})(g_2g_{2\theta'})^{-1}\hat{\mathbf{N}}_{g_2}. \tag{1.11}$$

In the above, g_θ was introduced as the rotation by θ about the spatial axis. Each element of the rotation group can be represented that way. Let g_i refer to the rotation axis $\hat{\mathbf{e}}_i$ and angle α_i. By passing to $\boldsymbol{\mu}_i=\hat{\mathbf{e}}_i\tan\alpha_i$, we arrive at a particularly convenient representation[7] of spatial rotations by 3×3 matrices $R(\boldsymbol{\mu})$:

$$R(\boldsymbol{\mu})_{ij}=[1/(1+\mu^2)][(1-\mu^2)\delta_{ij}+2\mu_i\mu_j-2\epsilon_{ijk}\mu_k], \tag{1.12}$$

with the composition rule

$$R(\boldsymbol{\mu}')R(\boldsymbol{\mu})=R(\boldsymbol{\mu}''),$$

$$\boldsymbol{\mu}''=(\boldsymbol{\mu}'+\boldsymbol{\mu}+\boldsymbol{\mu}'\times\boldsymbol{\mu})/(1-\boldsymbol{\mu}'\boldsymbol{\mu}), \tag{1.13}$$

allowing us to attribute to each rotation g in K' a respective spatial rotation $\tilde{g}$ in K,

$$g\mathbf{s}\leftrightarrow\hat{\mathbf{S}}_g=(gg_{\theta'})g_\theta^{-1}\hat{\mathbf{S}}\doteq\tilde{g}g_\theta^{-1}\hat{\mathbf{S}}. \tag{1.14}$$

Notice that, together with $\mathbf{S}$, we have automatically defined an orthogonal reference triad in K. The right screw convention for the vector product allows us to introduce the analogs of the x and y axes as $\mathbf{S}\times\mathbf{v}$ and $(\mathbf{S}\times\mathbf{v})\times\mathbf{S}$, respectively.

All rotations can be parametrized by Euler angles introduced in this frame. Respective parametrizations are not the same for $\{\tilde{g}g_{\theta}^{-1}\}$ and $\{\tilde{g}\}$. However, the very concept of the invariant integration on S_3 implies that a given g_0 displacement on a group $g\rightarrow gg_0$ does not affect the integration formulas, hence the respective Euler parametrization. We have[4] $dg=d(gg_0)$ and $\int f(g)dg=\int f(gg_0)dg$. Consequently, the effective image of the group of rotations in K' under the Lorentz transformation Λ^{-1} is

$$\mathcal{G}'\ni g\rightarrow\tilde{\mathcal{G}}\ni\tilde{g}=g\cdot g_{\theta'}, \tag{1.15}$$

which well fits with (1.11):

$$g_1\mathbf{n}=g_1g_2^{-1}(g_2\mathbf{n})\rightarrow\hat{\mathbf{N}}_{g_1}=\tilde{g}_1\tilde{g}_2^{-1}\hat{\mathbf{N}}_{g_2}. \tag{1.16}$$

If now the g's represent random rotations about the $\mathbf{s}$ polarized frame in K', then $\tilde{g}$'s are their images as random rotations about the $\mathbf{S}$ polarized frame in K.

Irrespective of whether we refer to K' or K, the previous discussion and arguments of Ref. 3, Sec. IV tell us that once we have fixed the polarization axis direction (i.e., the z axis of the orthonormal triad), then the induced Euler angle parametrization of S_3 allows us to determine the spin-$\frac{1}{2}$ SU(2) harmonics as eigenfunctions of the Laplace–Beltrami operator on S_3,

$$\mathbf{s}\rightarrow g=(\theta,\phi,\psi)\rightarrow\Delta_g f'(g)=\tfrac{3}{4}f'(g),$$

$$\mathbf{S}\rightarrow\tilde{g}=(\tilde{\theta},\tilde{\phi},\tilde{\psi})\rightarrow\Delta_{\tilde{g}}f(\tilde{g})=\tfrac{3}{4}f(\tilde{g}),$$

$$\Delta_{\tilde{g}}=\Delta_{(g\rightarrow\tilde{g})}, \tag{1.17}$$

$$\Delta_g=\frac{\partial^2}{\partial\theta^2}+\cos\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\psi^2}+\frac{\partial^2}{\partial\phi^2}\right)-2\frac{\cot\theta}{\sin\theta}\frac{\partial^2}{\partial\phi\,\partial\psi},$$

so that solutions acquire the same functional form, albeit with respect to entirely different parametrizations, thereby showing up different polarization directions in R^3.

