Fermion excitations of the nonlinear Schrödinger field in the attractive case

Piotr Garbaczewski
Institute of Theoretical Physics, University of Wroclaw, 50–205 Wroclaw, Poland[a] *and Indian Statistical Institute, Calcutta 700 035, India*

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The recent inverse scattering method analysis by L. Martínez Alonso [J. Math. Phys. **25**, 1935 (1984)] is extended to demonstrate that the Bose quantized (attractive) nonlinear Schrödinger field in 1 + 1 dimensions, admits fermion excitations in its (quantum soliton) spectrum.

I. QUANTUM SOLITON EXCITATIONS

The nonlinear Schrödinger field in 1 + 1 dimensions

$$i\psi_t = -\psi_{xx} + 2c\psi^*\psi\psi \tag{1}$$

is quantized according to Bose statistics

$$[\psi(x), \psi^*(y)]_- = \delta(x-y),$$
$$[\psi(x), \psi(y)]_- = 0 = [\psi^*(x), \psi^*(y)]_-, \tag{2}$$

since the choice of Fermi statistics would cancel the interaction term. Hence, *a priori* there is no room for fermions in this model, except for the specialized $c\to\infty$ regime in the repulsive ($c>0$) case. Then, indeed, the Bose model exhibits a metamorphosis into the free Fermi model, see, e.g., Ref. 1, which is accompanied by the collapse of the (Bose) Fock space $\mathscr{H}_{\rm B}$ into its proper subspace (of Fermi states) $\mathscr{H}_{\rm F} \subset \mathscr{H}_{\rm B}$.

The state space structure in the attractive ($c<0$) case is much more complicated[2] and does not reveal any apparent fermion (Fermi states of Bose systems[3]) content. The inverse scattering method involves here a passage from the Fock representation of the canonical commutation relations $\{\psi, \psi^*, |0)\}$ to a countable family of independent Bose fields $\{\phi_n, \phi_n^*, |0), n\geqslant 1\}$ such that $|0)$ is a common (cyclic vacuum) vector for both ψ, ψ^* and $\{\phi_n, \phi_n^*, n\geqslant 1\}$, while

$$[\phi_n(p), \phi_m^*(q)]_- = \delta_{nm}\,\delta(p-q),$$
$$[\phi_n(p), \phi_m(q)]_- = 0, \tag{3}$$

so that the extended Galilei group generators acquire the following form:[2]

$$M = \frac{1}{2}\int_{-\infty}^{+\infty} dx\, \psi^*\psi = \sum_{n\geqslant 1}\frac{n}{2}\int_{-\infty}^{+\infty} dp\, \phi_n^*(p)\,\phi_n(p),$$
$$H = \int_{-\infty}^{+\infty} dx(\psi_x^*\psi_x + c\psi^{*2}\psi^2)$$
$$= \sum_{n\geqslant 1}\int_{-\infty}^{+\infty} dp\left[\frac{p^2}{n} - \frac{c^2}{12}(n^3-n)\right]\phi_n^*(p)\,\phi_n(p),$$
$$P = \int_{-\infty}^{+\infty} dx\,\psi^*(-i\psi_x) = \sum_{n\geqslant 1}\int_{-\infty}^{+\infty} dp\, p\, \phi_n^*(p)\,\phi_n(p),$$
$$K = -\frac{1}{2}\int_{-\infty}^{+\infty} dx\, x\psi^*\psi$$
$$= -\sum_{n\geqslant 1}\frac{in}{2}\int_{-\infty}^{+\infty} dp\,\phi_n^*(p)\,\frac{\partial}{\partial p}\phi_n(p). \tag{4}$$

The eigenvectors of H due to $[H, N]_- = 0$, $N = \int_{-\infty}^{+\infty} dx\, \psi^*(x)\psi(x)$ in each n-particle sector have a standard (Bethe ansatz) form

$$|f) = \int dx_1\cdots\int dx_n\, \xi(x_1,\ldots,x_n)\, \psi^*(x_1)\cdots\psi^*(x_n)|0).$$

Nevertheless, as follows from (4) instead of the $\{\Pi_{i=1}^n \psi_i^*(x_i)|0)\}$ basis, another one can be used to generate the underlying state space. Namely

$$|p_1 n_1; p_2 n_2, \ldots, p_r\, n_r) = \phi_{n_1}(p_1)\cdots\phi_{n_r}^*(p_r)|0), \quad n_l\geqslant 1, \quad \forall l. \tag{5}$$

Since we have

$$[\phi_n^*(p)\,\phi_n(p), \mathscr{N}_k]_- = 0, \quad \forall\, n, k, \quad \forall\, p,$$
$$\mathscr{N}_k = \int_{-\infty}^{+\infty} dq\, \phi_k^*(q)\,\phi_k(q), \tag{6}$$

each operator $\mathscr{N}_k$ commutes with the generators M, K, P, and H of (4). Hence the single interacting Galilean (Bose) field ψ^*, ψ gives rise to a countable set of independent (free) Galilean bosons ϕ_n^* and ϕ_n with $H_n = \int dp\, \omega_n(p)\,\phi_n^*(p) \times \phi_n(p)$ and $\omega_n(p) = p^2/n - (c^2/12)(n^3-n)$.

