Fermi states of Bose systems in three space dimensions

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Recently an exact spectral solution was constructed by Sudarshan and Tata for the $(N\Theta)$ Fermi version of the Lee model. We demonstrate that it provides a partial solution for the related pure Bose spectral problems. Moreover, the $(N\Theta)$ Bose (Bolsterli–Nelson) version of the Lee model is shown to possess Fermi partners, both exhibiting the partial solubility interplay: finding solutions in the Fermi case would presumably be easier than in the original Bose model. Fermi states of the underlying Bose systems in three space dimensions are explicitly identified.

Let us study a specialized version of the Lee model[1–4] as considered by Sudarshan and Tata.[5] The model consists of two fermions N, Θ interacting with a boson V. If compared with the original version of the Lie field theory model, the momentum dependence of V and N is lost due to their (assumed) infinite mass, then V and N play the role of sources, while Θ is supposed to be massless. We have

$$H = m_0 V^* V + \int d^2k \cdot k \cdot a^*(k) a(k) + \int d^3k f(k) [V^* N a(k) + a^*(k) N^* V] \tag{1}$$

with the commutation rules of Ref. 5

$$\begin{aligned} &[N,N^*]_+ = [V,V^*]_- = 1, \\ &[a(k),a^*(p)]_+ = \delta^3(\mathbf{k}-\mathbf{l}), \\ &[N,N]_+ = [V,V]_- = [a(k),a(p)]_+ = 0, \\ &[N,a(k)]_+ = [N,a^*(k)]_+ = [N,V]_- = [N,V^*]_- \\ &\qquad = [a(k),V]_- = [a(k),V^*]_- = 0. \end{aligned} \tag{2}$$

Let us observe that irrespective of whether quantum objects V, N, Θ represent bosons or fermions, and irrespective of whether they mutually commute or anticommute, the following two operators are the constants of motion:

$$N_1 = N_V + N_N, \qquad N_2 = N_\Theta - N_N, \tag{3}$$

where

$$N_V = V^* V, \quad N_N = N^* N, \quad N_\Theta = \int d^3k\, a^*(k) a(k), \tag{4}$$

and upon assuming that N_V, N_N, N_Θ commute with any function of operators belonging to pairs of species $(N,\Theta), (V,\Theta), (V,N)$, respectively, we arrive at

$$\begin{aligned}[N_1,H]_- = \int d^3k f(k) \{ &V^*[V,V^*]_- N a(k) \\ &+ a^*(k) N^* [V^*,V]_- V \\ &+ V^*[N^*,N]_- N a(k) \\ &+ a^*(k) N^* [N,N^*]_- V \}, \end{aligned} \tag{5a}$$

$$\begin{aligned}[N_2,H]_- = &\int d^3k \int d^3p f(p) \{ V^* N [a^*(k),a(p)]_- a(k) \\ &+ a^*(k)[a(k),a^*(p)]_- N^* V \} \\ &- \int d^3k f(k) \{ V^*[N^*,N]_- N a(k) \\ &+ a^*(k) N^* [N,N^*]_- V \}. \end{aligned} \tag{5b}$$

If now to admit that each of the species obeys some canonical (commutation or anticommutation) rules, then the conservation laws

$$[N_1,H]_- = 0 = [N_2,H]_- \tag{6}$$

immediately follow.

The standard ansatz about the form of eigenfunctions for H is [1,2] that they should be superpositions of the bare states, i.e., eigenstates of

$$H_0 = m_0 V^* V + \int d^3k \cdot k \cdot a^*(k) a(k) \Rightarrow H = H_0 + H_{\rm int}.$$