The change of the local S_3 parametrization from (θ,ϕ,ψ) to $(\tilde{\theta},\tilde{\phi},\tilde{\psi})$ implies a replacement of the angular momentum vector $\mathbf{L}(g)$ by $\tilde{\mathbf{L}}(\tilde{g})$,

$$\mathbf{L}(g)=\alpha(g)\mathbf{e}_{\phi}+\beta(g)\mathbf{e}_{\psi}+\gamma(g)\mathbf{e}_{\theta},$$
$$\tilde{\mathbf{L}}(\tilde{g})=\alpha(\tilde{g})\mathbf{e}_{\tilde{\phi}}+\beta(\tilde{g})\mathbf{e}_{\tilde{\psi}}+\gamma(\tilde{g})\mathbf{e}_{\tilde{\theta}}, \tag{1.18}$$

where the vectors $\mathbf{e}$ indicate directions about which rotations by the respective angles are executed. Since the e' s are defined in the system frame (with the z axis given either by $\mathbf{s}$ or $\mathbf{S}$), the transformation from $\mathbf{e}_{\theta}$, $\mathbf{e}_{\phi}$, $\mathbf{e}_{\psi}$, to $\mathbf{e}_{\tilde{\theta}}$, $\mathbf{e}_{\tilde{\phi}}$, $\mathbf{e}_{\tilde{\psi}}$ is effected by the previously considered spatial rotation g_{θ}, taking $\mathbf{s}$ into $\mathbf{S}$. Accordingly,

$$\tilde{\mathbf{L}}(\tilde{g})=(g_{\theta}\mathbf{L})(g\rightarrow\tilde{g}), \tag{1.19}$$

i.e., the change of arguments is accompanied by the overall rotation of $\mathbf{L}$. We then have

$$\langle\tilde{\mathbf{L}}\rangle_i=\int\tilde{\mathbf{L}}(\tilde{g})\,|e_i(\tilde{g})|^2\,d\tilde{g}=\epsilon_i\frac{\hbar}{2}\hat{\mathbf{S}}=\epsilon_i\frac{\hbar}{2}g_{\theta}\mathbf{s}$$
$$=g_{\theta}\int\mathbf{L}(\tilde{g})\,|e_i(\tilde{g})|^2\,d\tilde{g}=g_{\theta}\int\mathbf{L}(g)\,|e_i(g)|^2\,dg. \tag{1.20}$$

Consequently, four stationary states of rotational diffusion, $\mathbf{e}_i(g)$, $g=(\theta,\phi,\psi)$ in K', can be mapped into four states of rotational diffusion again, while their polarization $\mathbf{s}$ is taken over to $\mathbf{S}$.

This map we shall study in more detail in connection with solutions of the Dirac equation.

II. ROTATIONAL DIFFUSIONS AS SEEN BY A RELATIVISTIC OBSERVER: CASE OF UNIFORM MOTION

The description of a stochastic process is usually confined to a fixed time interval, which eventually might be extended to an arbitrary size. Let us choose $[0,T']\ni t'$. A random variable $G(t')$ is represented by a rotational event g taking place at time t', while $\mathbf{x}'$ is the location of the origin of the rotating triad. Hence we deal with g at the space-time point $(ct',\mathbf{x}')$. By virtue of our previous considerations, $G(t')$ induces a random variable $\tilde{G}(t')$ in K that refers to a rotation $\tilde{g}$ taking place at the space-time point $(ct,\mathbf{x})$ in K,

$$\mathbf{x}=\mathbf{x}'+\boldsymbol{v}[(\gamma-1)(\boldsymbol{v}\mathbf{x})/v^2+\gamma t'],$$
$$t=\gamma\left(t'+\frac{\boldsymbol{v}\mathbf{x}'}{v^2}\right). \tag{2.1}$$

Since $\mathbf{x}'$ is fixed and the time label t' is only allowed to vary, we can rewrite (1.1) as $\mathbf{x}=\mathbf{x}'_0+\gamma\boldsymbol{v}t'$, $t=\gamma(t'_0+t')$. It tells us that the process is perceived in K as taking place in the time interval $[\gamma t'_0,\gamma(t'_0+T')]$ while the rotating triad origin is propagated uniformly with velocity $\boldsymbol{v}$ from the spatial location $\mathbf{x}=\mathbf{x}'_0$ to $\mathbf{x}_T=\mathbf{x}'_0+\gamma\boldsymbol{v}T$. Here $\mathbf{x}'_0$ is associated with the origin of the rotating frame; hence it is a valid assumption to consider $\mathbf{x}'_0=0$ only. This yields

$$\mathbf{x}=\gamma\boldsymbol{v}t', \quad t=\gamma t', \quad t\in[0,\gamma T']. \tag{2.2}$$

If $\boldsymbol{v}$ is the speed of the particle relative to the reference frame in which the time interval is measured, the notion of proper time comes from $\Delta\tau=\Delta t/\gamma$.

Apparently it gives rise to a proper time labeling of random variables,

$$K'\to K \Rightarrow G(t')\to\widetilde{G}(t')=\widetilde{G}(t/\gamma)=\widetilde{G}(\tau), \tag{2.3}$$

under the assumption (2.2).

Consider a small surface on S_3 with the area Δg centered about the point (rotation) g. Let $e_i(g)$ be the ith state of rotational diffusion. Then $|e_i(g)|^2\,\Delta g$ represents the probability with which rotations close to g are met along sample paths in the infinite sampling limit: then the frequency of an event approaches a probability of its occurrence.

Remark 1: The notion of randomness automatically induces the notion of sampling: a repeatable processing confined to a fixed time interval. In particular, the notion of sample paths[8] of a given stochastic process is of profound importance. In K' it amounts to representing the random propagation on S_3 by a collection of random trajectories: they are different realizations (samples) of a given random motion scenario in the time interval $[0,T']$, executed by the random variable $G(t')$.