II. QUANTUM SOLITONS AS FERMIONS

Despite the fact that in the above we deal with bosons only, the diagonal (with respect to ϕ_n^*, ϕ_n) structure of generators (4) of the extended Galilei group, together with (6), suggests the existence of state space vectors which respect the Pauli principle. After accounting for the analysis of Refs. 1, 3, and 4 it would indicate that the nonlinear Schrödinger field has Fermi states, and consequently gives rise to fermion excitations (paralleling the boson ones).

For this purpose, let us consider the following sequence $\{\Pi_n, n\geqslant 1\}$ of projection operators in the state space of our Bose system (compare, e.g., in this connection the general construction of Ref. 5):

$$\Pi_n = \sum_{s=0}^{\infty}\frac{1}{s!}\sum_{\alpha_1=1}^{n}\int dq_1\cdots\sum_{\alpha_s=1}^{n}\int dq_s\,[\sigma(\alpha_1 q_1;\ldots;\alpha_s q_s)]^2$$
$$\times\phi_{\alpha_1}^*(q_1)\cdots\phi_{\alpha_s}^*(q_s):\exp\left[-\sum_{\beta=1}^{n}\int dp\,\phi_\beta^*(p)\,\phi_\beta(p)\right]:$$
$$\times\phi_{\alpha_1}(q_1)\cdots\phi_{\alpha_s}(q_s), \tag{7}$$

where the (alternating) function $\sigma(\alpha_1 q_1;\ldots;\alpha_s q_s)$ is defined as follows:

[a] Permanent address.

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$$\sigma(\alpha_1 q_1;...;\alpha_s q_s) = \prod_{1<j<k\leq s} p_{jk},$$

$$p_{jk} = \delta_{\alpha_j\alpha_k}[\Theta(q_j - q_k) - \Theta(q_k - q_j)] + (1-\delta_{\alpha_j\alpha_k})(-1)^{1+\Theta(q_j-q_k)}, \tag{8}$$

provided $\Theta(q-p) = 1$, $q \geqslant p$, 0 otherwise.

Since $\sigma^3 = \sigma$, $\sigma = \pm 1$ depending on permutations of pairs (αq) of indices, and if coinciding pairs appear in the sequence, then $\sigma = 0$, and the analysis of Ref. 5 proves that $\forall\, n$, Π_n is a projection indeed. Moreover, if to denote $\mathscr{H}_{\rm B}$ the Hilbert space of the nonlinear Schrödinger system (3) and (4), then on its proper subspace $\mathscr{H}^n_{\rm F} = \Pi_n \mathscr{H}_{\rm B}$, the following fermion field operators (Fock representation of the CAR algebra) do automatically exist[5]:

$$b_\beta(p) = \sum_{s=0}^{\infty} \frac{(1+s)^{1/2}}{s!} \sum_{\alpha_1=1}^{n} \int dq_1 \cdots \sum_{\alpha_s=1}^{n} \int dq_s \times \sigma(\alpha_1 q_1;...;\alpha_s q_s) \times \sigma(\beta p;\alpha_1 q_1;...;\alpha_s q_s)\, \phi^*_{\alpha_1}(q_1)\cdots\phi^*_{\alpha_s}(q_s) \times :\exp\left[-\sum_{\gamma=1}^{n}\int dr\, \phi^*_\gamma(r)\,\phi_\gamma(r)\right]: \times \phi_\beta(p)\,\phi_{\alpha_1}(q_1)\cdots\phi_{\alpha_s}(q_s), \tag{9}$$

where $1 \leqslant \beta \leqslant n$ and

$$[b_\alpha(p), b^*_\beta(q)]_+ = \delta_{\alpha\beta}\,\delta(p-q)\,\Pi_n,$$
$$[b_\alpha(p), b_\beta(q)]_+ = 0, \quad 1 \leqslant \alpha, \quad \beta \leqslant n, \tag{10}$$

while $b_\alpha(p)|0) = 0$, $b^*_\alpha(p)|0) = \phi^*_\alpha(p)|0)$, $\forall\, \alpha, p$.

One should realize that each projection Π_n selects in $\mathscr{H}_{\rm B}$, its proper subspace $\mathscr{H}^n_{\rm F}$, on which the respective Bose variables (i.e., ϕ^*_α, ϕ_α, $1 \leqslant \alpha \leqslant n$) respect the Pauli principle. It means that the operator

$$\mathscr{P}_n = \sum_{\alpha=1}^{n} \mathscr{N}_\alpha(\mathscr{N}_\alpha - 1) \tag{11}$$

has the eigenvalue 0 on the whole of $\mathscr{H}^n_{\rm F}$. Because of (6), these Pauli-principle-saving subspaces, are the Galilei invariant sectors in $\mathscr{H}_{\rm B}$, thus giving rise to *the Galilean fermion excitations* in $\mathscr{H}_{\rm B}$.