Since we wish to solve a common (N_1,N_2,H) eigenvalue problem, it is natural to look for states $|a,b\rangle$ obeying

$$N_1|a,b\rangle = a|a,b\rangle, \quad N_2|a,b\rangle = b\,|a,b\rangle,$$

$$\begin{aligned}|a,b\rangle = \sum_{\substack{a=m+n \\ b=l-n}} \frac{1}{(m!n!l!)^{1/2}} &\int d^3k, \cdots \\ &\times \int d^3k_l\, \phi^{(m,n,l)}(k_1,\ldots,k_l) \\ &\times V^{*m} a^*(k_1) \cdots a^*(k_l) N^{*n} |0\rangle, \end{aligned} \tag{7}$$

which in the Fermi case (1) are restricted to summations over $n = 0,1$ while (N,V boson, Θ fermion) or (N,V,Θ bosons) allow $n = 0,1,2,\ldots$. In case of Θ fermionic, the coefficient function $\phi(k_1,\ldots,k_l)$ is antisymmetric with respect to the momentum variables, while in case of Θ bosonic, is symmetric. We demand $|a,b\rangle$ to be an eigenfunction of H, to be denoted $|\lambda\rangle = |\lambda,a,b\rangle$,

$$H|\lambda\rangle = \lambda\,|\lambda\rangle, \quad \lambda = \lambda_{(a,b)}. \tag{8}$$

To distinguish between the pure Bose version of (1) and the $(N\Theta)$ Fermi case of (1) we shall use the notation $H_{\rm B}, |\lambda\rangle_{\rm B}$ and H, $|\lambda\rangle$, respectively. In the pure Bose case by applying $H^{\rm B}_{\rm int}$ to $|\lambda\rangle_{\rm B}$, as given by (7) we arrive at

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$$H^{\mathrm{B}}_{\mathrm{int}}|\lambda,a,b\rangle_{\mathrm{B}}$$
$$= \sum_{\substack{a=m+n\\ b=l-n}} \Big\{\sqrt{(m+1)n\cdot l}\int d^3k_1\cdots\int d^3k_{l-1}\Big[\Big[\int d^3k f(k)\,\phi^{(m,n,l)}_{\mathrm{B}}(k,k_1,...,k_{l-1})\Big]\Big|m+1,n-1,k_1,...,k_{l-1}\rangle_{\mathrm{B}}$$
$$+\sqrt{m(n+1)(l+1)}\int d^3k_1\cdots\int d^3k_{l+1}\,f(k_1)\,\phi^{(m,n,l)}_{\mathrm{B}}(k_2,...,k_{l+1})|m-1,n+1,k_1,...,k_{l+1}\rangle_{\mathrm{B}}\,, \tag{9}$$

$$|m,n,k_1,...k_l\rangle_{\mathrm{B}} = \frac{1}{(m!n!l!)^{1/2}}V^{*m}b^*(k_1)\cdots b^*(k_l)N^{*n}_{\mathrm{B}}|0\rangle\,.$$

In the case of $(N\Theta)$ fermionic, we must have $n=0,1$, which implies that $|\lambda,a,b\rangle$ is a superposition of the two types of bare states only: $|m=a-1,n=1,l=b+1\rangle$ and $|m=a,n=0,l=b\rangle$ at a fixed choice of a,b. Consequently,

$$H_{\mathrm{int}}|\lambda,a,b\rangle = \int d^3k f(k)V^*Na(k)\int d^3k_1\cdots\int d^3k_{b+1}\,\phi^{(a-1,1,b+1)}(k_1,\cdots,k_{b+1})\cdot|a-1,1,k_1,\cdots,k_{b+1}\rangle$$
$$+\int d^3k f(k)a^*(k)N^*V\int d^3k_1\cdots\int d^3k_b\,\phi^{(a,0,b)}(k_1,\cdots,k_b)\cdot|a,0,k_1,\cdots,k_b\rangle$$
$$=(-1)^b\sqrt{a(b+1)}\int d^3k_1\cdots\int d^3k_b\,[f(k)\,\phi^{(a-1,1,b+1)}(k,k_1,...,k_b)]\,|a,0,k_1,\cdots,k_b\rangle$$
$$+(-1)^{b\pm1}\sqrt{a(b+1)}\int d^3k_1\cdots\int d^3k_{b+1}\,f(k_1)\,\phi^{(a,0,b)}(k_2,...,k_{b+1})|a-1,1,k_1,...,k_{b+1}\rangle\,. \tag{10}$$

