Quantization of spinor fields. IV. Joint Bose–Fermi spectral problems

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(Received 21 January 1983; accepted for publication 22 July 1983)

In this continued study of the connection between classical c-number spinor models and their quantized Fermi partners, we elaborate further necessary consequences of the *bosonization*. The genuine (c-number) path integral representation of tr $\exp(-iHt)$ is derived for the Fermi oscillator and simple lattice Fermi models. We find that the underlying Hamiltonian of the Fermi system H_F can be equivalently written as PH_BP, where H_B is the related Bose Hamiltonian, P is an appropriate projection in the state space of the Bose system, and $[\mathrm{P}, \mathrm{H}_B]_- = 0$, $H_F = PH_BP$. Grassmann algebras are not used. We prove further that both for the massive Thirring model (MT) and the chiral invariant Gross–Neveu model (CGN), the Bethe ansatz eigenstates for the Fermi Hamiltonians are exact eigenstates of the Bose MT and CGN Hamiltonians, so introduced that $H_F = PH_BP$, $[P, H_B]_- = 0$. As a consequence, through studying the c-number path integral representation for tr $\exp(-iH_Ft)$, we establish a class of classical (c-number) spinor solutions of the underlying field equations, which at the same time make stationary both the c-number Bose and c-number Fermi actions.

PACS numbers: 11.10.Cd

1. MOTIVATION

As is well known, for simple scalar field theory models, the U matrix of the scattering theory in the coherent state representation is identical with the conventional Feynman integral

$$\begin{aligned} U(t,t') &= (\varphi(t')|\exp[-iH(t'-t)]|\varphi(t)) \\ &= (\varphi(0)|\exp(iH_0t)\exp(-iH(t'-t)) \\ &\quad\times \exp(-iH_0t)|\varphi(0)) \\ &= \int\left[\frac{D\varphi}{\sqrt{2\pi}}\right]\exp\left[i\int_t^{t'} d^4x\mathscr{L}(\dot{\varphi},\varphi)\right]. \end{aligned} \tag{1.1}$$

However, in the case of Fermi fields the analogous formula exists only if the *classical* spinor fields to be used in the path integral are elements of the anticommuting function ring (Grassmann algebra). It created a folklore belief that the genuine classical c-number spinor fields have no meaning for the construction and then understanding of the canonical Fermi field theory. In fact, if one would follow the route (1.1) by using the c-number spinor field Lagrangian and the measure with respect to c-number spinor paths, the received result would certainly not coincide with the result known by other (Grassmann algebra) means to be correct for the respective Fermi model.

However, in the preceding series of papers[1–3] (see also Ref. 4), we have demonstrated that it is not meaningless to talk about a connection of classical c-number spinor fields with their quantized Fermi partners. Some other intuitions about a possible quantum meaning of classical c-number spinor fields can be drawn from Refs. 5–8. Our approach to the quantization of spinor fields problems originates from Klauder's[9] idea to avoid the use of Grassmann algebras in describing the canonically quantized Fermi fields, but is different from that work.

We have found previously that a formulation of the correspondence rule between quantized Fermi and classical c-number spinor fields is in principle possible, provided the Fermi model allows for *bosonization*. Different aspects of this problem were studied before in Refs. 1–3. However, a fundamental problem we were confronted with, the quantum meaning of classical c-number spinor models (with the Fermi–Dirac statistics in mind), has remained unsolved.

In the present paper we give a path integral reconstruction of tr $\exp(-iHt)$ in terms of genuine c-number trajectories for simple Fermi models, to prove that it is at all possible without any reference to Grassmann algebras. We do not, however, conclude (as Klauder did) that Grassmann algebras are an unnecessary addition to the mathematical physics. We believe they are extremely useful for explicit perturbative calculations. They suffer, however, from a serious drawback: they do not carry any true physical content, against the naive (but popular) expectations. In the light of the results presented below, it is possible to recover a new face (physical content) of Fermi models, the face which is completely obscure in the Grassmann algebra formulation.

For lattice Fermi models, the Jordan–Wigner transformation allows us to introduce the equivalent spin $\frac{1}{2}$ lattices. The corresponding Hamiltonians can be considered as projected Bose Hamiltonians (in the Hilbert space of the appropriate Bose system),

$$H_F = PH_BP, \tag{1.2}$$

where $[P, H_B]_- = 0$. One should notice that

$$\begin{aligned} 1 = 1_B = P + (1-P) &\Rightarrow \mathrm{tr}\exp(iH_Bt) \\ &= \mathrm{tr}\exp(iH_Ft) + \mathrm{tr}\exp(-(1-P)H_B(1-P)t), \end{aligned} \tag{1.3}$$

and the only problem is to extract the Fermi contribution to the Bose formula, which is known to be provided by using Grassmann algebra methods to compute tr $\exp(-iH_Ft)$.

If there exists a countable family of projections

 0022-2488/84/040862-10$02.50

$$\sum_k P_k = 1, \quad P_k P_l = \delta_{kl} P_k, \quad [P_k, H_B]_- = 0 \quad \forall\ k, \tag{1.4}$$

then

$$\text{tr} \exp(-iH_B t) = \sum_{k=1}^{\infty} \text{tr} \exp(-iH_F^k t),$$
$$H_F^k = P_k H_B P_k, \tag{1.5}$$

and the respective Bose model can be viewed as a (infinitely) reducible Fermi one or even as a *tower* of possibly distinct Fermi models, each one with its own Hamiltonian H_F^k.

For the (continuous) massive Thirring model, we prove that Bethe ansatz eigenvectors of the Fermi Hamiltonian are the exact eigenvectors (with the same eigenvalues) of the Bose massive Thirring model Hamiltonian, see also Ref. 4, i.e.,

$$H_F^{\text{MT}} = PH_B^{\text{MT}}P, \quad [P, H_B^{\text{MT}}]_- = 0. \tag{1.6}$$

Previously,[4] by using indirect (the inverse spectral transform) methods we have concluded that the mass spectrum of bound states is the same for the Bose and Fermi massive Thirring models. If to combine it with our Bethe ansatz observation, we find supported the conjecture of Ref. 4 that the Bose massive Thirring model can be equivalently rewritten as the (infinitely) reducible Fermi one. Next, we deduce the spin $\frac{1}{2}$ type version of H_F^{MT} for which a path integral representation of tr $\exp(-iH_F t)$ is constructed in terms of c-number trajectories. We prove that spinor trajectories such that $\sigma_i = \phi_i^* \phi_i \leqslant A < \infty$, $i = 1,2$, A being arbitrary, give exactly the same contributions to both Bose and Fermi path integrals via the c-number massive Thirring model action. Its stationary *points* are the classical spinor solutions of the respective field equations satisfying $\sigma_i \leqslant \text{A} < \infty$.

