Boson approximants for lattice Fermi systems

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It is demonstrated that Bose systems can be used to simulate properties of their Fermi partners in thermal bath, provided the spectral property $H_B = PH_BP + (1-P)H_B(1-P)$, $H_F = PH_BP$ can be proved in the state space of the Bose system. The approximation accuracy can be made arbitrarily good by varying the (free) coupling parameter $\lambda \in (0,\infty)$, and permits studying of fermionic partition or correlation functions in a finite volume by means of the standard techniques (Bose:path integral, Monte Carlo, etc).

I. MOTIVATION

Among many researchers studying numerical (Monte Carlo) simulation of Fermi systems,[1–9] the so-called pseudofermion method has gained some popularity.[3–5] Though called pseudofermions, the objects used to replace the traditional Grassmann algebra elements are the commuting c-number functions. Since the commuting function ring is used to quantize Bose systems via path integration, in the pseudofermion approach we have in fact the Bose system attributed to the Fermi one of interest.

The customary prejudice is that if no Bose–Fermi (e.g., Coleman's) equivalence can be established, then Bose and Fermi systems are considered disjointedly. On the other hand, our investigations on quantization of spinor fields[10] have shown that in principle each fermion system (irrespective of the space-time dimensionality) can be embedded in the related (mother) Bose system. In application to lattice models, this embedding observation suggests looking for such Bose systems and such physical situations (temperature and coupling constant regimes) which allow for an unambiguous separation of Fermi contributions from all relevant characteristics of the Bose system. This idea underlies the use of boson expansion methods in the study of finite spin or Fermi lattices,[11,12] and this of the spin-$\frac{1}{2}$ approximation concept for lattice bosons.[13]

Our aim is to investigate a family of Fermi models in one space dimension (for which the Monte Carlo tests are usually performed) to prove that their partition and correlation functions can be arbitrarily well approximated by means of those for the related Bose systems. Since this Bose approximation scheme is considered for a family of solvable models, the reliability of the method can be easily tested against the exact results.

II. BOSE–FERMI INTERPLAY FOR THE SIMPLEST FERMION MODELS ON A ONE-DIMENSIONAL LATTICE

Let us consider the family of fermion models for which the Monte Carlo simulation was attempted.[8,9] We shall be interested in the spinless fermion hopping problem

$$H = -J\sum_k (c^*_{k+1}c_k + c^*_k c_{k+1}),$$
$$[c_k, c^*_l]_+ = \delta_{kl}, \quad [c_k, c_l]_+ = 0 \tag{2.1}$$

and its modifications received from (2.1) by adding the different density–density interaction terms

$$V_1 = V\sum_k n_{k+1}n_k, \quad n_k = c^*_k c_k,$$
$$V_2 = V\sum_k \left(n_k - \frac{1}{2}\right)\left(n_{k+1} - \frac{1}{2}\right), \tag{2.2}$$

plus (one-flavor case) the lattice Gross–Neveu model

$$H = \sum_k \{ -J(c^*_k c_{k+1} + c^*_{k+1}c_k) + \Delta(-1)^k c^*_k c_k - \tfrac{1}{2}V(n_k - n_{k+1})^2\}. \tag{2.3}$$

The particle number operator N is conserved for each of these models $[H,N]=0$, $N = \Sigma_j n_j$, hence the general form of the eigenvectors is immediate

$$N|f) = n|f),$$
$$|f) = \sum_{(i)} f_{i_1,\ldots,i_n} c^*_{i_1}\cdots c^*_{i_n}|0), \quad c_j|0) = 0, \quad \forall_j. \tag{2.4}$$

The eigenvalue equations for the problem (2.1) can be written as follows:

$$n=0, \quad H|0) = 0,$$
$$n=1, \quad H|f) = -J\sum_{j=1}^{M}(f_j c^*_{j+1} + f_{j+1}c^*_j|0) = -2J\sum_j \frac{1}{2}(f_{j-1} + f_{j+1})c^*_j|0),$$
$$n=2, \quad H|f) = -2J\sum_{i<j}\frac{1}{2}[(f_{i-1j} + f_{i+1j}) + (f_{ij-1} + f_{ij+1})]c^*_i c^*_j|0), \tag{2.5}$$
$$\vdots$$
$$n=k, \quad H|f) = -2J\sum_{i_1<\cdots<i_k}\sum_{j=i_1}^{i_k}\frac{1}{2}(f_{i_1\cdots j-1\cdots i_k} + f_{i_1\cdots j+1\cdots i_k})c^*_{i_1}\cdots c^*_j\cdots c^*_{i_k}|0).$$

We assume that our fermions are embedded in the canonical commutation relations (CCR) algebra generated[10] by operators