The particular structure of the interaction term H_{int} of H as given by (1) has intriguing consequences in the Bose case. Namely, $|\lambda,a,b\rangle_{\mathrm{B}}$ for all b but with the value of a restricted not to exceed 1: $a\leqslant1$, can always be composed as a superposition of vectors taken from pairwise orthogonal Hilbert space sectors, each sector being spanned by vectors of the (shorthand) form

$$|a-k,k,b+k\rangle,\quad |a-k+1,k-1,b+k-1\rangle,\quad k\leqslant a\leqslant1\,. \tag{11}$$

In particular let us consider the following contribution to $|\lambda,a,b\rangle_{\mathrm{B}}$:

$$|\lambda,a,b\rangle^{(0)}_{\mathrm{B}} := \int d^3k,\cdots\int d^3k_{b+1}\,\phi^{(a-1,1,b+1)}_{\mathrm{B}}(k_1,...,k_{b+1})\cdot|a-1,1,k_1,...,k_{b+1}\rangle_{\mathrm{B}}$$
$$+\int d^3k_1\cdots\int d^3k_b\,\phi^{(a,0,b)}_{\mathrm{B}}(k_1,...,k_b)\cdot|a,0,k_1,...,k_b\rangle_{\mathrm{B}}\,. \tag{12}$$

The action of $H^{\mathrm{B}}_{\mathrm{int}}$ on (12) reads as follows:

$$H^{\mathrm{B}}_{\mathrm{int}}|\lambda,a,b\rangle^{(0)}_{\mathrm{B}} = \sqrt{a(b+1)}\int d^3k_1\cdots\int d^3k_b\Big[\int d^3k f(k)\cdot\phi^{(a-1,1,b+1)}_{\mathrm{B}}(k,k_1,\cdots,k_b)\Big]|a,0,k_1,...,k_b\rangle_{\mathrm{B}}$$
$$+\sqrt{a(b+1)}\int d^3k_1\cdots\int d^3k_{b+1}\,f(k_1)\,\phi^{(a,0,b)}_{\mathrm{B}}(k_2,...,k_{b+1})\cdot|a-1,1,k_1,\cdots,k_{b+1}\rangle_{\mathrm{B}}\,. \tag{13}$$

Any domain spanned by vectors of the form (11) is in fact left invariant by $H^{\mathrm{B}}_{\mathrm{int}}$.

Remark: Let us observe that if to abandon the restriction $a\leqslant1$, then the action of $H^{\mathrm{B}}_{\mathrm{int}}$ on $|\lambda,a,b\rangle^{(0)}_{\mathrm{B}}$ would produce an additional additive term in (13) following from the application of (a^*N^*V) to $|a-1,1,k_1,...,k_{b+1}\rangle$. The resulting $|a-2,2,k_1,...,k_{b+2}\rangle$ contribution can be eliminated from further discussion, but the price paid is the modification of the pure Bose Hamiltonian to the form $PH^{\mathrm{B}}_{\mathrm{int}}P$ with $P=\,:\exp(-N^*_{\mathrm{B}}N_{\mathrm{B}}):+N^*_{\mathrm{B}}:\exp(-N^*_{\mathrm{B}}N_{\mathrm{B}}):N_{\mathrm{B}}$.

It corresponds to the replacement in $H^{\mathrm{B}}_{\mathrm{int}}$ of the pure Bose variable N_{B} by the spin-$\frac{1}{2}$ Pauli operator variable $\sigma^-=PN_{\mathrm{B}}P$, $(\sigma^+=PN^*_{\mathrm{B}}P)$. Hence, we in fact pass then from the pure Bose model to the $(V\Theta)$ Bose, N Fermi version of the Lee model. It is worth emphasizing that though the whole subsequent analysis is made for the pure Bose model with the restriction $a\leqslant1$ on state vectors, all the arguments apply without any change (up to minor modifications in H^{B}_{int}) to the above-mentioned $(V\Theta)$ Bose N Fermi system, where a and b are completely arbitrary. It means that the spectral solution for the $(N\Theta)$ Fermi case produces this as well for the (N) Fermi case. It is also instructive to mention that the interaction of two static fermions with the scalar boson was studied in Ref. 4 in the $N_1=N^*N+V^*V=a=1$ state (sub) space. The subsequent analysis establishes the Θ Fermi partner for this case.