Finally the analogous properties are established for the chiral invariant Gross–Neveu model. Our analysis is confined to 1 + 1 dimensions where explicit solutions for the spectral problems are available. We expect, however, that the Bose–Fermi interplay described here will prove useful in 1 + 3 dimensions as well, see, e.g., also Ref. 2.

2. QUANTUM OSCILLATOR PROBLEM: BOSE VERSUS FERMI

The Bose oscillator

$$\hat{L}_B = ib^*(t)\dot{b}(t) - \omega b^*(t)b(t) = b^*(t)\left(i\frac{d}{dt} - \omega\right)b(t) \tag{2.1}$$

is determined by using the CCR algebra generators (equal time variables omitted)

$$[b, b^*]_- = 1_B, \quad b|0\rangle = 0. \tag{2.2}$$

The Hamiltonian reads

$$\hat{h}_B = \omega b^* b; \tag{2.3}$$

hence the infinitesimal propagator of the model is given by

$$\hat{U}_B(\Delta t) = \exp(-i\Delta t \hat{h}_B) \cong 1 - i\Delta t \hat{h}_B. \tag{2.4}$$

In the coherent state representation, the infinitesimal kernel[10] of $\hat{U}_B(t)$ reads

$$U_B(\Delta t) = \exp(\beta^*\beta - ih_B^{\text{cl}}\Delta t),$$
$$h_B^{\text{cl}} = \langle\beta|\hat{h}_B|\beta\rangle = \omega\beta^*\beta, \tag{2.5}$$
$$|\beta\rangle = \exp(\beta b^* - \beta^* b)|0\rangle,$$

so that through the standard arguments,[11] we arrive at the following (formal continuum t limit) path integral representation of tr $\exp(-i\hat{h}_B t)$:

$$I_B = \text{tr} \exp(-i\hat{h}_B t) = \int [d\beta][d\beta^*] \exp i \int_0^t \{i\beta^*(t)\dot{\beta}(t) - \omega\beta^*(t)\beta(t)\}dt = \int [d\beta][d\beta^*] \exp i \int_0^t L_B(t)dt. \tag{2.6}$$

One should realize that I_B is given with the accuracy up to the normalization factor.

Let us define the Fermi oscillator

$$\hat{L}_F = a^*(t)\left(i\frac{d}{dt} - \omega\right)a(t) \tag{2.7}$$

by using the Fock representation of the CAR algebra, which is completely embedded in the CCR algebra as follows:

$$[a, a^*]_+ = 1_F, \quad a^*|0\rangle = |1\rangle = b^*|0\rangle, \quad a|0\rangle = 0,$$
$$a^{*2} = 0 = a^2,$$
$$a^* = b^*{:}\exp(-b^*b){:}, \quad a = {:}\exp(-b^*b){:}a, \tag{2.8}$$
$$1_F = {:}\exp(-b^*b){:} + b^*{:}\exp(-b^*b){:}b.$$

The CAR generators act invariantly on a proper subspace $h_F = 1_F h$ of the Bose oscillator Hilbert space, but nevertheless, allow a trivial extension to the whole of h. Consequently the Fermi propagator $\hat{U}_F(\Delta t) = \exp(-i\Delta t\hat{h}_F)$ can be represented in the Hilbert space of the Bose oscillator, thus allowing one to follow the previous path integration route. We have

$$\hat{U}_F(\Delta t) = \exp(-i\hat{h}_F\Delta t) \cong 1_F - i\Delta t\hat{h}_F = 1_F(1 - i\Delta t\hat{h}_B)1_F$$
$$= {:}\exp(-b^*b){:} + b^*{:}\exp(-b^*b){:}b - i\omega\Delta t b^*{:}\exp(-b^*b){:}b, \tag{2.9}$$

so that the infinitesimal kernel reads

$$U_F(\Delta t) \cong \langle\beta|1_F - i\Delta t\hat{h}_F)|\beta\rangle \cdot \exp(\beta^*\beta)$$
$$= 1 + \beta^*\beta - i\omega\Delta t\beta^*\beta = (1 + \beta^*\beta)[1 - i\omega\Delta t\beta^*\beta/(1 + \beta^*\beta)]$$
$$\cong \exp[\ln(1 + \beta^*\beta) - i\omega\Delta t\beta^*\beta/(1 + \beta^*\beta)]. \tag{2.10}$$

Furthermore

$$I_F = \mathrm{tr}\exp(-i\hat{h}_F t) = \int [d\beta][d\beta^*] \exp i \int_0^t \frac{i\beta^*\dot{\beta} - \omega\beta^*\beta}{1+\beta^*\beta} dt = \int [d\beta][d\beta^*] \exp i \int_0^t dt \frac{L_B(t)}{1+\beta^*(t)\beta(t)}, \tag{2.11}$$

which is a c-number alternative for the usual,[6] Grassmann algebra path integral.

Remark: The semiclassical quantization procedure for the continuous Heisenberg system[12] resulted in the introduction of the spin path integral with respect to genuine (i.e., c-number, non-Grassmann) paths in the phase space of the classical spin system. The genuine c-number path notion is also inherent in the approach of Refs. 13–16 based on the SU(2) phase variables, and making use[14–16] of spin coherent states, these being well known in the many-body physics. The general method of Ref. 16 to construct measures for spin variable path integrals can be immediately adopted to either a single spin $\frac{1}{2}$ or to the many-body spin $\frac{1}{2}$ problem.

Attempts to introduce probabilistic ideas (i.e., probabilistic measures) to the study[17,18] of Fermi fields start from an appealing assumption. Consider the classical harmonic oscillator problem. View its equations of motion as stochastic differential equations and then add information that one is considering a two-level Fermi system instead of the ordinary Bose one. It results in specifying the class of stochastic processes in which solutions of the would-be classical oscillator equations of motion are to be found. The underlying processes are Markov processes with values in Z_2 which demonstrates that, except for the form of the equations of motion, the *classical* paths of the Fermi system are as unrelated to the Bose oscillator paths as the Grassmann algebra *paths* would be. An analogous line is followed in Ref. 19, though in a different Poisson processes framework.