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$$[a_k, a_l^*]_- = \delta_{kl}\,, \quad [a_k, a_l]_- = 0 = [a_k^*, a_l^*]_-\,, \tag{2.6}$$
$$a_k|0) = c_k|0) = 0\,, \quad \forall k\,, \quad a_k^*|0) = c_k^*|0)\,,$$

which, according to Refs. 10 and 11, implies that all Fermi states of the Bose system belong to a proper subspace [including $|0)$] $\mathscr{H}_{\mathrm{F}}$ of the Bose state space $\mathscr{H}_{\mathrm{B}}$, so that

$$|f) = |f)_{\mathrm{F}} = \sum_{i_1<\cdots<i_m} f_{i_1\cdots i_m}\, c_{i_1}^*\cdots c_{i_n}^*|0)$$
$$= \sum_{i_1<\cdots<i_m} f_{i_1\cdots i_m}\cdot \epsilon_{i_1\cdots i_m}\cdot a_{i_1}^*\cdots a_{i_m}^*|0)\,, \tag{2.7}$$

where $\epsilon_{i_1\cdots i_m}$ is the completely antisymmetric (Levi-Civita) tensor taking values $0, \pm 1$. Consequently the tensor coefficient $\overset{1}{f} = f_{i_1\cdots i_n}\cdot\epsilon_{i_1\cdots i_n}$ is symmetric and vanishes if any two indices coincide.

Let us now enact the following Bose Hamiltonian:

$$H_{\mathrm{B}} = -J\sum_k (a_k^* a_{k+1} + a_{k+1}^* a_k)\,, \tag{2.8}$$

on vectors of the form $|f)$. For $n = 2$ we have

$$H_{\mathrm{B}}|f)$$
$$= -J\sum_{i<j} \overset{1}{f}_{ij}(a_{i+1}^* a_j^* + a_{j+1}^* a_i + a_{i-1}^* a_j^* + a_{j-1}^* a_i^*)|0)$$
$$= -J\sum_{i<j} (\overset{1}{f}_{i-1\,j} + \overset{1}{f}_{ij-1} + \overset{1}{f}_{i+1\,j} + \overset{1}{f}_{ij+1}) a_i^* a_j^*|0)\,, \tag{2.9}$$

where $\overset{1}{f}_{ik} = \epsilon_{ik} f_{ik}$. Now it is enough to observe that given (i,j), $i<j$ implies either $\epsilon_{ik} = \epsilon_{i-1\,j} = \epsilon_{ij-1} = \epsilon_{i+1\,j} = \epsilon_{ij+1}$, which corresponds to $i<j$, $i+1<j$, $i<j-1$, or $\epsilon_{ik} = \epsilon_{i-1\,j} = \epsilon_{ij+1}$, which corresponds to $i+1 = j$, i.e., $i = j-1$ (then $\overset{1}{f}_{ij-1} = 0 = \overset{1}{f}_{i+1\,j}$). Consequently, the coefficient function $\overset{1}{f}_{ij}$, if multiplied by ϵ_{ij} (note that $\epsilon_{ij}^2 = 0,1$), exactly coincides with this appearing in $\Sigma_{i<j} f_{ij} c_i^* c_j^*|0) = |f)$ provided we exploit the identity

$$\epsilon_{ij} a_i^* a_j^*|0) = c_i^* c_j^*|0) \tag{2.10}$$

implied by the "bosonization" discussions in Refs. 10 and 11.

A generalization to arbitrary n is obvious with the result that eigenstates of $H = H_{\mathrm{F}}$ may happen to be those of H_{B} (the joint Bose–Fermi spectral problem of Ref. 14)

$$f_{iM+1} = f_{i1}\,, \quad f_{M+1,j} = f_{1j}$$
$$\Rightarrow H_{\mathrm{F}}|f) = \epsilon|f) \equiv H_{\mathrm{B}}|f) = H_{\mathrm{F}}|f) = \epsilon|f)\,. \tag{2.11}$$

It is useful to observe that the operator unit 1_{F} of the Fermi algebra plays in $\mathscr{H}_{\mathrm{B}}$ the role of the projection operator

$$1_{\mathrm{F}}\mathscr{H}_{\mathrm{B}} = \mathscr{H}_{\mathrm{F}}\,,$$
$$1_{\mathrm{F}}|f) = |f) \Leftrightarrow |f) \in \mathscr{H}_{\mathrm{F}}\,, \tag{2.12}$$

and because of (2.11) we may expect to have

$$[H_{\mathrm{B}}, 1_{\mathrm{F}}]_- = 0\,,$$

$$H_{\mathrm{B}} = 1_{\mathrm{F}} H_{\mathrm{B}} 1_{\mathrm{F}} + (1-1_{\mathrm{F}}) H_{\mathrm{B}} (1-1_{\mathrm{F}})$$
$$= H_{\mathrm{F}} + (1-1_{\mathrm{F}}) H_{\mathrm{B}} (1-1_{\mathrm{F}})\,, \quad H_{\mathrm{F}} = 1_{\mathrm{F}} H_{\mathrm{B}} 1_{\mathrm{F}}\,, \tag{2.13}$$

which is an orthogonal decomposition since $1_{\mathrm{F}}(1-1_{\mathrm{F}}) = 0$, 1 being the operator unit of the Bose algebra. As a straightforward consequence, we realize then that

$$\mathscr{Z}_{\mathrm{B}} = \mathrm{tr}\exp(-\beta H_{\mathrm{B}})$$
$$= \mathrm{tr}\exp(-\beta H_{\mathrm{F}}) + \mathrm{tr}\exp[-\beta(1-1_{\mathrm{F}}) H_{\mathrm{B}}(1-1_{\mathrm{F}})]$$
$$\doteq \mathscr{Z}_{\mathrm{F}} + R\,. \tag{2.14}$$

Hence the Bose trace formula includes the Fermi trace formula as a well-defined, but to be extracted, contribution.