On the basis of Ref. 5 we know how to establish the eigenvalues and eigenvectors (i.e., $\phi^{(a-1,1,b+1)}$, $\phi^{(a,0,b)}$) for the $(N\Theta)$ Fermi problem. At this point we are guided by our earlier studies of the $(1+1)$-dimensional models,[6] and the joint Bose–Fermi spectral problems arising there. For the exactly soluble Fermi model of Ref. 5 we wish to establish its pure Bose partner, such as the joint spectral problem makes sense.

Let us make use of Refs. 7–9, where relations between linear spaces of symmetric and antisymmetric functions were investigated. In application to our problem, the formal realization of the isomorphism, invented in Ref. 7 by means of the Friedrichs–Klauder antisymmetric symbol, is best suited. The symbol reads

$$\sigma(k_1,...,k_n) = \pm 1,0 \quad (14)$$

depending on even (+) or odd (−) permutations of momenta, the value 0 occurring if any two momenta coincide. Then

$$\sigma^3 = \sigma, \quad \sigma(1-\sigma^2) = 0, \quad (15)$$

and any symmetric function $f_s(k_1,...,k_n)$ allows[7] for a decomposition

$$f_s(k_1,...,k_n) = [\sigma^2 f_s + (1-\sigma^2) f_s](k_1,...,k_n) := (\overset{1}{f} + \overset{2}{f})(k_1,...,k_n) \quad (16)$$

with the property that

$$\sigma f_s = \sigma \overset{1}{f} = f_a \quad (17)$$

is an antisymmetric function of n-momentum variables. The formula (17) establishes an isomorphism between symmetric functions $\overset{1}{f}_s$ (they respect the Pauli exclusion principle since $\overset{1}{f}_s$ vanishes if any two momenta coincide), and their antisymmetric partners f_a.

The above isomorphism has been exploited in Ref. 8 to construct an embedding of the CAR algebra representation with generators $[a(p),a^*(q)]_+ = \delta^3(\mathbf{p}-\mathbf{q})$, $[a(k), a(p)]_+ = 0$ in the representation of the CCR algebra generated by $[b(p),b^*(q)]_- = \delta^3(\mathbf{p}-\mathbf{q})$, $[b(k),b(p)]_- = 0$, provided the representation spaces are constructed about the same (generating in the GNS construction sense) Hilbert space vector. We refer to Ref. 8 for the explicit "bosonization" formulas valid in the Fock case (see also Ref. 9). For our purposes the following identity resulting from the CAR- = CAR(CCR) construction of Ref. 8 is necessary:

$$\begin{aligned}|k_1,...,k_n\rangle_F &= (1/\sqrt{n!})a^*(k_1)\cdots a^*(k_n)|0\rangle \\ &= \sigma(k_1,..., k_n)(1/\sqrt{n!})b^*(k_1)\cdots b^*(k_n)|0\rangle \\ &= \sigma(k_1,..., k_n)|k_1,...,k_n\rangle_B . \quad (18)\end{aligned}$$

Since in (16) we deal with an object N^*

$$\begin{aligned}N^*|k_1,...,k_n\rangle_F &= (-1)^n|1,k_1,..., k_n\rangle_F \\ &= ((-1)^n/\sqrt{n!})a^*(k_1)\cdots a^*(k_n)N^*|0\rangle , \quad (19)\end{aligned}$$

an appropriate realization for $N^* = N^*_F$ is necessary. We define

$$N^*_F = (-1)^{\int d^3k\, a^*(k)a(k)} N^*_B : \exp(-N^*_B N_B):, \quad (20)$$

which has all the necessary properties, i.e., $N^{*2}_F = 0$ [notice that $:\exp(-N^*_B N_B):$ is a projection on the vacuum state for the boson $[N_B,N^*_B]_- = 1$], and anticommutes with the $a^*(k)$'s in (19). Instead of $(-1)^{\int d^3k\, a^*(k)a(k)}$ one can obviously use $\exp i\pi \int d^3k\, a^*(k)a(k)$.