The same process, but seen from another inertial frame K, induces sample paths as rotational events, which are continuously distributed along the relativistic path. The K' eigenvalue problem (1.17) refers to stationary solutions of the Schrödinger equation on S_3,

$$\begin{aligned} &i\hbar\partial_{t'}f'(g,t)=(-\hbar^2/2I)\Delta_g f'(g,t'),\\ &f'(g,t')=f'(g)\exp(-mc^2t'/\hbar), \end{aligned} \tag{2.4}$$

where we set $I=3\hbar^2/8mc^2$ to deal with spin-$\frac{1}{2}$. The respective eigenvalue problem in K is given by the Schrödinger equation on S_3, but with a proper time τ instead of t',

$$\begin{aligned} &i\hbar\partial_\tau f(\widetilde{g},\tau)=(-\hbar^2/2I)\Delta_{\widetilde{g}}f(\widetilde{g},\tau),\\ &f(\widetilde{g},\tau)=f(\widetilde{g})\exp(-mc^2\tau/\hbar). \end{aligned} \tag{2.5}$$

Then (2.4) is related to the stochastic process with a random variable $G(t')$ while (2.5) induces $\widetilde{G}(\tau)$.

On the other hand, the manifestly covariant form $p_\mu x^\mu$ of mc^2t' emerges by setting $p=\gamma m v$, $p^0=\gamma mc=E/c$, $\boldsymbol{v}=c\mathbf{p}/E$, $\boldsymbol{\beta}=\mathbf{p}/E$, i.e.,

$$mc^2t'=-\mathbf{px}+p^0ct=p_\mu x^\mu, \tag{2.6}$$

with $\mathbf{x}=\mathbf{x}(\mathbf{x}',t')$, $t=t(\mathbf{x}',t')$ given by (1.1). It means, however, that $f'(g,t')$ should take in K the form

$$f(\widetilde{g},t)=f(\widetilde{g})\exp[-i(p^0ct-\mathbf{px})/\hbar], \tag{2.7}$$

which is characteristic of plane wave solutions of the Dirac equation, except for the explicit $\widetilde{g}$ dependence of the coefficient $f(\widetilde{g})$.

At this point it is quite instructive to invoke an exhaustive discussion of Ref. 4 on the determination of the rest mass and spin of the particle in the context of relativistic invariant wave equations.

Usually one deals with arbitrary plane wave solutions and attempts to extract their rest frame properties. We have proceeded in reverse, while having a detailed rest frame picture (of random phenomena) in hand. Let us view (2.7) as an arbitrary plane wave, i.e., allow t and $\mathbf{x}$ to take any value. We can always pass[4] to the rest frame of the wave and recover a corresponding stationary plane wave, which is (2.5) in our case. Indeed, the Lorentz transformation Λ: $(\gamma c,\gamma\boldsymbol{v})\to(c,0,0,0)$ implies

$$\begin{aligned} &f(\widetilde{g})\exp(-ip_\mu x^\mu/\hbar)\to f'(g)\exp(-imc^2t'/\hbar),\\ &mc^2t'=p_0'x_0', \end{aligned} \tag{2.8}$$

for all $\mathbf{x}$ and t. Although the spatial image $\mathbf{x}'$ of $\mathbf{x}$ under Λ is not manifestly present in (2.8), it is implicitly there, since in our framework spatial rotations g take place as rotations about this point.

The transformation Λ affects only the longitudinal component $x_\parallel'$ of $\mathbf{x}'$,

$$\begin{aligned} &x_\parallel=\gamma(x_\parallel'+\boldsymbol{v}t'),\\ &t=\gamma(t'+x_\parallel'/\boldsymbol{v}), \quad x_\parallel=\boldsymbol{v}\mathbf{x}/|\boldsymbol{v}|, \end{aligned} \tag{2.9}$$

while on the other hand, $p_\mu x^\mu=p_0x_0-|\mathbf{p}|\mathrm{x}_\parallel$; hence at each given time instant x_0/c the plane wave effectively describes a transversal plane (wave front) labeled by $x_\parallel$. The formula (2.9) maps the $x_\parallel'$ plane in K' into the $x_\parallel$ plane in K. The space-time location of the plane in K is uniquely defined by a corresponding time instant t' of the rest frame evolution. By (2.9) the one-parameter family of wave fronts at rest (in K') is perceived in K as the one-parameter family of traveling surface (planes): at $t'=0$ we have $x_\parallel^0=\gamma x_\parallel'$, $t_0=\gamma x_\parallel'/v$.

Consider a plane $x_\parallel^0$ at t_0. Each point of this transversal plane is uniquely mapped into the respective point of the image plane $x_\parallel^T$ at time T by following the uniform motion path: paths do not intersect and the moving surface traces their flow in configuration space. In the framework of rotational diffusions, the plane wave thus accounts for all alternative (purely classical) motion scenarios to be followed by the origin of the rotating frame in the sampling series. Consequently, they are much akin to the Hamilton waves of classical mechanics[9] and not at all to what we usually call traveling waves (Ref. 10 addresses the issue in more detail).