Quite analogously, in the case of the bosonic correlation functions

$$\mathscr{Z}_{\mathrm{B}}\rho_k[(m_1,\beta_1),\ldots,(m_k,\beta_k)]$$
$$= \mathrm{tr}[\phi_{m_1}(\beta_1)\cdots\phi_{m_k}(\beta_k)\exp(-\beta H_{\mathrm{B}})]\,, \tag{2.15}$$
$$\beta_1 \leqslant \cdots \leqslant \beta_k \leqslant \beta\,, \quad \phi_m(\beta) = \exp(\beta H_{\mathrm{B}})\cdot\phi_m\cdot\exp(-\beta H_{\mathrm{B}})\,,$$

we arrive at

$$(\mathscr{Z}_{\mathrm{F}} + R)\rho_k[(m_1\beta_1),\ldots,(m_k,\beta_k)]$$
$$= \mathrm{tr}[\sigma_{m_1}^1(\beta_1)\cdots\sigma_{m_k}^1(\beta_k)\exp(-\beta H_{\mathrm{F}})]$$
$$+ R\,[(m_1,\beta_1),\ldots,(m_k,\beta_k)]\,, \tag{2.16}$$
$$\sigma_m^1(\beta) = \exp\beta H_{\mathrm{F}}\cdot\sigma_m^1\cdot\exp(-\beta H_{\mathrm{F}})\,,$$
$$\sigma_m^1 \doteq 1_{\mathrm{F}}\phi_m 1_{\mathrm{F}}\,, \quad \phi_m = 2^{-1/2}(a_m^* + a_m)\,.$$

Hence, upon dividing both sides of (2.16) by Z_{F}, the fermionic contribution explicitly arises in the general formula

$$(\mathscr{Z}_{\mathrm{B}}/\mathscr{Z}_{\mathrm{F}})\rho_{\mathrm{B}} = \rho_{\mathrm{F}} + R/\mathscr{Z}_{\mathrm{F}}\,. \tag{2.17}$$

One problem which remains is that the observation (2.17) is not that useful, unless the spin-$\frac{1}{2}$ lattice (fermionic) contribution can be viewed as dominant.

There is also another big problem (I would like to thank the referee for pointing out the issue): the discussion (2.9)–(2.11) does not yet provide the guarantee that the projection P with the properties $H_{\mathrm{F}} = PH_{\mathrm{B}}P$ and $H_{\mathrm{B}} = PH_{\mathrm{B}}P + (1-P)H_{\mathrm{B}}(1-P)$ exists in the state space of the general Bose system. At this point it is not useless to mention that the Kac–Moody algebra which is developed in Refs. 15 and 16 is the infinite-dimensional generalization of the Lie algebra. This gives a recursive procedure for writing equations like Eq. (2.17) in terms of initial data, say q and r and all derivatives $q_x, q_{xx}, \ldots, r_x, r_{xx}, \ldots$ (see, e.g., Sec. 7 of Ref. 16, where the original bosonization of Skyrme[17,18] and others[19] is shown for Kac–Moody algebras). Presumably this observation can be used to argue that the projection operator P must exist when the Kac–Moody algebra is well defined.

In the next section we shall give an explicit construction of the projection P for the particular lattice Bose model. From the technical point of view one essential difference, if compared with Refs. 16 and 20, must be emphasized. Namely, it is that in our construction of the Fermi system in the (mother) Bose system, each lattice Bose degree is mapped into the corresponding lattice Fermi degree. This is not the case in Refs. 16 and 20, where it is essential that the so-called

integral lattice is subdivided into the odd and even sublattices. Then, while the lattice boson is defined everywhere, the Fermi operators are attributed to the odd sublattice merely. In connection with different aspects of Fermi–Bose relationships see, e.g., Ref. 21.

III. EXACT SPECTRAL SOLUTION FOR THE LATTICE HOPPING

We study (alternative procedure) the model (2.1), both in its Fermi and Bose versions $\{H_F, c_k^*, c_k\}$ and $\{H_B, a_k^*, a_k\}$, respectively. Our aim is to relate them through considering the joint Bose–Fermi spectral problem in the same state space (this of the Bose system). We assume the periodic boundary conditions, which implies that both hopping Hamiltonians can be rewritten in the form

$$H = -J\sum_{ij=1}^{n} A_i^* W_{ij} A_j\,,$$
$$W_{ij} = \delta_{ij-1} + \delta_{ij+1z}, \quad i,j = 1,\ldots,n\,, \tag{3.1}$$

where the square $n\times n$ matrix W can be given as follows:

$$W = \sum_{l=0}^{n-1} c_l \gamma^{l-1}\,,$$
$$c_l = \delta_{l1} + \delta_{ln-1}\,,$$
$$\gamma^n = 1\,, \tag{3.2}$$
$$\gamma = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where n indicates the number of sites in the chain.