A nice property of the realization (20) is that a Bose representation for (19) is immediate:

$$\begin{aligned}|1,k_1,..., k_n\rangle_F &= (1/\sqrt{n!})\sigma(k_1,...,k_n)b^*(k_1)\cdots b^*(k_n)\cdot N^*_B|0\rangle \\ &= \sigma(k_1,...,k_n)|1,k_1, ..., k_n\rangle_B . \quad (21)\end{aligned}$$

The notion of Fermi states of the Bose system acquires thus a meaning in three space dimensions.

A straightforward application of (18) and (21), if combined with (15)–(17), allows us to rewrite formula (16) as follows:

$$\begin{aligned}H_{\rm int}|\lambda,a,b\rangle = \sqrt{a(b+1)} &\int d^3k_1 \cdots \int d^3k_b \Big\{(-1)^b \\ &\times \Big[\int d^3k f(k)\,\phi^{(a-1,1,b+1)}(k,k_1,..., k_b)\Big] \\ &\times \sigma(k_1,..., k_b)\Big\}\, |a,0,k_1,..., k_b\rangle_B \\ &+ \sqrt{a(b+1)} \int d^3k_1 \cdots \int d^3k_{b+1} \\ &\times \{(-1)^{b+1} f(k_1)\,\phi^{(a,0,b)}(k_2,..., k_{b+1}) \\ &\times \sigma(k_1,...,k_{b+1})\}\cdot|a-1,1,k_1,...,k_{b+1}\rangle_B . \quad (22)\end{aligned}$$

Since in (13) and (22) we deal with superpositions of the Bose (bare) basis vectors, the respective expansion coefficients (with respect to this basis system) can be compared.

The formula (13) implies

$$\begin{aligned}{}_B\langle a,0,k_1, \cdots, k_b|H^B_{\rm int}|\lambda,a,b\rangle^{(0)}_B &= \sqrt{a(b+1)} \int d^3k f(k)\,\phi^{(a-1,1,b+1)}_B(k,k_1,..., k_b) \quad (23)\end{aligned}$$

and

$$\begin{aligned}{}_B\langle a-1,1,k_1,...,k_{b+1}|H^B_{\rm int}|\lambda,a,b\rangle^{(0)}_B &= \sqrt{a(b+1)}({\rm sym})[f(k_1)\,\phi^{(a,0,b)}_B(k_2,...,k_{b+1})] , \quad (24)\end{aligned}$$

where

$$({\rm sym}) = S_{b+1} = \frac{1}{(b+1)!}\sum_P P_{b+1}$$

is a symbol of symmetrization with respect to all momentum variables, $\Sigma_P P$ stands for a sum over all permutations.

Quite analogously, from (22) we arrive at

$$\begin{aligned}{}_B\langle a,0,k_1,..., k_b|H_{\rm int}|\lambda,a,b\rangle &= \sqrt{a(b+1)}(-1)^b \int d^3k f(k)\,\phi^{(a-1,1,b+1)} \\ &\times (k,k_1,...,k_b)\sigma(k_1,...,k_b) \quad (25)\end{aligned}$$

and

$$\begin{aligned}{}_B\langle a-1,1,k_1,...,k_{b+1})H_{\rm int}|\lambda,a,b\rangle &= \sqrt{a(b+1)}(-1)^{b+1} \\ &\times ({\rm sym})[f(k_1)\,\phi^{(a,0,b)}(k_2,...,k_{b+1}) \\ &\times \sigma(k_1,...,k_{b+1})] . \quad (26)\end{aligned}$$

In addition to (sym), let us introduce the antisymmetrization operation

$$({\rm asym}) = A_{b+1} = \frac{1}{(b+1)!}\sum_P (-1)^P P_{b+1} .$$

Both S and A are examples of the Young's idempotent operators Y_n, allowing for a decomposition of any n-point function with respect to different types of symmetry

$$f_n = \sum_Y Y_n f_n \,.$$

We shall exploit a property (Ref. 7, Theorem 2.7), which connects Young's operators Y_n with their duals Y_n^d:

$$Y_n \sigma_n = \sigma_n \, Y_n^d \,. \tag{27}$$

In particular $S_n^d = A_n$, $A_n^d = S_n$, hence $S_n \sigma_n = \sigma_n A_n$. It means that (26) acquires the form of