3. SPIN $\frac{1}{2}$ LATTICES, LATTICE FERMIONS, AND c-NUMBER PATH INTEGRALS

It is well known that at least in $1+1$ dimensions, the lattice Fermi systems can be equivalently described as lattices of spins $\frac{1}{2}$, and conversely. It happens so due to the Jordan–Wigner (JW) transformation, realizing fermions as *strings* of spins $\frac{1}{2}$ in the linear chain. An easy example is here in the Ising model in $1+1$ dimensions, whose Hamiltonian,

$$H = J\sum_i \sigma_i^x \sigma_{i+1}^x, \tag{3.1}$$

after making the JW transformation, and then Fourier transforming the image Fermi variables, goes over to the one[20] which can be unitarily transformed into

$$\begin{aligned} H &= \sum_i \epsilon_q(\xi_q^*\xi_q - \tfrac{1}{2}), \\ \epsilon_q &= \cosh^{-1}\{\cosh 2(J-J') \\ &\quad + (1-\cos q)\sinh 2J' \sinh 2J\}, \\ J' &= \tanh^{-1}\exp(-2J), \end{aligned} \tag{3.2}$$

with

$$\begin{aligned} &[\xi_q, \xi_{q'}^*]_+ = \delta_{qq'}, \quad [\xi_q, \xi_{q'}]_+ = 0, \\ &q = (2\pi/N)p, \quad p = 0, \pm 1..., \pm (N-2)/2, N/2. \end{aligned} \tag{3.3}$$

Since $\xi_q^*\xi_q$ is a particle number operator of the qth mode with eigenvalues 0,1, we can replace it by the equivalent one given in terms of Pauli operators

$$\begin{aligned} &\xi_q^*\xi_q \equiv \sigma_q^+\sigma_q^- = b_q^*{:}\exp(-b_q^*b_q){:}\, b_q, \\ &[b_q, b_{q'}^*]_- = \delta_{qq'}, \quad [b_q, b_{q'}]_- = 0, \\ &b_q|0\rangle = 0 = \xi_q|0\rangle = \sigma_q^-|0\rangle \quad \forall\ q, \\ &b_q^*|0\rangle = \xi_q^*|0\rangle = \sigma_q^+|0\rangle. \end{aligned} \tag{3.4}$$

The operator unit contributing the $\frac{1}{2}$ term to H equals $1_F = \Pi_q 1_q$, with $1_q = {:}\exp(-b_q^*b_q){:} + b_q^*{:}\exp(-b_q^*b_q){:}\, b_q$. The Ising Hamiltonian acquires thus the Bose form

$$\begin{aligned} H &= \sum_q \frac{\epsilon_q}{2}[b_q^*{:}\exp(-b_q^*b_q){:}\, b_q - {:}\exp(-b_q^*b_q){:}] \\ &= \sum_q H_q, \\ &[H_q, H_{q'}]_- = 0. \end{aligned} \tag{3.5}$$

Consequently, the infinitesimal propagator

$$\begin{aligned} \hat{U}_F(\Delta t) &= \exp(-iH\Delta t) \\ &\cong \prod_q (1_q - i\Delta t H_q) = 1_F - i\Delta t H \\ &\cong \prod_q \exp(-iH_q\Delta t), \\ 1_F &= \prod_q 1_q \end{aligned} \tag{3.6}$$

allows for the following infinitesimal kernel:

$$\begin{aligned} U_F(\Delta t) &\cong \Big(\exp\sum_q \beta_q^*\beta_q\Big)\cdot\langle\beta|\prod_q(1_q - i\Delta t H_q)|\beta\rangle \\ &= \prod_q \left\{(1+\beta_q^*\beta_q)\left[1 - \frac{i\Delta t\epsilon_q}{2}\frac{\beta_q^*\beta_q - 1}{1+\beta_q^*\beta_q}\right]\right\} \\ &\cong \exp\sum_q\left[\ln(1+\beta_q^*\beta_q) - \frac{i\Delta t \epsilon_q}{2}\frac{\beta_q^*\beta_q - 1}{1+\beta_q^*\beta_q}\right], \end{aligned} \tag{3.7}$$

where

$$|\beta\rangle = \exp\sum_q(\beta_q b_q^* - \beta_q^* b_q)|0\rangle, \tag{3.8}$$

and so

$$I_F = \mathrm{tr}\exp(-iHt)$$
$$= \int [d\beta][d\beta^*]$$
$$\times \exp i \int_0^t \sum_q \frac{i\beta_q^*\dot{\beta}_q - (\epsilon_q/2)(\beta_q^*\beta_q - 1)}{1+\beta_q^*\beta_q} dt. \quad (3.9)$$

The lattice Fermi model of Ref. 21,

$$H_F = -i\sum_n \{\psi_n^*\psi_{n+1} - \psi_{n+1}^*\psi_n\},$$
$$[\psi_n, \psi_m]_+ = 0, \quad [\psi_n, \psi_m^*]_+ = \delta_{nm}, \quad (3.10)$$
$$\dot{\psi}_n = -\tfrac{1}{2}(\psi_{n+1} - \psi_{n-1}),$$

can be equivalently rewritten in terms of lattice spins $\frac{1}{2}$:

$$H_F = -\frac{1}{2}\sum_n \{\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+\}. \quad (3.11)$$

However, a straightforward application of the previous *bosonization* recipe to the spin $\frac{1}{2}$ generators $\sigma_m^\pm$ is not efficient at all. It is much more reliable to make first a lattice Fourier transformation of the original Fermi variables,

$$\psi_n = N^{-1/2}\sum_q c_q \exp(iqn),$$
$$q = (2\pi/N)p, \quad p = 0, \pm 1, \ldots, \pm (N-2)/2, \; N/2, \quad (3.12)$$

where the familiar lattice identity (N is large)

$$\frac{1}{N}\sum_q \exp[-iq(n-m)] = \delta_{nm} \quad (3.13)$$

guarantees that both ψ_n, ψ_n^*, and c_q, c_q^* are the CAR algebra generators, and moreover

$$H_F = -\frac{i}{2}\sum_n \frac{1}{N}$$
$$\times \sum_{p,q} \{e^{-iqn}e^{i(n+1)p}c_q^*c_p - e^{ipn}e^{-iq(n+1)}c_q^*c_p\}$$
$$= -\frac{i}{2}\sum_{q,p}\left(\frac{1}{N}\sum_n e^{-in(q-p)}\right)[e^{ip}c_q^*c_p - e^{-iq}c_q^*c_p]$$
$$= -\frac{i}{2}\sum_q [e^{iq}c_q^*c_q - e^{-iq}c_q^*c_q] = \sum_q (\sin q)c_q^*c_q. \quad (3.14)$$