Since $W = \gamma + \gamma^{n-1}$ and $\gamma^n = 1$ the spectral problem for W is solved immediately by making use of

$$\gamma f_k = \lambda_k f_k\,,$$
$$\lambda_k = \exp i(2\pi/n)k = \varphi^k, \quad \varphi = \exp i(2\pi/n)\,,$$
$$k = 0,1,\ldots,n-1\,, \tag{3.3}$$
$$f_k = \{f_{k\alpha}\}, \quad \alpha = 1,2,\ldots,n\,,$$
$$f_{k1} = 1, \quad f_{k2} = \varphi^k, \quad f_{k3} = \varphi^{2k},\ldots, \quad \varphi_{kn} = \varphi^{(n-1)k},$$

which yields

$$Wf_k = w_k f_k\,,$$
$$W_k = \lambda_k + \lambda_k^{n-1} = \varphi^{-k}(1+\varphi^{2k}) = \varphi^{-k} + \varphi^k \tag{3.4}$$
$$= 2\cos(2\pi/n)k\,.$$

Vectors $\{g_k = f_k/\sqrt{n}\}$ form an orthonormal eigensystem for W

$$\frac{1}{n}(f_k, f_l) = \frac{1}{n}\sum_{q=0}^{n-1} \exp i\frac{2\pi}{n}(k-l)q = \delta_{kl}\,, \tag{3.5}$$

hence a passage to a new set of conjugate variables is possible

$$\xi_k = \sum_{\alpha=1}^{n} \bar{g}_{k\alpha} a_k\,, \quad \xi_k^* = \sum_{\alpha=1}^{n} g_{k\alpha} a_\alpha^*\,,$$
$$[\xi_k, \xi_l^*]_- = \delta_{kl}\,, \quad [\xi_k, \xi_l]_- = 0\,,$$
$$\eta_k = \sum_{\alpha=1}^{n} \bar{g}_{k\alpha} c_\alpha\,, \quad \eta_k^* = \sum_{\alpha=1}^{n} g_{k\alpha} c_\alpha^*\,, \tag{3.6}$$
$$[\eta_k, \eta_l^*]_+ = \delta_{kl}\,, \quad [\eta_k, \eta_l]_+ = 0\,,$$

which implies

$$H_B = \sum_k \left(-2J\cos\frac{2\pi}{n}k\right)\xi_k^*\xi_k\,,$$
$$H_F = \sum_k \left(-2J\cos\frac{2\pi}{n}k\right)\eta_k^*\eta_k\,. \tag{3.7}$$

The respective eigenvectors belong to the n-body Fock space of the Bose and Fermi chains, respectively,

$$|m_1,\ldots,m_n)_B = \xi_1^{*m_1}\cdots\xi_n^{*m_n}|0)_B\,,$$
$$|p_1,\ldots,p_n)_F = \eta_1^{*p_1}\cdots\eta_n^{*p_n}|0)_F\,. \tag{3.8}$$

In the boson case we shall compose the product of two level projections

$$P = \prod_k p_k\,, \quad p_k = :\exp(-\xi_k^*\xi_k): + \xi_k^*:\exp(-\xi_k^*\xi_k):\xi_k\,, \tag{3.9}$$

which has the following properties:

$$[H_B, P]_- = 0\,, \quad H_B = PH_BP + (1-P)H_B(1-P)\,,$$
$$P\xi_k^*P \equiv \sigma_k^+\,, \quad P\xi_kP \equiv \sigma_k^-\,, \quad [\sigma_k^-, \sigma_k^+]_+ = p_k\,,$$
$$[\sigma_k^\#, \sigma_l^\#]_- = 0\,, \quad k \neq l\,, \tag{3.10}$$
$$P\xi_1^{*k_1}\cdots\xi_n^{*k_n}|0)_B = (\sigma_1^+)^{k_1}\cdots(\sigma_n^+)^{k_n}|0)_B\,, \quad \text{or } 0\,,$$
$$(\sigma_j^\pm)^k = 0\,, \quad k > 1\,.$$

One should here notice the identity

$$(\sigma_1^+)^{k_1}\cdots(\sigma_n^+)^{k_n}|0)_B = \xi_1^{*k_1}\cdots\xi_n^{*k_n}|0)_B\,, \quad k_i \leqslant 1\,. \tag{3.11}$$

On the other hand, if we start from the fermion case, then either by using the Jordan–Wigner transformation or by exploiting the embedding of the Fermi (CAR) algebra in the Bose (CCR) algebra (see, e.g., Refs. 10–14), we are able to identify all fermion eigenvectors in the state space of the Bose system (as its Fermi states)

$$\eta_1^{*p_1}\cdots\eta_n^{*p_n}|0)_F = (\sigma_1^+)^{p_1}\cdots(\sigma_n^+)^{p_n}|0)_B$$
$$= \xi_1^{*p_1}\cdots\xi_n^{*p_n}|0)_B\,, \quad |0)_F = |0)_B\,. \tag{3.12}$$

It automatically follows that

$$H_B\eta_1^{*p_1}\cdots\eta_n^{*p_m}|0)_B = PH_BP\eta_1^{*p_1}\cdots\eta_n^{*p_n}|0)_B$$
$$= H_F\eta_1^{*p_1}\cdots\eta_n^{*p_n}|0)_B\,, \tag{3.13}$$

and in the range of the projection P there holds $PH_BP = H_F$.