$$\begin{aligned}&\sqrt{a(b+1)}(-1)^{b+1}\sigma(k_1,\ldots,k_{b+1})\\ &\quad\times\{(\text{asym})[f(k_1)\,\phi^{(a,0,b)}(k_2,\ldots,k_{b+1})]\}\\ &=\sqrt{a(b+1)}(-1)^{b+1}\sigma^2(k_1,\ldots k_{b+1})\\ &\quad\times\{(\text{sym})[f(k_1)\,\phi_s^{(a,0,b)}(k_2,\ldots,k_{b+1})]\}\,,\end{aligned} \tag{28}$$

where

$$\phi_s^{(a,0,b)}(k_2,\ldots,k_{b+1}) = \sigma(k_2,\ldots,k_{b+1})\cdot\phi^{(a,0,b)}(k_2,\ldots,k_{b+1})\,. \tag{29}$$

Let us now make an identification,

$$\phi_{\rm B}^{(m,n,l)}(k_1,\ldots,k_l) = \sigma(k_1,\ldots,k_l)\,\phi^{(m,n,l)}(k_1,\ldots,k_l)\,, \tag{30}$$

relating the pure Bose and the $(N\Theta)$ Fermi expansion coefficients in the above. By virute of (18) and (21) it implies that the Bose vectors (12) upon (30) satisfy

$$\begin{aligned}|\lambda,a,b\rangle_{\rm B}^{(0)} &= |\lambda,a-1,1,b+1\rangle_{\rm B} + |\lambda,a,0,b\rangle_{\rm B}\\ &= |\lambda,a-1,1,b+1\rangle + |\lambda,a,0,b\rangle = |\lambda,a,b\rangle\,,\end{aligned} \tag{31}$$

i.e., coincide with the respective Fermi vectors in the Fock space. Furthermore, the pure Bose expression (23) reads

$$\begin{aligned}&\sqrt{a(b+1)}\int d^3k\,f(k)\,\phi_{\rm B}^{(a-1,1,b+1)}(k,k_1,\ldots,k_b)\\ &=\sqrt{a(b+1)}\int d^3k\,f(k)\,\phi^{(a-1,1,b+1)}\\ &\quad\times(k,k_1,\ldots,k_b)\cdot\sigma(k,k_1,\ldots,k_b)\\ &=\sqrt{a(b+1)}\int d^3k\,f(k)\,\phi^{(a-1,b+1)}\\ &\quad\times(k,k_1,\ldots,k_b)\cdot\sigma(k_1,\ldots,k_b)\,,\end{aligned} \tag{32}$$

which by a factor $(-1)^b$ differs from the corresponding $(N\Theta)$ Fermi expression $(-1)^b(23) = (25)$.

As a result of (30) and (28) the following formula holds true for the $(N\Theta)$ Fermi model expression (26):

$$\begin{aligned}&(-1)^{b+1}\sqrt{a(b+1)}(\text{sym})[f(k_1)\,\phi^{(a,0,b)}\\ &\quad\times(k_2,\ldots,k_{b+1})\sigma(k_1,\ldots,k_{b+1})]\\ &=\sqrt{a(b+1)}(-1)^{b+1}\sigma^2(k_1,\ldots,k_{b+1})\\ &\quad\times\{(\text{sym})[f(k_1)\,\phi_B^{(a,0,b)}(k_2,\ldots k_{b+1})]\}\,,\end{aligned} \tag{33}$$

which upon dropping out a factor $(-1)^{b+1}$ is exactly the σ^2F contribution to the decomposition formula $[\sigma^2F + (1-\sigma^2)F]$ valid for the pure Bose expression $(25) = F$. By virtue of (15) the decomposition is orthogonal.

Since, because of (31) we have

$$H_0^{\rm B}|\lambda,a,b\rangle = H_0|\lambda,a,b\rangle\,; \tag{34}$$

the relevant information about the relationships between the Bose and Fermi spectral problems comes from the interaction terms.