Now the previous route applies apparently, and then

$$I_F = \mathrm{tr}\exp(-iH_F t)$$
$$= \int [d\beta][d\beta^*]$$
$$\times \exp i\int_0^t \sum_q \frac{i\beta_q^*\dot{\beta}_q - (\sin q)\beta_q^*\beta_q}{1+\beta_q^*\beta_q} dt. \quad (3.15)$$

The path integral formulation for simplest Fermi models on the lattice, in terms of genuine c–number paths is thus possible, though certainly not well suited for less trivial examples, like, e.g., the lattice version of the massive Thirring model,[22]

$$H_{\mathrm{MT}} = \sum_{n=-N+1}^{N} \left\{\frac{iv}{2a}(\psi_n^*\psi_{n+1} - \psi_{n+1}^*\psi_n)\right.$$
$$+ (-1)^n \frac{m}{2}(\psi_n^*\psi_{n+1}^* + \psi_{n+1}\psi_n)$$
$$\left. - \frac{g}{2a}\left(\psi_n^*\psi_n - \frac{1}{2}\right)\left(\psi_{n+1}^*\psi_{n+1} - \frac{1}{2}\right)\right\} - E_0, \quad (3.16)$$

a being the lattice spacing. By using the JW transformation, one arrives at the equivalent spin $\frac{1}{2}$ *xyz* model Hamiltonian (with cyclic boundary conditions),

$$H_{\mathrm{MT}} = -\frac{1}{2}\sum_n \left\{\left(\frac{v}{2a} + \frac{m}{2}\right)\sigma_n^x\sigma_{n+1}^x\right.$$
$$\left. + \left(\frac{v}{2a} - \frac{m}{2}\right)\sigma_n^y\sigma_{n+1}^y + \frac{g}{2a}\sigma_n^z\sigma_{n+1}^z\right\} - E_0, \quad (3.17)$$

which is also

$$H_{\mathrm{MT}} = 1_F H_{\mathrm{MT}}^B 1_F,$$
$$H_{\mathrm{MT}}^B = -\frac{1}{2}\sum_n \left\{\frac{v}{2a}(b_n^*b_{n+1} + b_{n+1}^*b_n)\right.$$
$$+ \frac{m}{2}(b_n^*b_{n+1}^* + b_nb_{n+1})$$
$$\left. - \frac{g}{2a}\left(b_n^*b_n - \frac{1}{2}\right)\left(b_{n+1}^*b_{n+1} - \frac{1}{2}\right)\right\} - E_0,$$
$$1_F = \prod_n [:\exp(-b_n^*b_n): + b_n^*:\exp(-b_n^*b_n):b]$$
$$= \prod_n P_n. \quad (3.18)$$

Here H_B is considered as an operator in the Hilbert space $\mathscr{H}_N = h^{\otimes 2N}$, while H_F in $1_F\mathscr{H}_N = h_F^{\otimes 2N}$, $h_F = Ph$. Because 1_F is a projection operator in $\mathscr{H}_N$, we have

$$H_B = H_F + (1-1_F)H_B 1_F + 1_F H_B(1-1_F)$$
$$+ (1-1_F)H_B(1-1_F), \quad (3.19)$$

and consequently for any $|\psi\rangle = 1_F|\psi\rangle \in \mathscr{H}_N$ we find

$$H_B|\psi\rangle = H_F|\psi\rangle + (1-1_F)H_B|\psi\rangle. \quad (3.20)$$

Hence the necessary and sufficient condition to fulfill

$$H_B|\psi\rangle = H_F|\psi\rangle, \quad 1_F|\psi\rangle = |\psi\rangle \quad (3.21)$$

is that

$$H_B|\psi\rangle \in 1_F\mathscr{H}_N \quad \text{if } |\psi\rangle \in 1_F\mathscr{H}_N. \quad (3.22)$$

Then if $H_F|\psi\rangle = \epsilon|\psi\rangle$ we get automatically $H_B|\psi\rangle = \epsilon|\psi\rangle$ and conversely (provided $1_F|\psi\rangle = |\psi\rangle$).

Since we work in the Hilbert space $\mathscr{H}_N$, the respective spectra are discrete, and $\mathscr{H}_N$ is spanned by a complete (countable) eigenfunction system of H_B. 1_F is a projection in $\mathscr{H}_N$ and $1_F\mathscr{H}_N$ is spanned by these eigenvectors of H_B which obey $1_F|\psi\rangle = |\psi\rangle$. Notice that if $H_F|\psi\rangle = \epsilon|\psi\rangle$, then necessarily

$$H_B|\psi\rangle = H_B \sum_{(\alpha)} a_\alpha |\psi,\alpha\rangle = \sum_{(\alpha)} a_\alpha \epsilon_\alpha |\psi,\alpha\rangle$$

$$= H_F|\psi\rangle = \epsilon|\psi\rangle = \epsilon \sum_{(\alpha)} a_\alpha |\psi,\alpha\rangle,$$

$$\text{i.e., } \epsilon = \epsilon_\alpha \quad \forall \ \alpha \in (\alpha). \tag{3.23}$$

It means that once any H_B is obtained for which $H_F = 1_F H_B 1_F$ for a given Fermi model H_F, then automatically the eigenfunction system of H_F is a subsystem in this of H_B, and moreover the respective eigenvalues of H_B and H_F do coincide. It then follows that

$$\operatorname{tr} \exp(-iH_B t) = \operatorname{tr} \exp(-iH_F t) + \operatorname{tr} \exp[-(1-1_F)H_B(1-1_F)]. \tag{3.24}$$

In the N-particle linear chain, the operator $1_F = \Pi_n P_n$ is composed from projections on two lowest levels in each single site Schrödinger problem. Obviously, there is an infinity of other choices. In particular, if to imagine a sequence

$$\left\{ 1_F^k = \prod_n P_n^k \quad \text{such that} \quad \sum_k 1_F^k = 1, \ 1_F^k \cdot 1_F^l = \delta_{kl} 1_F^k, \right.$$

then

$$\operatorname{tr} \exp(-iH_B t) = \sum_k \operatorname{tr} \exp(-iH_F^k t), \qquad H_F^k = 1_F^k H_B 1_F^k, \tag{3.25}$$

provided $[H_B, 1_F^k]_- = 0 \ \forall \ k = 1,2,\ldots$. In principle any two states at each site can be used for our construction. However, the need for the last commutation rule may play the role of the constraint capable of removing a nonuniqueness appearing in the construction of 1_F. The Bose system may thus happen to be equivalent either to a reducible Fermi one, or to the whole *tower* of might-be-distinct Fermi systems. A speculation on the possibly infinite *tower* of fundamental, say, leptons generated by the relatively simple Bose system, does not seem to be hopeless.