III. DIRAC EQUATION IN STOCHASTIC MECHANICS: REVIVAL OF SOME OLD IDEAS

The stochastic implementation[1,2,11,12] of the quantum spin-$\frac{1}{2}$ system involves four distinct states of rotational diffusion, which reflect[13] the existence of left and right representations of the SU(2) on $\mathscr{H}_{1/2}$. In K' we have

$$(8\pi^2)^{1/2}e_1(g)=i\cos\frac{\theta}{2}\exp\frac{i}{2}(\psi+\phi)\sim d^{1/2}_{1/2\,1/2}(g),$$

$$\langle\mathbf{L}\rangle_1=\frac{\hbar}{2}\,\mathbf{k},$$

$$(8\pi^2)^{1/2}e_2(g)=i\sin\frac{\theta}{2}\exp\frac{1}{2}(\phi-\psi)\sim d^{1/2}_{1/2-1/2}(g),$$

$$\langle\mathbf{L}\rangle_2=\frac{\hbar}{2}\,\mathbf{k},$$

$$(8\pi^2)^{1/2}e_3(g)=i\sin\frac{\theta}{2}\exp\frac{i}{2}(-\phi+\psi)\sim d^{1/2}_{-1/2\,1/2}(g), \tag{3.1}$$

$$\langle\mathbf{L}\rangle_3=-\frac{\hbar}{2}\,\mathbf{k},$$

$$(8\pi^2)^{1/2}e_3(g)=i\cos\frac{\theta}{2}\exp\frac{i}{2}(-\phi-\psi)\sim d^{1/2}_{-1/2-1/2}(g),$$

$$\langle\mathbf{L}\rangle_4=-\frac{\hbar}{2}\,\mathbf{k},$$

where d^s_{mn} is the standard notation for SU(2) harmonics. The respective stochastic processes are determined by computing the angular velocity $\boldsymbol{\omega}(g)$ induced by the rotation g and $\boldsymbol{\omega}(g)$ is a sum of the current $\boldsymbol{\omega}_v$ and osmotic $\boldsymbol{\omega}_u$ contributions behaving differently under time reversal. Namely $t'\to -t'$ implies $\boldsymbol{\omega}_u\to\boldsymbol{\omega}_u$ while $\boldsymbol{\omega}_v\to -\boldsymbol{\omega}_v$.

As a consequence (compare, e.g., Sec. IV of Ref. 2), we arrive at

$$\begin{aligned}&\boldsymbol{\omega}^1_v\to\boldsymbol{\omega}^4_v,\quad \boldsymbol{\omega}^2_v\to\boldsymbol{\omega}^3_v,\\ &\boldsymbol{\omega}^1_u\to\boldsymbol{\omega}^4_u,\quad \boldsymbol{\omega}^2_u\to\boldsymbol{\omega}^3_u,\end{aligned} \tag{3.2}$$

which amounts to the map

$$\begin{aligned}&e_1(g)\to e_4(g),\quad e_2(g)\to e_3(g),\\ &f'(g,t')\to f'(g,-t')=e(g)\exp(mc^2t'/\hbar).\end{aligned} \tag{3.3}$$

Remark 2: Four states of rotational diffusion (3.1) were introduced in connection with the forward propagation. Apparently, the discussion[14] of how to describe effects of time reversal in stochastic mechanics as a forward propagation again may be adopted here. Usually the reverse process is viewed as the random propagation in the backward direction, which allows one to reproduce past (statistical) data of the process given the present, hence as a mere mathematical artifice. It appears that in the case of spin-$\frac{1}{2}$ diffusions it is no longer so. The arguments of Ref. 14, the Introduction, tell us that for Markovian diffusions we can define a forward process that is the exact time reversal of another forward process, and the diffusions underlying (3.1) provide us with explicit examples.

Let us recall[1-3] that the SU(2) labeling of eigenfunctions (3.1) is provided by the eigenvalues of the operators M_3, N_3, where $\mathbf{M}$ is the generator of left rotations while $\mathbf{N}$ is the (abnormal) generator of right rotations. We have[4] $\mathbf{M}^2=\mathbf{N}^2=-\hbar^2\Delta_g$ on the S_3 manifold, and $M_3=-i\hbar\partial/\partial\phi$, $N_3=-i\hbar\partial/\partial\psi$. The eigenvalues of M_3 correspond to expectation values $\langle\mathbf{L}\rangle_i$ of the angular momentum (spin) arising due to the rotational diffusion.

The ordering (e_1,e_3,e_2,e_4) of the basis system refers to a $(+,-,+,-)$ sequence of the M_3 eigenvalues and to $(+,+,-,-)$ for N_3. Analogously, (e_2,e_4,e_1,e_3) refers to $(+,-,+,-)$ for M_3 and $(-,-,+,+)$ for N_3. In view of this, formulas (2.8) and (3.3) give rise to two distinct evolution equations in K' that encompass the time reversal in a manifest way. Namely, $e_j(g)\exp(-imc^2t'/\hbar)$, $j=1,3$ and $e_k(g)\exp(imc^2t'/\hbar)$, $k=2,4$ form a set of independent solutions of the equation

$$i\hbar\partial_{t'}f'(g,t')=(2/\hbar)mc^2N'_3f'(g,t'), \tag{3.4}$$

while $e_k(g)\exp(-imc^2t'/\hbar)$, $k=2,4$ and $e_j(g)\times\exp(imc^2t'/\hbar)$, $j=1,3$ for

$$i\hbar\partial_{t'}f'(g,t')=-(2/\hbar)mc^2N'_3f'(g,t'). \tag{3.5}$$

The "positive energy" solutions of (3.4) and (3.5) constitute the orthonormal set in $\mathscr{H}_{1/2}$. The prime refers to the rest frame Euler parametrization.