By virtue of (31) we arrive at

$$\begin{aligned}&H_{\rm int}^{\rm B}|\lambda,a,b\rangle_{\rm B}^{(0)}\\ &\quad= H_{\rm int}^{\rm B}|\lambda,a,b\rangle\\ &\quad=(-1)^b\int d^3k\,f(k)V^*Na(k)|\lambda,a-1,1,b+1\rangle\\ &\qquad+(-1)^{b+1}\int d^3k\,f(k)a^*(k)\\ &\qquad\times N^*V|\lambda,a,0,b\rangle + (-1)^{b+1}|R\rangle\,,\end{aligned} \tag{35}$$

where

$$\begin{aligned}|R\rangle = &\int d^3k_1\cdots\int d^3k_{b+1}\,\sqrt{a(b+1)}\\ &\times[1-\sigma^2(k_1,\ldots,k_{b+1})]\\ &\times\{(\text{sym})[f(k_1)\,\phi_{\rm B}^{(a,0,b)}(k_2,\ldots,k_{b+1})]\}\\ &\times|\lambda,a,0,k_1,\ldots,k_{b+1}\rangle_{\rm B}\,.\end{aligned} \tag{36}$$

Let us however, recall that because of (30), $\phi_{\rm B} = \sigma\cdot\phi$ and that

$$(\text{sym}) = \frac{1}{(b+1)!}\sum_P P_{b+1}\,,$$

so that in (36) we encounter products of the form

$$[1-\sigma^2(k_1,\ldots,k_{b+1})]\cdot\sigma(k_{i_1},\ldots,k_{i_b}) \tag{37}$$

with k_i's taken from the set $(k_1,\ldots,k_{1+b})$. But (37) either identically vanishes, or gives a nonzero contribution to (36) on the set of measure zero only. Hence, $|R\rangle = 0$.

If we introduce the notation

$$\begin{aligned}H_\pm^{\rm B} &= \int d^3k\,f(k)[V^*Nb(k) + b^*(k)N^*V]\,,\\ H_\pm^{\rm F} &= \int d^3k\,f(k)[V^*Na(k) + a^*(k)N^*V]\,,\end{aligned} \tag{38}$$

then (35) appears as an example of a few more relations between Bose and Fermi Hamiltonians

$$\begin{aligned}H_+^{\rm B}|\lambda,a,b\rangle &= (-1)^b H_-^{\rm F}|\lambda,a,b\rangle\,,\\ H_-^{\rm B}|\lambda,a,b\rangle &= (-1)^b H_+^{\rm F}|\lambda,a,b\rangle\,,\quad a\leqslant 1\,.\end{aligned} \tag{39}$$

After accounting for the contribution of H_0, the complete Hamiltonians of the form $H_0 \pm H_{\rm int}$ become related as follows: $N_1 \leqslant 1$, $N_2 = b$,

b even,

$$\begin{aligned}(H_0^{\rm B} + H_+^{\rm B})|\lambda,a,b\rangle &= (H_0^{\rm F} + H_-^{\rm F})|\lambda,a,b\rangle\,,\\ (H_0^{\rm B} + H_-^{\rm B})|\lambda,a,b\rangle &= (H_0^{\rm F} + H_+^{\rm F})|\lambda,a,b\rangle\,,\end{aligned} \tag{40}$$

b odd,

$$\begin{aligned}(H_0^{\rm B} \pm H_+^{\rm B})|\lambda,a,b\rangle &= (H_0^{\rm F} \mp H_-^{\rm F})|\lambda,a,b\rangle\,,\\ (H_0^{\rm B} \pm H_-^{\rm B})|\lambda,a,b\rangle &= (H_0^{\rm F} \mp H_+^{\rm F})|\lambda,a,b\rangle\,.\end{aligned} \tag{41}$$

In this number the pure Bose problem H_B of Refs. 3 and 4 is identified with $H_0^{\rm B} + H_+^{\rm B}$ and the $(N\Theta)$ Fermi problem of Ref. 5 with $H_0^{\rm F} + H_+^{\rm F}$.