Moreover, projections $\{\Pi_n, n \geqslant 1\}$ form a decreasing sequence

$$\Pi_n\,\Pi_{n+1} = \Pi_{n+1}. \tag{12}$$

But then, according to the standard knowledge: (1) there exists a strong limit $\Pi = $ s-lim Π_n, which is a projection on $\mathscr{H}_{\rm B}$, (2) the property $\Pi_n\,\Pi = \Pi$ holds true for all n, and (3) for any vector $|f) \in \mathscr{H}_{\rm B}$ for which $\lim \Pi_n\,|f) \neq 0$, upon setting $|\psi) = \lim \Pi_n\,|f)$ we have $|\psi) \neq 0$ and $\Pi_n\,|\psi) = |\psi)$, $\forall\, n$. On the respective subspace $\Pi\,\mathscr{H}_{\rm B} = \mathscr{H}_{\rm F}$ of $\mathscr{H}_{\rm B}$ the operator

$$\mathscr{P} = \sum_{\alpha=1}^{\infty} \mathscr{N}_\alpha(\mathscr{N}_\alpha - 1)$$

has the eigenvalue 0, and the generalization of the formula (9) to $n \to \infty$ is possible. Then, however, we arrive at the conclusion that the Bose quantized nonlinear Schrödinger field with attractive coupling, in addition to bearing the infinite set of Galilean bosons, gives rise as well to the infinite set of Galilean fermions

$$\Pi\, M\, \Pi = \sum_{n\geqslant 1} \frac{n}{2} \int_{-\infty}^{+\infty} dp\, b^*_n(p)\, b_n(p),$$

$$\Pi\, H\, \Pi = \sum_{n\geqslant 1} \int_{-\infty}^{+\infty} dp \left[\frac{p^2}{n} - \frac{c^2}{12}(n^3 - n)\right] \times b^*_n(p)\, b_n(p),$$

$$\Pi\, P\, \Pi = \sum_{n\geqslant 1} \int_{-\infty}^{+\infty} dp\, p\, b^*_n(p)\, b_n(p),$$

$$\Pi\, K\, \Pi = -\sum_{n\geqslant 1} \frac{in}{2} \int_{-\infty}^{+\infty} dp\, b^*_n(p) \frac{\partial}{\partial p} b_n(p), \tag{13}$$

which live in the Hilbert space of our Bose system.

Since, *a priori*, each field $\phi_n(p)$ can be given as a function of the primary interacting fields $\psi^*(x)$, $\psi(x)$, it happens so in the case of fermions $b^*_n(p)$, $b_n(p)$. However we cannot present the corresponding formulas. As well, we do not know how the primary fields $\psi(x)$, $\psi^*(x)$ act on the Pauli-principle-saving domain $\Pi\,\mathscr{H}_{\rm B} = \mathscr{H}_{\rm F}$. Nevertheless, since

$$|\alpha_1 q_1;..., \alpha_s q_s)_{\rm F} \doteq b^*_{\alpha_1}(q_1)\cdots b^*_{\alpha_s}(q_s|0) = \sigma(\alpha_1 q_1;...;\alpha_s q_s)\, \phi^*_{\alpha_1}(q_1)\cdots\phi^*_{\alpha_s}(q_s|0), \tag{14}$$

the analysis of Ref. 2 apparently can be applied to determine the scalar products

$$\frac{1}{\sqrt{n!}}(x_1,..., x_n|\alpha_1 q_1;...;\alpha_s q_s)_{\rm F},$$
$$n = n_1 + \cdots + n_s,$$
$$|x_1,..., x_n) = \psi^*(x_1)\cdots\psi^*(x_n)|0). \tag{15}$$

It is, however, quite transparent that unlike our previous investigations[3,4] the property $[H, \Pi]_- = 0$ does not suffice to convert the Bose Hamiltonian $H = H(\psi^*, \psi)$ of (4) into the (Fermi) Hamiltonian $H_{\rm F} = \Pi\, H\, \Pi$, where the primary bosons ψ^*, ψ are simply replaced by the respective fermions. In the present case, the fermion content of the model becomes manifest on another level of the theory. Albeit, the basic (boson–fermion unduality) mechanism $H_{\rm B} = P H_{\rm B} P + (1-P) H_{\rm B}(1-P)$, $P H_{\rm B} P = H_{\rm F}$ is still the same as previously, see Refs. 1, 3, and 4. A more detailed study of the issue in connection with the boson and fermion Fock space unification can be found in Refs. 6 and 7.

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