Remark 3: The above observation, if combined with the previous Remark 2, lends weight to Barut's conjecture[15] that perhaps there is no real need to invoke the hole theory or the notion of backward propagation in time to describe antiparticles.

Let us address the question of how the rest frame evolution (respectively, the eigenvalue problem for N_3) equations (3.4) and (3.5) are seen in another Lorentz frame.

In accordance with the standard rules of the game[16] the Lorentz transformation Λ: $K\to K'$ should imply a *nonunitary* map in the function space, replacing the K frame data by the K' ones,

$$f'(x',g)=(T_\Lambda f)(x,\tilde{g}). \tag{3.6}$$

We shall investigate the outcome of (3.6) in the Hilbert space spanned by $e_i(\tilde{g})$ with the $L^2(S_3)$ scalar product $\int f_1(\tilde{g})f_2(\tilde{g})d(\tilde{g})=(f_1,f_2)$ valid in K. Let us consider a transformation,

$$\Lambda:K\to K'\Rightarrow f(\tilde{g})=(T_\Lambda f)(\tilde{g}), \tag{3.7}$$

where $T_\Lambda=T_\Lambda(\mathbf{M},\mathbf{N})$ is given by (this formula was first introduced to represent Lorentz transformations in Ref. 13)

$$T_\Lambda=\cosh\frac{\eta}{2}\left[c+\frac{1}{E+mc^2}\left(\frac{2}{\hbar}\right)^2 N_1(\mathbf{pM})\right], \tag{3.8}$$

with

$$\cos\frac{\eta}{2}=\left(\frac{E+mc^2}{2mc^2}\right)^{1/2},\quad p^0=\frac{E}{c}=(\vec{p}^2+m^2c^2)^{1/2}. \tag{3.9}$$

In the above, $\mathbf{N}$ and $\mathbf{M}$ are the previously defined differential operators on S_3 whose explicit form displays the local $(\tilde{\theta},\tilde{\phi},\tilde{\psi})$ parametrization.

By exploiting the formulas[4] valid for an irreducible representation of the group of rotations in $L^2(S_3)$,

$$\begin{aligned}(M_1\pm iM_2)d^s_{mn}&=\hbar[(s\mp m)(s\pm m+1)]^{1/2}d^s_{m\pm 1n},\\ (N_1\mp iN_2)d^s_{mn}&=\hbar[(s\mp m)(s\pm m+1)]^{1/2}d^s_{mn\pm 1},\end{aligned} \tag{3.10}$$

and specializing them to spin-$\frac{1}{2}$, we obtain

$$\begin{aligned}&N_1d^{1/2}_{1/2n}=(\hbar/2)d^{1/2}_{-1/2n},\quad M_2d^{1/2}_{1/2n}=-i(\hbar/2)d^{1/2}_{-1/2n},\\ &N_1d^{1/2}_{m1/2}=(\hbar/2)d^{1/2}_{m-1/2},\quad N_2d^{1/2}_{m1/2}=-i(\hbar/2)d^{1/2}_{m-1/2}.\end{aligned} \tag{3.11}$$

This entails an immediate evaluation of the action of T_Λ on any of the e_i's. Let us introduce the notation

$$(e_1,e_3,e_2,e_4)=(\phi_1,\phi_2,\phi_3,\phi_4). \tag{3.12}$$

Then we arrive at

$$(T_\Lambda\phi_i)(\tilde{g})=\sum_{k=1}^4(T_\Lambda)_{ik}\phi_k=\sum_{k=1}^4 S^T_{ik}\phi_k, \tag{3.13}$$

where one recognizes S^T to be a transposed bispinor transformation matrix [see (3.7) in Ref. 16],

$$\begin{aligned}&S=\exp\left(-\frac{\eta}{2}\frac{\alpha v}{|v|}\right)\quad \gamma^i=\gamma^0\alpha_i\\ &\psi'(x')=\psi'(\Lambda x)=S(\Lambda)\psi(x)=S(\Lambda)\psi(\Lambda^{-1}x').\end{aligned} \tag{3.14}$$

Accordingly, the 4×4 matrix S comes out by evaluating matrix elements of the operator T_Λ in the $\{\phi_i(\tilde{g})\}$ rotational basis.