Let us mention that the expansion of the Bose trace into a sum of Fermi traces resembles the expansions which appear in the path integral quantization of spin systems.[13] Then the Green's function is known to propagate all spins simultaneously. Nevertheless the recovery of the usual Pauli spinors is possible by projecting to a specific spin subspace which is propagated into itself. One deals then with a Bose system, which describes a *particle* allowed to live in several (infinity in fact) spin states.

To compute a particular Fermi trace, we may not refer to the original (Bose) phase-space variables, but instead we can use the conventional Grassmann algebra tools. They prove to be successful, indeed, once remaining in the particular Fermi sector of the Bose model.

4. MASSIVE THIRRING MODEL: FROM FERMIONS TO BOSONS

A spectral problem for the Fermi model,

$$H_F = \int dx[-i(\psi_1^*\partial_x\psi_1 - \psi_2^*\partial_x\psi_2) + m(\psi_1^*\psi_2 + \psi_2^*\psi_1) + 2g\psi_1^*\psi_2^*\psi_2\psi_1], \tag{4.1}$$

has been solved by means of the Bethe ansatz in Ref. 23. One assumes to work with a Fock representation of the CAR algebra,

$$[\psi_i(x), \psi_j^*(y)]_+ = \delta_{ij}\delta(x-y),$$
$$[\psi_i(x), \psi_j(y)]_+ = 0, \quad \psi_i(x)|0\rangle = 0,$$
$$\forall \ i = 1,2, \ x \in R^1. \tag{4.2}$$

Then after introducing

$$|\alpha_1,\ldots,\alpha_n\rangle = \int dx_1 \cdots \int dx_n \chi(x_1,\ldots,x_n,\alpha_1,\ldots,\alpha_n) \times \prod_{i=1}^n \psi^*(x_i,\alpha_i)|0\rangle,$$

$$\chi(x_1,\alpha) = \exp\left(im\sum_i x_i \sinh\alpha_i\right) \times \prod_{1<i<j<n} [1 + i\lambda(\alpha_i,\alpha_j)\epsilon(x_i - x_j)], \tag{4.3}$$

$$\psi(x,\alpha) = e^{\alpha/2}\psi_1(x) + e^{-\alpha/2}\psi_2(x),$$
$$\lambda(\alpha_i,\alpha_j) = -\tfrac{1}{2} g \tanh \tfrac{1}{2}(\alpha_i - \alpha_j),$$

one demonstrates (see, e.g., the Appendix of Ref. 23) that

$$H_F|\alpha_1,\ldots,\alpha_n\rangle = \left(\sum_i m\cosh\alpha_i\right)|\alpha_1,\ldots,\alpha_n\rangle. \tag{4.4}$$

Let us now assume that the fields entering H_F satisfy not the CAR algebra relations (4.2), but the CCR algebra ones,

$$H_F \to H_B = H_F(\psi \to \phi),$$
$$[\phi_i(x), \phi_j^*(y)]_- = \delta_{ij}\delta(x-y), \quad [\phi_i(x), \phi_j(y)]_- = 0, \tag{4.5}$$

provided the CAR algebra (4.2) is constructed in the Fock representation of the CCR algebra, according to Ref. 24. It means that if to introduce the antisymmetric function of Ref. 24,

$$\sigma^3 = \sigma, \quad x_k \in R^1, \quad i_k = 1,2, \quad \forall \ k = 1,2,\ldots,n,$$
$$\sigma = \sigma(x_1,i_1,\ldots,x_j,i_j,\ldots,x_k,i_k,\ldots,x_n,i_n)$$
$$= -\sigma(\ldots x_k,i_k,\ldots,x_j,i_j,\ldots), \tag{4.6}$$

where $\sigma(x_1,i_1,\ldots,x_n,i_n) = 0, \ \pm 1$ depending on the choice of the (x,i) sequence; then the operator

$$\psi_k(x) = \sum_n \frac{\sqrt{1+n}}{n!} \times \sum_{i_1,\ldots,i_n} \int dy_1 \cdots \int dy_n \sigma(y_1,i_1,\ldots,y_n,i_n)$$
$$\times \sigma(x,k,y_1,i_1,\ldots,y_n,i_n) \cdot \phi_{i_1}^*(y_1) \cdots \phi_{i_n}^*(y_n).$$
$$\times : \exp\left\{-\sum_{i=1}^2 \int dz \phi_i^*(z)\phi_i(z)\right\}:$$
$$\times \phi_k(x)\phi_{i_1}(y_1)\ldots\phi_{i_n}(y_n), \quad x \in R^1, \quad k = 1,2 \tag{4.7}$$

is the CAR algebra generator of (4.2), and together with its adjoint, obeys

$$\phi_i^*(x)|0\rangle = \psi_i^*(x)|0\rangle, \quad \phi_i(x)|0\rangle = 0 = \psi_i(x)|0\rangle \quad \forall\ i, x,$$

$$\psi_{i_1}^*(x_1)\cdots\psi_{i_n}^*(x_n)|0\rangle = \sigma(x_1, i_1,..., x_n, i_n)\phi_{i_1}^*(x_1)\cdots\phi_{i_n}^*(x_n)|0\rangle,$$

$$[\psi_i(x), \psi_j^*(y)]_+ = \delta_{ij}\delta(x-y)1_F, \tag{4.8}$$

with

$$1_F = \sum_n \frac{1}{n!}\sum_{i_1,...,i_n}\int dx_1\cdots\int dx_n \sigma^2(x_1, i_1,..., x_n, i_n) \times\phi_{i_1}^*(x_1)\cdots\phi_{i_n}^*(x_n) : \exp\left\{-\sum_{i=1}^2\int dz\phi_i^*(z)\phi_i(z)\right\} : \times\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n), \tag{4.9}$$

which is a projection in the Hilbert space of the Bose system selecting the Fermi subspace in it.