The relevant observation at this point is that the projection P, though defined in terms of $\{\overset{*}{\xi}_k,\xi_k\}$ (and only through a^*,a expansions of $\overset{*}{\xi}_k\xi_k$, in terms of the initial operators $\{a_k^*,a_k\}$), is nevertheless a projection on the state (sub-)space $\mathscr{H}_F$ in $\mathscr{H}_B$, including all possible Fermi states of the Bose (CCR) algebra constructed about the Bose vacuum. In fact, either $\{a_k^*,a_k\}_{1<k<n}$ or $\{\overset{*}{\xi}_k,\xi_k\}_{1<k<n}$ can be used to construct the basis system in the (very same) subspace of $\mathscr{H}_B: P(\overset{*}{\xi},\xi)\mathscr{H}_B = \mathscr{H}_F = P(a^*,a)\mathscr{H}_B$.

IV. FERMI STATES OF THE BOSE SYSTEM: ON SYMMETRY PROPERTIES OF THE FERMI GROUND STATE

It should be emphasized that in the above construction Fermi states of the Bose system correspond to the lowest excitation levels of the latter (i.e., the mother one). This property is quite important with respect to the Bose approximation idea[13] in application to Fermi systems. The ground state for the Bose and Fermi systems is the same here. On the other hand, the traditional way of thinking requires that the ground state of the Fermi system exhibit an antisymmetry property, while that for bosons is symmetric. Before embarking on the Bose approximant problem, let us clarify this point, by analyzing properties of the harmonic chain

$$H=\sum_j \frac{p_j^2}{2m}+\sum_{i<j}\frac{m\omega^2}{2}(q_i-q_j)^2\,,$$
$$[q_i,p_j]_- = i\delta_{ij}\,,\quad [q_i,q_j]_- = 0 = [p_i,p_j]_-\,, \tag{4.1}$$

where

$$P=\sum_{i=1}^N p_i\,,$$
$$Q=\frac{1}{N}\sum_{i=1}^N q_i \Rightarrow \sum_{1<i<j<N}(q_i-q_j)^2 = N\sum_{i=1}^N q_i^2 - N^2Q^2\,, \tag{4.2}$$
$$[H,P]_- = 0$$

allow us to replace H by the new operator ($m=1$)

$$H_0 = H-(2N)^{-1}P^2 = \sum_{i=1}^N\left(\frac{p_i^2}{2}+\frac{N}{2}\omega^2 q_i^2\right) - \left(\frac{1}{2N}P^2+\frac{1}{2}N^2\omega^2Q^2\right). \tag{4.3}$$

We use the N-site harmonic oscillator basis

$$\Omega = \overset{N}{\underset{i=1}{\otimes}}(f_0)_i\,,$$
$$n_k\Omega = f_0\otimes\cdots\otimes(nf_0)_k\otimes\cdots\otimes f_0 = 0\,,\quad \forall k\,, \tag{4.4}$$
$$n_k = a_k^*a_k\,,\quad nf_0 = a^*af_0 = 0\,,$$
$$q_j = (a_j^*+a_j)(2\omega\sqrt{N})^{-1/2}\,,\quad p_j = i\left(\frac{\omega\sqrt{N}}{2}\right)^{1/2}(a_j^*-a_j)\,,$$

so that

$$\omega\sqrt{N}(\hat{\eta}+\tfrac{1}{2}) = \frac{1}{2N}\sum_{ij}p_ip_j+\frac{\omega^2}{2}\sum_{ij}q_iq_j = \omega\sqrt{N}\left\{\frac{1}{2}+\frac{1}{N}\left[\sum_{i=1}^N n_i+\sum_{i<j}(a_i^*a_j+a_j^*a_i)\right]\right\} \tag{4.5}$$

and

$$\hat{\eta} = \frac{1}{N}\left[\sum_{i=1}^N n_i+\sum_{i<j}(a_i^*a_j+a_j^*a_i)\right], \tag{4.6}$$
$$[a_i,a_j^*]_- = \delta_{ij}\,,\quad [a_i,a_j]_- = 0\,,\quad a_j\Omega = 0\,,\quad \forall j\,,$$

where

$$\left[\sum_i n_i\,,\sum_{i<j}(a_i^*a_j+a_j^*a_i)\right]_- = 0\,.$$

We observe that (compare, e.g., the lattice-hopping problem)

$$\sum_{i<j}(a_i^*a_j+a_j^*a_i) = \sum_{i,j}a_i^*\omega_{ij}a_j\,,\quad \omega_{ij} = (1-\delta_{ij})\,,$$
$$\omega = \sum_{l=0}^{N-1}c_l\gamma^l\,,\quad c_l = (1-\delta_{0l})\,, \tag{4.7}$$
$$\gamma f_k = \lambda_k f_k\,,\quad g_k = (1/\sqrt{n})f_k\,,$$