Remark 4: Let us emphasize that our analysis is carried out in a four-dimensional vector space, which is a natural module for the compact group $SU(2)_L\times SU(2)_R$. In order to establish a relativistic description (rather to exploit what is known[16] about the relativistic covariance of the Dirac equation) the *same* space is required to act as a module for the noncompact group $SL(2,C)$. The latter action does not seem to arise that naturally, except for rather conspicous affinity (dimension four) with the standard bispinor transformations induced by the Lorentz mapping. In fact, T_Λ (3.8), in view of its irreducible action on the four-dimensional carrier space, is equivalent (matrix form!) to the well-known mappings: (3.13) and (3.14) should be compared with the formula (3.7) of Ref. 16. Our procedure should not be confused with the general SO(4) complexification problem. In fact, this point makes the original Dahl's proposal[13] indigestible: the $SL(2,C)$ covariance cannot be naively replaced by the SO(4) covariance. Although the SO(4) covariant $(SU(2)\times SU(2)/Z_2)$ spin-$\frac{1}{2}$ system is our starting point, we pass to a new $SL(2,C)$ covariant spin-$\frac{1}{2}$ system built on the carrier (representation) space of the former. This task is accomplished by means of the *nonunitary* representation of $SL(2,C)$ for which finite-dimensional realizations are known to exist. The role of $\mathbf{M},\mathbf{N}$ generators is different in the $SL(2,C)$ case if compared with SO(4). One of them, instead of generating rotations, gives rise to Lorentz boosts. This feature is completely revealed by formulas (3.8)–(3.14).

Let $w^r=w^r(\mathbf{p})$ be the rth column of the matrix S. We can rewrite (3.13) as follows:

$$(T_\Lambda\phi_r)(\tilde{g})=\sum_{k=1}^4 w^r_k(\mathbf{p})\phi_k(\tilde{g})=\phi'_r(\mathbf{p},\tilde{g}). \tag{3.15}$$

The $L^2(S_3)$ orthonormality relations imply here (we use the bispinor normalization identity in the second step)

$$(\phi'_r(\epsilon_r\mathbf{p}),\phi'_{r'}(\epsilon_{r'}\mathbf{p}))=w^{r'*}(\epsilon_{r'}\mathbf{p})w^r(\epsilon_r\mathbf{p})=\frac{E}{mc^2}\delta_{rr'}, \tag{3.16}$$

$$\epsilon_r=+1,\quad r=1,2,\quad \epsilon_r=-1,\quad r=3,4,$$

and allow us to introduce a new orthonormal basis system in $\mathscr{H}_{1/2}$ encompassing the effects of the Lorentz transformation $K\to K'$,

$$e_1'(\tilde{g})=\phi_1'(\mathbf{p},\tilde{g})\left(\frac{mc^2}{E}\right)^{1/2},\quad e_3'(\tilde{g})=\phi_2'(\mathbf{p},\tilde{g})\left(\frac{mc^2}{E}\right)^{1/2},$$

$$e_2'(\tilde{g})=\phi_3'(-\mathbf{p},\tilde{g})\left(\frac{mc^2}{E}\right)^{1/2}, \tag{3.17}$$

$$e_4'(\tilde{g})=\phi_4'(-\mathbf{p},\tilde{g})\left(\frac{mc^2}{E}\right)^{1/2},$$

where $(e_i',e_j') = \int d\tilde{g}\, e_i'(\tilde{g})e_j'(\tilde{g}) = \delta_{ij}$ is a positive definite sesquilinear form.

More explicitly,

$$\begin{aligned}
e_1'(\tilde{g})&=\left(\frac{E+mc^2}{2E}\right)^{1/2}\left\{e_1(\tilde{g})+\frac{c}{E+mc^2}\times[p_z e_2(\tilde{g})+p_+e_4(\tilde{g})]\right\},\\
e_3'(\tilde{g})&=\left(\frac{E+mc^2}{2E}\right)^{1/2}\left\{e_3(\tilde{g})+\frac{c}{E+mc^2}\times[p_- e_2(\tilde{g})-p_z e_4(\tilde{g})]\right\},\\
e_2'(\tilde{g})&=\left(\frac{E+mc^2}{2E}\right)^{1/2}\left\{e_2(\tilde{g})+\frac{c}{E+mc^2}\times[-p_z e_1(\tilde{g})-p_+e_3(\tilde{g})]\right\},\\
e_4'(\tilde{g})&=\left(\frac{E+mc^2}{2E}\right)^{1/2}\left\{e_4(\tilde{g})+\frac{c}{E+mc^2}\times[-p_- e_1(\tilde{g})+p_z e_3(\tilde{g})]\right\},\\
\mathbf{p}&=(p_x,p_y,p_z),\quad p_\pm=p_x\pm ip_y,
\end{aligned} \tag{3.18}$$

where $e_i(\tilde{g})$ have the form (3.1), except for the replacement of θ,ϕ,ψ by $\tilde{\theta},\tilde{\phi},\tilde{\psi}$.

It was demonstrated for the first time in Ref. 13 that the functions

$$f_j(\tilde{g},x)=(T_\Lambda\phi_j)\exp(-\epsilon_j ip_\mu x^\mu/\hbar), \tag{3.19}$$

solve the evolution equation

$$i\hbar\,\partial_t f_j(\tilde{g},x)=\left\{\frac{2}{\hbar}mc^2N_3+\frac{4c}{\hbar^2}N_1[\mathbf{M}(-i\hbar\nabla)]\right\}f_j(\tilde{g},x). \tag{3.20}$$

Its matrix form in the $\{\phi_j(\tilde{g})\}$ basis is the familiar Dirac equation

$$[mc^2\beta+c\alpha(-i\hbar\nabla)]\psi=i\hbar\,\partial_t\psi. \tag{3.21}$$

The image of (3.5) under Λ is obtained through replacing m by $-m$ in the above.