Consequently the Bethe ansatz states for the original Fermi model can be generated from the vacuum by using the Bose operators. We have

$$[e^{\alpha_1/2}\psi_1^*(x_1) + e^{-\alpha_1/2}\psi_2^*(x_1)]\cdots [e^{\alpha_n/2}\psi_1^*(x_n) + e^{-\alpha_n/2}\psi_2^*(x_n)]|0\rangle = \sum_{i_1\cdots i_n}\Big\{\sigma(x_1, i_1,..., x_n, i_n)\phi_{i_1}^*(x_1)\cdots\phi_{i_n}^*(x_n)|0\rangle \times \exp\sum_{k=1}(-1)^{i_k+1}\alpha_k/2\Big\}, \tag{4.10}$$

where $i_k = 1,2$, and obviously

$$\prod_{k=1}^n [e^{\alpha_k/2}\psi_1^*(x_k) + e^{-\alpha_k/2}\psi_2^*(x_k)]|0\rangle = \sum_{i_1,...,i_n}\left[\exp\sum_{k=1}^n(-1)^{i_k+1}\alpha_k/2\right]\prod_{k=1}^n\psi_{i_k}^*(x_k)|0\rangle. \tag{4.11}$$

Hence

$$|\alpha_1,..., \alpha_n\rangle = \int dx_1\cdots\int dx_n\chi(x_1,...,x_n, \alpha_1,..., \alpha_n) \times\sum_{i_1\cdots i_n}\sigma(x_1i_1,..., x_n, i_n)\left[\exp\sum_k(-1)^{i_k+1}\alpha_k/2\right] \times\phi_{i_1}^*(x_1)\cdots\phi_{i_n}^*(x_n)|0\rangle = \int dx_1\cdots\int dx_n\sum_{\{i\}}\eta(x, \alpha, i)\phi_{i_1}^*(x_1)\cdots\phi_{i_1}^*(x_n)|0\rangle, \tag{4.12}$$

with

$$\eta(x, \alpha, i) = \chi(x,\alpha)\sigma(i,x)\exp\sum_{k=1}^n(-1)^{i_k+1}\alpha_k/2. \tag{4.13}$$

In Ref. 4 we have proved via the inverse spectral transform method, a close connection between the Bose and Fermi versions of the massive Thirring model. To make this connection more explicit, we shall demonstrate that vectors (4.13) are eigenvectors of the Hamiltonian H_B arising as $H_B = H_F(\psi^*\rightarrow\phi^*, \psi\rightarrow\phi)$. For the kinetic term of H_B, $\Sigma_{i=1}^2\phi_i^*(x)\partial_x\phi_i(x)(-1)^{i+1}$ by commuting it through the product of ϕ^*'s, and then integrating by parts we arrive at

$$H_B^{\rm kin}|\alpha_1,..., \alpha_n\rangle = \int dx_1\cdots\int dx_n\sum_{i_1\cdots i_k}\times\sum_{k=1}^n\Big\{(-1)^{i_k+1}[(-i\nabla_k\chi)\sigma + \chi(-i\nabla_k\sigma)]\exp\Big(\sum_{j=1}^n(-1)^{i_j+1}\alpha_j/2\Big) \times\phi_{i_1}^*(x_1)\cdots\phi_{i_n}^*(x_n)\Big\}|0\rangle$$
$$= \int dx_1\cdots\int dx_n\sum_{k=1}^n(-i\nabla_k\chi)\psi^*(\alpha_1, x_1)\cdots \times(e^{\alpha_k/2}\psi_1^*(x_k) - e^{-\alpha_k/2}\psi_2^*(x_k))\cdots\psi^*(\alpha_n, x_n)|0\rangle$$
$$= \int dx_1\cdots\int dx_n\chi(x_1,..., x_n, \alpha_1,..., \alpha_n)\sum_{i_1\cdots i_n}\times\sum_{k=1}^n(-1)^{i_k+1}(-i\nabla_k\sigma) \times\Big(\exp\sum_{j=1}^n(-1)^{i_j+1}\alpha_j/2\Big)\phi_{i_1}^*(x_1)\cdots\phi_{i_n}^*(x_n)|0\rangle$$
$$= H_F^{\rm kin}|\alpha_1,..., \alpha_n\rangle + |A\rangle. \tag{4.14}$$

An exact form of $H_F^{\rm kin}|\alpha_1,... \alpha_n\rangle$ was given in Ref. 23, and we shall concentrate on the spurious term $|A\rangle$ in $H_B^{\rm kin}|\alpha_1,..., \alpha_n\rangle$. By virtue of (4.6), we have formally

$$(-i\nabla_k)\sigma(x_1i_1,..., x_n, i_n) = (-i\nabla_k)\sigma^{2n+1}(x_1i_1,..., x_n, i_n) = (2n+1)\sigma^{2n}(-i\nabla_k)\sigma = (2n+1)\sigma^2(-i\nabla_k)\sigma = (-i\nabla_k)\sigma, \quad n = 1,2,..., \tag{4.15}$$

which holds true for all integers and all possible choices of sequences $\{(x,i)\}$ while inserted in (4.14). Hence the identity (4.15) can be satisfied if and only if (up to a set of measure zero)

$$(-i\nabla_k)\sigma(x_1i_1,..., x_n, i_n) \equiv 0, \quad \forall\ k = 1,2,..., n. \tag{4.16}$$

It however means that $|A\rangle = 0$ which proves the property

$$H_B^{\rm kin}|\alpha_1,..., \alpha_n\rangle = H_F^{\rm kin}|\alpha_1,..., \alpha_n\rangle. \tag{4.17}$$

Remark: As an example of $\sigma = \sigma^3$ used in the above, one can take

$$\sigma(x_1,i_1,..., x_n, i_n) = \prod_{1\leqslant j<k\leqslant n} p_{jk}, \quad p_{jk} = \delta_{i_ji_k}[\Theta(x_j - x_k) - \Theta(x_k - x_j)] + |i_k - i_j|(-1)^{\Theta(x_j - x_k)}, \tag{4.18}$$

where $\Theta(x-y) = 1$, $x\geqslant y$, 0, otherwise.

For $i_j = i_k$ we have

$$p_{jk} = \Theta(x_j - x_k) - \Theta(x_k - x_j), \tag{4.19}$$

which equals either 0 or ± 1.