which implies

$$\xi_k = \sum_{\alpha=1}^N \bar{g}_{k\alpha}a_\alpha\,,\quad \overset{*}{\xi}_k = \sum_{\alpha=1}^N g_{k\alpha}a_\alpha^*\,,$$
$$[\xi_k,\overset{*}{\xi}_l]_- = \delta_{kl}\,,\quad \sum_{\alpha\beta}a_\alpha^*\omega_{\alpha\beta}a_\beta = \sum_k\left(\sum_l c_l\varphi^{k\cdot l}\right)\overset{*}{\xi}_k\xi_k\,, \tag{4.8}$$
$$k\neq 0\Rightarrow\sum_l c_l\varphi^{k\cdot l} = \sum_{l\neq 0}\varphi^{k\cdot l} = -\varphi^0 = -1\,,$$
$$k=0\Rightarrow\sum_l c_l\varphi^{k\cdot l} = N-1\,,$$

i.e.,

$$\sum_{\alpha\beta}a_\alpha^*\omega_{\alpha\beta}a_\beta = -\sum_{k\neq 0}\overset{*}{\xi}_k\xi_k+(N-1)\overset{*}{\xi}_0\xi_0 = -\sum_{k=0}^{N-1}\overset{*}{\xi}_k\xi_k+N\overset{*}{\xi}_0\xi_0\,.$$

After accounting for

$$\sum_{k=0}^{N-1}\overset{*}{\xi}_k\xi_k = \sum_{\alpha=1}^N a_\alpha^*a_\alpha = \sum_{\alpha=1}^N n_\alpha\,, \tag{4.9}$$

we arrive at a complete spectral solution for $H_0|E) = E|E)$ in terms of

$$H_0 = \omega\sqrt{N}\left[\sum_{k=0}^{N-1}\left(\overset{*}{\xi}_k\xi_k+\frac{1}{2}\right)-\left(\overset{*}{\xi}_0\xi_0+\frac{1}{2}\right)\right],$$
$$|n_1,\ldots,n_N) = \overset{*}{\xi}_0^{n_1}\overset{*}{\xi}_1^{n_2}\cdots\overset{*}{\xi}_{N-1}^{n_N}\Omega\,, \tag{4.10}$$
$$H_0\Omega = \omega\sqrt{N}\tfrac{1}{2}(N-1)\Omega\,,$$

the eigenvectors being given up to normalization and the so-called permutation degeneracy (it is a real surprise to the author that this complete solution was not produced in pa-

pers devoted to the harmonically coupled chains[22–24]).

In the form

$$H_0=\omega\sqrt{N}\left[\sum_{k=0}^{N-1}\left(\xi^*_k\xi_k+\frac{1}{2}\right)-\left(\xi^*_0\xi_0+\frac{1}{2}\right)\right]$$

or $H_0=H_0(a^*,a)$ [(4.3) and (4.5)] the harmonic chain can be considered either in its Bose or Fermi version by using the methods of Sec. III. Indeed, by using a product of two-level projections as done previously, we can replace the Bose problem H_0, (4.10) by its spin-$\frac{1}{2}$ (Fermi) relative

$$PH_0P=\omega\sqrt{N}\left[\sum_{k=0}^{N-1}\left(\sigma_k^+\sigma_k^-+\frac{1}{2}\right)-\left(\sigma_0^+\sigma_0^-+\frac{1}{2}\right)\right]\doteq H_F\,, \tag{4.11}$$

$$P\xi^*_kP=\sigma_k^+\,,\quad P\xi_kP=\sigma_k^-\,,\quad [P,H_0]_-=0\,.$$

On the other hand, if we recall that each eigenvector of H_0 can be expanded with respect to the N-site harmonic oscillator basis, we find out that the choice of the Schrödinger representation converts $|n_1,...,n_N\rangle$ into the N-point function of space variables $f_{n_1,...,n_N}(x_1,...,x_N)$. It is obvious that with respect to symmetrization or antisymmetrization of such a function, the eigenvalue problem is quite insensitive except that the permutation degeneracy of energy levels is removed. In symbolic notation we have the commutation relations $[H_0,S]_-=0=[H_0,A]_-$. However, the antisymmetrization cannot be applied blindly since the lowest-energy eigenvector which persists in this operation is the well-known one

$$|0,1,2,...,N-1\rangle=\xi^*_1\xi^*_2\cdots\xi^*_{N-1}|0\rangle\,, \tag{4.12}$$

$$H_0|0,1,...,N-1\rangle=\frac{\omega\sqrt{N}}{2}(N^2-1)|0,1,...,N-1\rangle\,.$$

The respective vector after antisymmetrization is traditionally identified as the ground state of the Fermi oscillators subject to the harmonic couplings. But in our opinion this Fermi chain notion acquires a meaning only if we refer to the field theory model

$$H=-\frac{1}{2m}\int dx\,\nabla\phi^*\nabla\phi+\frac{1}{2}\int dx\int dy\,\phi^*(x)\phi^*(y)V(x,y)\phi(y)\phi(x)\,,$$

$$[\phi(x),\phi^*(y)]_-=\delta(x-y)\,,\quad [\phi(x),\phi(y)]_-=0\,, \tag{4.13}$$

$$\phi(x)|0\rangle=0\,,\quad \forall x\in R'\,,$$

where, depending on the (anti-) commutation relations choice, either symmetric or antisymmetric wave functions are necessary

$$|f\rangle=\int dx_1\cdots\int dx_N\,f(x_1,...,x_N)\phi^*(x_1)\cdots\phi^*(x_N)|0\rangle\,.$$