The equation (3.21) is known to be Lorentz invariant. Then what about (3.20)? By setting

$$L^0=I,\quad L^i=-N_1M_i,\quad i=1,2,3, \tag{3.22}$$

(3.20) can be cast in the manifestly covariant form

$$\frac{2}{\hbar}mc^2N_3f_j=i\hbar L^\mu\,\partial_\mu f_j. \tag{3.23}$$

The standard[16] Lorentz covariance arguments require that (3.6) be a map of the K frame data into the K' frame ones. Accordingly,

$$N_3=T_\Lambda N_3'T_\Lambda^{-1}, \tag{3.24}$$

reflects merely the change of the Euler parameters from $\tilde{\theta},\tilde{\phi},\tilde{\psi}$ to θ,ϕ,ψ as a result of the Lorentz transformation, while there holds

$$T_\Lambda L^\mu T_\Lambda^{-1}a_\mu^\nu\,\partial_\nu' f_i'(g,x')=L'^\nu\,\partial_\nu' f_j'(g,x'),$$

$$a_\mu^\nu=\frac{\partial x'^\nu}{\partial x^\mu}. \tag{3.25}$$

The K' frame version of (3.20),

$$\frac{2}{\hbar}mc^2N_3'f_j'(g,x')=i\hbar L'^\nu\,\partial_\nu' f_j'(g,x'), \tag{3.26}$$

reduces to (3.4) in the case of plane wave solutions.

As a consequence of (3.6), we realize that stationary plane wave solutions $\theta_j(g)\exp(-mc^2t'/\hbar)$ of (3.4) are represented in terms of the K frame data by the solutions (3.17) of the evolution equation (3.20). A serious problem comes here from the covariant normalization statement[16]

$$\bar{w}^r(\mathbf{p})w^{r'}(\mathbf{p})=\delta_{rr'}\epsilon_r. \tag{3.27}$$

Before, the plane waves were found to refer to four distinct stochastic rotational processes in K'. Because of the improper (negative) normalization, the $r=3$ and 4 images of random motions in K' do not admit any reasonable probabilistic meaning in K, hence they cannot be perceived as stochastic processes in K. It is the normalization identity (3.15) that allows us to introduce an orthonormal basis system (3.16) with prospects for a correct probabilistic content (due to a positive normalization). Apparently (3.16) arises only if we consider a complete set of "positive energy" solutions of both (3.4) and (3.5). *Both* these evolution equations *are in-*

dispensable for a covariant transformation of the orthonormal basis given in K' into an orthonormal basis in K. In fact, the formulas (3.16) identify these functions in $\mathscr{H}_{1/2}$, which provide us with the K frame image (via Lorentz transformation) of four distinct stochastic processes in K'. This map allows us to perceive certain K' frame diffusion as genuine diffusion processes in the inertial frame K.

Since all $e_i(\widetilde{g})$ solve the eigenvalue problem $\Delta_{\widetilde{g}} f(\widetilde{g}) = \frac{3}{4} f(\widetilde{g})$ the basis functions (3.16) solve it as well. A complete stochastic decoding of (3.16) amounts to a repetition of Dankel's strategy[1] once the e's are cast in the canonical (Madelung) form $e' = \exp(R+iS)$. Now $|e'_i(\widetilde{g})|^2$ represents the probability distribution of the ith stationary diffusion as perceived in K. The respective random variable is labeled by the proper time.

We may now formulate a definite answer to the question raised in the Introduction. What is perceived in K as a stochastic rotational diffusion is no longer a diffusion associated with the forward time development exclusively, i.e., $e_i(g)\exp(-imc^2t'/\hbar)$ for *all* $i=1,2,3,4$. *The answer is positive* if we go over to the rest frame diffusions associated with the evolutions $e_j(g)\exp(-imc^2t'/\hbar)$ for $j=1,3$ and $e_k(g)\exp(imc^2t'/\hbar)$ for $k=2,4$. The backward evolution for $k=2,4$ can be represented as a forward evolution again by invoking the arguments of Ref. 14, but is irreducibly different from the one associated with $e_k(g)\exp(-imc^2t'/\hbar)$.

Our analysis allows one to associate diffusions on S_3 with plane wave solutions of the Dirac equation, which is possible due to the implicit validity of the proper time Schrödinger equation on S_3. We then deal with rotational fluctuations that are intrinsic to a particle in uniform motion. There is no essential difficulty in extending the arguments to cases covered by the semiclassical regime for solutions of the Dirac equation in the presence of external electromagnetic fields (inhomogeneities included). The proper time evolution governed by the Bargmann–Michel–Telegdi equation[5] amounts to a purely rotational diffusion process, which is effected along a space-time trajectory of the particle. The motions show the same feature: randomness is exclusively intrinsic and does not affect the space-time path followed by the origin of the rotating frame (on the contrary, it is rather that the spin precession is strongly path dependent).

The problem we have left aside at the moment is the probabilistic analysis of general wave packet solutions of the Dirac equation, where a nontrivial input of the random process affecting a particle velocity (extrinsic randomness) is expected to show up.