For $i_j \neq i_k$ we arrive at

$$p_{jk} = (-1)^{\Theta(x_j - x_k)}, \tag{4.20}$$

which equals ± 1. Consequently $p_{jk} = 0, \pm 1$ and in addition to the manifest antisymmetry property, $(i_jx_j)\leftrightarrow(i_kx_k)\Rightarrow\sigma\rightarrow-\sigma$, we have satisfied the property $\sigma^3 = \sigma$ as required by our previous definition. Notice that σ vanishes if and only if a pair (i, x) appears more than once in the sequence $\{(i, x)\}$.

For the mass term of H_B we get

$$H_B^m = m \int dx [\phi_1^*(x)\phi_2(x) + \phi_2^*(x)\phi_1(x)],$$

$$H_B^m|\alpha_1,\dots,\alpha_n\rangle$$
$$= m\int dx_1 \cdots \int dx_n \chi(x_1,\dots,x_n,\alpha_1,\dots,\alpha_n)$$
$$\times \sum_{i_1\cdots i_n} \sigma(x_1,i_1,\dots,x_n,i_n)\exp\Big(\sum_{j=1}^n(-1)^{i_j+1}\alpha_j/2\Big)$$
$$\times\Big[\sum_{k=1}^n \exp(-1)^{i_k}\alpha_k\Big]\phi_{i_1}^*(x_1)\cdots\phi_{i_n}^*(x_n)|0\rangle$$
$$= m\int dx_1\cdots\int dx_n\chi(x_1,\dots,x_n,\alpha_1,\dots,\alpha_n)$$
$$\times\sum_{k=1}^n [\psi^*(\alpha_1,x_1)\cdots\{e^{-\alpha_k/2}\psi_1^*(\alpha_k,x_k)$$
$$+ e^{\alpha_k/2}\psi_2^*(\alpha_k,x_k)\}\cdots\psi^*(\alpha_n,x_n)]|0\rangle$$
$$= H_F^M|\alpha_1\cdots\alpha_n\rangle, \tag{4.21}$$

and for the interaction term

$$H_B^{\rm int}|\alpha_1,\dots,\alpha_n\rangle$$
$$= 2g\int dx_1\cdots\int dx_n\chi(x_1,\dots,x_n,\alpha_1,\dots,\alpha_n)$$
$$\times\sum_{i_1\cdots i_n}\Big[\exp\Big(\sum_{j=1}^n(-1)^{i_j+1}\alpha_j/2\Big)\Big]$$
$$\times\sum_k\sum_{l\neq k}\delta(x_k-x_l)[\delta_{1i_k}\delta_{2i_l}+\delta_{1i_l}\delta_{2i_k}]$$
$$\times\psi_{i_1}^*(x_1)\cdots\psi_{i_n}^*(x_n)|0\rangle = 4g\int dx_1\cdots$$
$$\int dx_n\chi(x_1,\dots,x_n,\alpha_1,\dots,\alpha_n)$$
$$\times\sum_k\sum_{l\neq k}\delta(x_k-x_l)(-1)^{k-l+1}\epsilon(k-l)$$
$$\times\sinh\tfrac{1}{2}(\alpha_k-\alpha_l)\psi_1^*(x_k)\psi_2^*(x_l)$$
$$\times\psi^*(\alpha_1,x_1)\cdots\psi^*(\alpha_n x_n)|0\rangle = H_F^{\rm int}(\alpha_1,\dots,\alpha_n), \tag{4.22}$$

and by virtue of Ref. 23 we thus arrive at

$$(H_F^{\rm kin}+H_F^m+H_F^{\rm int})|\alpha_1,\dots,\alpha_n\rangle = H_F|\alpha_1,\dots,\alpha_n\rangle$$
$$= H_B|\alpha_1,\dots,\alpha_n\rangle. \tag{4.23}$$

Needless to say,

$$1_F|\alpha_1,\dots,\alpha_n\rangle = |\alpha_1,\dots,\alpha_n\rangle, \tag{4.24}$$

i.e., indeed

$$H_F = 1_F H_B 1_F, \quad [H_B, 1_F]_- = 0 \tag{4.25}$$

holds true for the continuum field theory, the massive Thirring model.

Let us notice that in contrast to lattice Fermi models we have not made any explicit transformation of Fermi variables into spin $\frac{1}{2}$ type variables. The formula (4.24) proves that such a transformation exists.

Remark 1: By Ref. 4 the spectrum of H_B and H_F is the same in the physical Hilbert space (which is not the Fock space), but the spectrum of H_B appears as infinitely degenerate. As a conjecture we have suggested that the Bose massive Thirring model can be rewritten as a reducible Fermi one.

From the heuristic point of view we find evidence that

$$H_B = \sum_{k=1}^\infty 1_F^k H_B 1_F^k = \sum_{k=1}^\infty H_F^k,$$
$$1_F^k 1_F^l = \delta_{kl}1_F^k, \quad \mathscr{H}_B = \bigoplus_{k=1}^\infty \mathscr{H}_F^k, \quad \mathscr{H}_F^k = 1_F^k\mathscr{H}_B, \tag{4.26}$$
$$\operatorname{tr}\exp(-iH_Bt) = \sum_{k=1}^\infty \operatorname{tr}\exp(-iH_F^kt).$$

Remark 2: A relationship between the c-number (classical) and Fermi massive Thirring models is thus established as follows.

(1) Take $H_B \equiv H_F(\phi^*,\phi)$ and a coherent state of the Bose spinor field $|\varphi\rangle$ such that $\langle\varphi|\phi|\varphi\rangle = \langle 0|\phi+\varphi|0\rangle = \varphi$ is a classical c-number solution of the MT model field equations.

(2) Construct a separable Hilbert space IDPS($|\varphi\rangle$) by using the CCR algebra generators $\{\phi^*,\phi\}$.

(3) Check (see Refs. 3 and 25) whether $|\varphi\rangle$ allows for the existence of Fermi states in IDPS($|\varphi\rangle$), i.e., that $1_F|\psi\rangle = |\psi\rangle \in$ IDPS($|\varphi\rangle$). If so, then look for eigenvectors of H_B which are Fermi vectors, they are then the eigenvectors of H_F. So the Fermi model appears and the (irreducible) Fermi fields can be introduced.