V. LATTICE FERMIONS IN TERMS OF BOSONS

When dealing with Fermi systems, except for $\mathscr{Z}_F$, one tries to compute thermal averages of distinct quantities, e.g., the simplest correlation functions

$$\langle c_k^*(\beta_1)c_j(\beta_2)\rangle=(1/\mathscr{Z}_F)\mathrm{tr}\{c_k^*\exp(\beta_1-\beta_2)H_F\times c_j\exp(-\beta_1+\beta_2-\beta)H_F\}\,, \tag{5.1}$$

$$\langle c_k^*c_j\rangle=(1/\mathscr{Z}_F)\mathrm{tr}\{c_k^*c_j\exp(-\beta H_F)\}\,,$$

$$\langle c^*_{k_1}\cdots c^*_{k_p}c_{j_1}\cdots c_{j_p}\rangle=\frac{1}{\mathscr{Z}_F}\mathrm{tr}\{c^*_{k_1}\cdots c^*_{k_p}c_{j_1}\cdots c_{j_p}\exp(-\beta H_F)\}\,,$$

the computation of which, according to the standard methods, involves Grassmann algebra functional integrals.

A passage from Fermi variables to Bose variables can be accomplished by first "defermionizing" the system by means of the Jordan–Wigner formulas and then constructing boson approximants for so-received spin-$\frac{1}{2}$ lattice quantities. The procedure is easy for the equal temperature correlations, since then, for example,

$$\langle c_k^*c_{k+1}\rangle=\langle\sigma_k^+\sigma_{k+1}^-\rangle\,, \tag{5.2}$$

while for distant correlations we have

$$\langle c_k^*c_l\rangle=\left\langle\sigma_k^+\left(\exp i\pi\sum_{j=k+1}^{l-1}\sigma_j^+\sigma_j^-\right)\cdot\sigma_j^-\right\rangle=\left\langle\sigma_k^+\prod_{j=k+1}^{l-1}(1-2n_j)\cdot\sigma_j^-\right\rangle\,, \tag{5.3}$$

$$n_j=c_j^*c_j=\sigma_j^+\sigma_j^-\,,$$

which implies that the fermion correlations can be established by using the sequence of boson approximants

$$\begin{aligned}&\langle a_k^*a_l\rangle\,,\quad \langle n_{k+p}a_k^*a_l\rangle\,,\quad k<p<l\,,\\ &\langle n_{k+p}n_{k+q}a_k^*a_l\rangle\,,\quad k<p<q<l\,,\\ &\langle n_{k+p}n_{k+q}a_k^*a_l\rangle\,,\quad k<p<q<r<l,...\,.\end{aligned} \tag{5.4}$$

The underlying boson approximation procedure amounts to modifying the Bose Hamiltonian

$$H_B=PH_BP+(1-P)H_B(1-P)\,,$$

$$H_F=PH_BP\,,\quad P\equiv 1_F\,, \tag{5.5}$$

by adding to it the operator λL

$$H_B\to H_B(\lambda)=H_B+\lambda L;\lambda\gg 1\,,\quad \lambda\in(0,\infty)\,, \tag{5.6}$$

$$L=\sum_j n_j(n_j-1)\,,\quad n_j=a_j^*a_j\,.$$

If we insert $H_B(\lambda)$ in the place of H_B in all the thermal formulas, we realize that because of $PLP=0$ and $\lambda\gg 1$, the spectral problem for $H_B(\lambda)$ is determined by solving the stationary state perturbation problem with the degenerate spectrum for the operator

$$H'_B(\lambda)\doteq(1/\lambda)H_B(\lambda)=L+(1/\lambda)H_B\,, \tag{5.7}$$

where L plays the role of the initial (unperturbed) operator. The eigenvalues of L we denote $l=\Sigma_j n_j(n_j-1)\geqslant 0$ and the respective eigenvectors we denote $|l,\alpha\rangle$, where α enumerates the pairwise orthogonal eigenvectors of L corresponding to the eigenvalue l. In their linear span, we can always find another orthogonal set $\{|l,a\rangle\}$ such that $\langle l,a|H_B|l,a'\rangle=0$, $a\neq a'$ and $|l,\alpha\rangle=\Sigma_a f_{\alpha a}|l,a\rangle$.