Since random paths of stochastic mechanics are quite akin to Feynman paths,[14,17] it should, in principle, be possible to establish a unifying framework for an increasing number of path integral approaches to the description of Dirac particles in the *non-Grassmann vein*.[12,18,19] It especially pertains to random walk representations of the Dirac propagator,[20–22] where one generally assumes that at each step of the random walk executed by the spinning particle in Minkowski space, its quantization axis is rotated by a certain angle. Compare, e.g., our discussion of the Introduction, where momentum change induces a well-defined rotation of the polarization.

References to numerous relativization attempts in the context of Nelson's stochastic mechanics can be found in the recent papers;[23,24] also see Refs. 25 and 26. A problem worth a deeper exploration in the presented probabilistic framework is the physical meaning of different notions of position invented for the Dirac particle, and also of Zitterbewegung, which from our perspective is definitely *not* the intrinsic mechanism[27,28] implementing the electron spin. On the other hand, the recent magnetic top model,[29] albeit devoid of any explicit randomness, shows up all basic features discussed in the present paper.

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[1] T. Dankel, Jr., Arch. Rat. Mech. Anal. **37**, 192 (1970).
[2] P. Garbaczewski, Fortschr. Phys. **38**, 447 (1990), see also Ref. 35.
[3] F. Guerra and R. Marra, Phys. Rev. Lett. **50**, 1715 (1983).
[4] I. M. Gelfand, R. A. Milnos, and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and Their Applications* (Pergamon, New York, 1963).
[5] V. Bargmann, L. Michel, and V. L. Telegdi, Phys. Rev. Lett. **2**, 435 (1959).
[6] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).
[7] M. Rivas, J. Math. Phys. **30**, 318 (1989).
[8] K. Ito and H. P. McKean, Jr., *Diffusion Processes and Their Sample Paths* (Academic Press, New York, 1965).
[9] L. Brillouin, *Tensors in Mechanics and Elasticity* (Academic Press, New York, 1965).
[10] B. Diu, Eur. J. Phys. **1**, 231 (1980).
[11] L. de la Peña, J. Math. Phys. **12**, 453 (1971).
[12] L. Schulman, Phys. Rev. **176**, 1558 (1968).
[13] J. P. Dahl, Kon. Dansk. Vid. Selsk. Mat Fys. **39**, 12 (1977).

[14] P. Garbaczewski, Phys. Lett. A **143**, 85 (1990).
[15] A. O. Barut, Phys. Rev. Lett. **20**, 893 (1968).
[16] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
[17] P. Garbaczewski, Phys. Lett. A **147**, 168 (1990).
[18] P. Garbaczewski, J. Math. Phys. **23**, 442 (1983).
[19] P. Garbaczewski, *Classical and Quantum Field Theory of Exactly Soluble Nonlinear Systems* (World Scientific, Singapore, 1985).
[20] T. Nakagomi, Prog. Theor. Phys. **81**, 916 (1989), see also Refs. 36–43.
[21] J. Ambjorn, B. Durhuus, and T. Jonsson, Nucl. Phys. B **330**, 509 (1990).
[22] T. Jaroszewicz and P. S. Kurzepa, Ann. Phys. NY, **210**, 255 (1991).
[23] M. Serva, Ann. IHP **49**, 415 (1988).
[24] G. F. De Angelis, J. Math. Phys. **31**, 1408 (1990).
[25] T. Zastawniak, J. Math. Phys. **30**, 1354 (1989).
[26] P. Garbaczewski, Phys. Lett. A **164**, 6 (1992).
[27] A. O. Barut and A. J. Bracken, Phys. Rev. D **23**, 2454 (1981).
[28] A. O. Barut and A. J. Bracken, J. Math. Phys. **26**, 2515 (1985).
[29] A. O. Barut, M. Bozić, and Z. Marić, Ann. Phys. NY **214**, 53 (1992).
[30] J. C. Zambrini, Phys. Rev. A **33**, 1532 (1986).
[31] J. C. Zambrini, J. Math. Phys. **27**, 2307 (1986).
[32] M. Nagasawa, Prob. Theor. Relativ. Fields **82**, 109 (1989).
[33] P. Garbaczewski, Phys. Lett. A **162**, 129 (1992), see also Refs. 44 and 45.
[34] J. C. Zambrini (private communication).
[35] T. C. Wallstrom, Trans. Am. Math. Soc. **318**, 749 (1990).
[36] A. O. Barut and I. H. Duru, Phys. Rep. **172**, 1 (1989).
[37] A. O. Barut and I. H. Duru, Phys. Rev. Lett. **53**, 2355 (1984).
[38] M. Stone, Nucl. Phys. B **314**, 557 (1989).
[39] M. S. Plyushchay, Phys. Lett. B **236**, 291 (1990).
[40] A. Yu. Alekseev and S. L. Shatashvili, Mod. Phys. Lett. A **3**, 1551 (1988).
[41] T. Kashiwa, Int. J. Mod. Phys. A **5**, 375 (1990).
[42] P. Orland, Int. J. Mod. Phys. A **4**, 3615 (1989).
[43] K. Johnson, Ann. Phys. NY **192**, 104 (1989).
[44] P. Garbaczewski and J. P. Vigier, Phys. Lett. A (in press).
[45] P. Garbaczewski and J. P. Vigier, Phys. Rev. A (in press).