Relations (40) and (41) prove that for the family of four $(N\Theta)$ Fermi models, there is a corresponding family of pure Bose models, with the property that in the state space of the Bose system there exists a projection Π such that the eigenvalue problem for h_F can be solved in the range of Π, and

$$[h_{\rm B},\Pi]_- = 0, \quad h_{\rm F} \equiv \Pi h_{\rm B}\,\Pi,$$

$$h_{\rm B} = \Pi h_{\rm B}\,\Pi + (1-\Pi)h_{\rm B}(1-\Pi)\,. \tag{42}$$

Here $h_{\rm B}$ stands for the Bose, while $h_{\rm F}$ stands for the respective Fermi Hamiltonian. Complementary studies of $(1+1)$-dimensional field theory models sharing the property (42) can be found in Refs. 10, 11, and 6.

The results (40) and (41) mean in particular that the pure Bose model

$$h_{\rm B} = m_0 V^*V + \int d^3k\, k\cdot b^*(k)b(k) + \int d^3k f(k)[V^*Nb(k) - b^*(k)N^*V] \tag{43}$$

has eigenvectors and eigenvalues common with the $(N\Theta)$ Fermi model solved by Sudarshan and Tata[5]: all b even eigenvectors of $h_{\rm F}$ of (1) are exact eigenvectors with the same eigenvalues for the pure Bose Hamiltonian (43). The odd eigenvectors of $h_{\rm F}$ are shared with

$$h'_{\rm B} = m_0\, V^*V + \int d^3k\, k\cdot b^*(k)b(k) - \int d^3k f(k)[V^*Nb(k) - b^*(k)N^*V]\,. \tag{44}$$

One should also notice that upon solving the eigenvalue problem for the Fermi Hamiltonians $(H_0^{\rm F} \pm H_-^{\rm F})$ we would have received a partial spectral solution for the pure Bose model of the Bolsterli–Lee type.[1–3] Unfortunately the Bose Hamiltonian $(H_0^B \pm H_+^{\rm B})$ is related to the Fermi Hamiltonian $(H_0^{\rm F} \mp H_-^{\rm F})$ and likewise $(H_0^{\rm B} \pm H_-^{\rm B})$ is related to $(H_0^{\rm F} \mp H_+^{\rm F})$. Thus, the spectral solutions of Sudarshan and Tata cannot be used to obtain the solution of the spectrum of the Bolsterli–Lee model: This entails the solution of the problem for $(H_0^{\rm F} \pm H_-^{\rm F})$. For this form of the fermion Hamiltonian, however, the simple form of Eq. (3.2b) in Ref. 5 does not arise since the right-hand side now entails the operator $V^*V - N^*N$ instead of the eigenoperator $V^*V + N^*N = N_1$. (It was the eigenoperator structure that led to the simple solution in Ref. 5.) This is exactly the structure for the corresponding equation that would occur if one were directly dealing with the Hamiltonian for the Bolsterli–Lee model.

One more problem arises in connection with the (formal) non-self-adjointness of operators $H_-^{\rm B,F}$. However, since we relate them to self-adjoint operators via (42) it appears that projections Π identify the appropriate (Hermicity) domains.

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[1] T. D. Lee, Phys. Rev. **95**, 1329 (1954).
[2] S. Schweber, *Introduction to Relativistic Quantum Field Theory* (Row, Evanston, IL, 1961).
[3] M. Bolsterli, Phys. Rev. **166**, 1760 (1968); and C. A. Nelson, J. Math. Phys. **13**, 1051 (1972).
[4] D. Fivel, J. Math. Phys. **11**, 699 (1970).
[5] E. C. G. Sudarshan and X. Tata, J. Math. Phys. **24**, 1810 (1983).
[6] P. Garbaczewski, J. Math. Phys. **25**, 862 (1984).
[7] P. Garbaczewski and J. Rzewuski, Rep. Math. Phys. **6**, 431 (1974).
[8] P. Garbaczewski, Commun. Math. Phys. **43**, 131 (1975).
[9] P. Garbaczewski, J. Math. Phys. **19**, 619 (1978).
[10] P. Garbaczewski, J. Math. Phys. **24**, 651 (1983).
[11] P. Garbaczewski, Nucl. Phys. B **218**, 321 (1983).