The stationary state perturbation theory says then that in the presence of the perturbation, the group of states $\{|l,a\rangle\}$ is replaced by the new group $\{|\mathbf{l},\mathbf{a}\rangle\}$, whose elements in the

first order have the well-known form

$$|\mathbf{l},\mathbf{a}) = |l,a) + \frac{1}{\lambda}\sum_k{}' \frac{|k)(k|H_B|l,a)}{l-k}, \tag{5.8}$$

where Σ' indicates that summations run over all eigenstates of L except for those which form the l th eigenvector.

The (la)th energy level to the second order reads

$$\epsilon'_{la} = l + \frac{1}{\lambda}(l,a|H_B|l,a) + \left(\frac{1}{\lambda}\right)^2 \sum_k{}' \frac{|(k|H_B|l,a)|^2}{l-k}, \tag{5.9}$$

and ϵ'_{la} is the eigenvalue of $H'_B(\lambda)$: $\epsilon_{la} = \lambda\epsilon'_{la}$. Because of (5.8) and $\lambda \gg 1$, the spectrum of $H_B(\lambda)$ is characterized by a large energy gap opening between the lowest ($l = 0$) group of eigenlevels and the others. In the $l = 0$ sector we have $H_B(\lambda) = H_B = PH_BP = H_F$. As a consequence, the replacement of H_B by $H_B(\lambda)$ in the thermal formulas allows us to view the $l = 0$ contribution as dominant, and hence to approximate the thermal characteristics of the spin-$\frac{1}{2}$ (Fermi) system by using the Bose formulas with $H_B(\lambda)$ instead of H_B. The approximation accuracy can be made arbitrarily good with the growth of λ. For $\lambda \gg 1$ there holds

$$[1/\mathscr{Z}_B(\lambda)]\,\mathrm{tr}[\phi_{m_1}(\beta_1)\cdots\phi_{m_k}(\beta_k)\exp(-\beta H_B(\lambda))]$$
$$\cong \frac{1}{\mathscr{Z}_F}\,\mathrm{tr}[\sigma^1_{m_1}(\beta_1)\cdots\sigma^1_{m_k}(\beta_k)\exp(-\beta H_F)], \tag{5.10}$$
$$\beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_k \leqslant \beta.$$

Let us notice that the Jordan–Wigner transformation makes it possible to get a Bose approximation scheme for the genuine Fermi variables (see, e.g., the previous discussion). In the case of the short range order, the simple formula

$$\langle c_k^* c_{k+1}\rangle = \langle \sigma_k^+ \sigma_{k+1}^-\rangle \cong \langle a_k^* a_{k+1}\rangle \tag{5.11}$$

holds true.

[1] J. Kogut, Rev. Mod. Phys. **55**, 755 (1983).
[2] F. Fucito, E. Marinari, G. Parisi, and C. Rebbi, Nucl. Phys. B **180**, 369 (1981).
[3] F. Fucito and E. Marinari, Nucl. Phys. B **190**, 369 (1981).
[4] G. Bhanot and U. Heller, Phys. Lett. B **129**, 440 (1983).
[5] D. Zwanziger, Phys. Rev. Lett. **50**, 1886 (1983).
[6] S. Otto and M. Randeria, Nucl. Phys. B **220**, 479 (1983).
[7] T. Burkitt, Nucl. Phys. B **220**, 4314 (1983).
[8] R. Blankenbecler and R. Sugar, Phys. Rev. D **27**, 1304 (1983).
[9] J. E. Hirsch, R. Sugar, D. Scalapino, and R. Blankenbecler, Phys. Rev. B **26**, 5033 (1983).
[10] P. Garbaczewski, J. Math. Phys. **19**, 642 (1978); **22**, 442 (1982); **24**, 341 (1983).
[11] P. Garbaczewski, Phys. Rep. C **36**, 65 (1978).
[12] J. Dobaczewski, Nucl. Phys. A **369**, 219 (1981).
[13] P. Garbaczewski, J. Math. Phys. **21**, 2670 (1980); **22**, 574 (1981); **24**, 651 (1983).
[14] P. Garbaczewski, J. Math. Phys. **25**, 862 (1984).
[15] D. I. Olive and N. Turok, Nucl. Phys. B **220**, 491 (1983).
[16] P. Goddard and D. I. Olive, "Algebras, lattices and strings," DAMPT83 22, Cambridge preprint.
[17] T. H. R. Skyrme, Proc. R. Soc. London, Ser. A **247**, 260 (1958).
[18] T. H. R. Skyrme, Proc. R. Soc. London, Ser. A **262**, 237 (1961).
[19] R. F. Streater and I. Wilde, Nucl. Phys. B **24**, 561 (1970).
[20] A. Luther and K. D. Schotte, Nucl. Phys. B **242**, 407 (1984).
[21] P. Garbaczewski, *Classical and Quantum Field Theory of Exactly Soluble Nonlinear Systems* (World Scientific, Singapore, 1985).
[22] J. M. Levy-Leblond, Phys. Lett. A **26**, 540 (1968).
[23] H. R. Post, Proc. Phys. Soc. London, Ser. A **66**, 649 (1953).
[24] F. Calogero, J. Math. Phys. **12**, 419 (1971); A. Perelomov, Teor. Mat. Fiz. **6**, 364 (1971).