Remark 3: One must realize that Fermi states of the Bose system are allowed to exist in a very restrictive subset of the set of all non-Fock sectors of the Bose model.[3,25] This restriction follows from the assumption that Fermi states are admitted to arise in Hilbert spaces (in fact in the incomplete direct product ones of von Neumann) which are generated about coherent states of the Bose system. The latter are necessary to satisfy the (weak) correspondence principle

$$\langle\varphi|:H_B(\phi^*,\phi):|\varphi\rangle = H(\varphi^*,\varphi) = H_{\rm classical}, \tag{4.27}$$

where $H_{\rm classical}$ is a classical Hamiltonian of the c-number spinor field satisfying the field equations of the massive Thirring model.[26]

5. MASSIVE THIRRING MODEL: c-NUMBER PATH INTEGRAL REFORMULATION AND ALL THAT

To strengthen the above introduced links between the classical c-number and Fermi massive Thirring models, we shall try to derive the path integral expression for $\operatorname{tr}\exp(-iH_Ft)$ in terms of genuine c-number trajectories. We are not aiming at any practical application (for which the Grassmann algebra formulation perfectly suffices); our problems are rather of the foundational nature. The observation that $H_F = 1_F H_B 1_F$ suggests that Bose operators in H_B should be replaced by spin $\frac{1}{2}$ objects like in the lattice cases. The lattice spins suitable for our purposes are given by

$$\sigma_i^+(k) = \phi_i^*(k):\exp(-\phi_i^*(k)\phi_i(k)):,$$
$$\sigma_i^-(k) = :\exp(-\phi_i^*(k)\phi_i(k)):\phi_i(k),$$
$$k = 0, \pm 1,\dots, \quad i = 1,2, \tag{5.1}$$
$$[\sigma_i^-(k),\sigma_i^+(k)]_+ = 1_F^i(k),$$
$$1_F(k) = \prod_{i=1}^2 1_F^i(k), \quad 1_F = \prod_k 1_F(k),$$

where we assume

$$\phi_i^\#(k) = (1/\sqrt{\delta}) \int_{R^1} dx\, \chi_k(x) \phi_i^\#(x),$$

$$\chi_k(x) = \begin{cases} 1, & x \in \Delta_k, \\ 0, & \text{otherwise}, \end{cases} \quad [\phi_i(k), \phi_j^*(p)]_- = \delta_{kp}\delta_{ij}, \tag{5.2}$$

δ being the length of each interval Δ_k. Since we assume $\delta \ll 1$, one can formally write $\phi_i^\#(k) \cong \sqrt{\delta}\phi_i^\#(x_k)$, i.e., a reasonable continuum limit would arise after rescaling fields $\phi_i^\#(k)$ by $1/\sqrt{\delta}$ as then $\delta_{kp} \to \delta_{kp}/\sqrt{\delta} \to \delta(x-y)$. It is inconvenient to work explicitly on the continuum level of quantum field theory, hence to avoid inconsistencies we shall use the appropriately discretized model and the transition to continuum will be investigated after achieving the c-number level of the theory.

The mass and interaction terms of H_B we discretize as follows:

$$H_B \to \sum_k H_B(k),$$

$$H_B^m(k) = m[\phi_1^*(k)\phi_2(k) + \phi_2^*(k)\phi_1(k)], \tag{5.3}$$

$$H_B^{\rm int}(k) = (2g/\delta)\phi_1^*(k)\phi_2^*(k)\phi_2(k)\phi_1(k),$$

so that for $\delta \ll 1$ we would formally have

$$H_B^m(k) \cong \delta H_B^m(x_k),$$

$$H_B^{\rm int}(k) \cong \delta H_B^{\rm int}(x_k). \tag{5.4}$$

The main difficulty, as usual with Fermi models, comes from the kinetic term, and we shall use a trick which differs from those used in the literature, see for example Ref. 21.

We introduce

$$\phi_i(k,\alpha) = \frac{1}{\sqrt{\delta}} \int_{R^1} \chi_k(x)\phi_i(x+\alpha),$$

$$\frac{\partial}{\partial\alpha}\phi_i^*(k)\phi_i(k,\alpha)|_{\alpha=0} = \frac{1}{\sqrt{\delta}}\phi_i^*(k)\int dx\, \chi_k(x)\partial_x\phi_i(x), \tag{5.5}$$

which allows for the following computation:

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\sigma_i^*(k)\sigma_i(k,\alpha)|_{\alpha=0}:
&= \phi_i^*(k): \exp(\phi_i^*(k)\phi_i(k)): \\
&\quad \times \frac{\partial}{\partial\alpha}\{: \exp(-\phi_i^*(k,\alpha)\phi_i(k,\alpha)) : \phi_i(k,\alpha)\}|_{\alpha=0} \\
&= \phi_i^*(k): \exp(-\phi_i^*(k)\phi_i(k)): \partial_\alpha\phi_i(k,\alpha)|_{\alpha=0} \\
&\quad - \phi_i^*(k): \exp(-\phi_i^*(k)\phi_i(k)): \{(\partial_\alpha\phi_i^*(k,\alpha)|_{\alpha=0}) \\
&\quad \times : \exp(-\phi_i^*(k)\phi_i(k)) : \phi_i^2(k) + \phi_i^*(k) \\
&\quad \times : \exp(-\phi_i^*(k)\phi_i(k)) : (\partial_\alpha\phi_i(k,\alpha)|_{\alpha=0})\phi_i(k)\} \\
&\equiv \phi_i^*(k): \exp(-\phi_i^*(k)\phi_i(k)) : \partial_\alpha\phi_i(k,\alpha)|_{\alpha=0}.
\end{aligned} \tag{5.6}$$

Consequently we introduce

$$H_B^{\rm kin}(k) = -i[\phi_1^*(k)\partial_\alpha\phi_1(k,\alpha) - \phi_2^*(k)\partial_\alpha\phi_2(k,\alpha)]|_{\alpha=0}. \tag{5.7}$$

With such H_B, the formula $H_B(k) \to 1_F(k)H_B(k)1_F(k)$ arises through replacing all Bose operators by appropriate spin $\frac{1}{2}$ operators:

$$\begin{aligned}
H_B(k) \to H_F(k)
&= -i[\phi_1^*(k): \exp(-\phi_1^*(k)\phi_1(k)) : \partial_\alpha\phi_1(k,\alpha)|_{\alpha=0} \\
&\quad - \phi_2^*(k): \exp(-\phi_2^*(k)\phi_2(k)) : \partial_\alpha\phi_2(k,\alpha)|_{\alpha=0}] \\
&\quad + m\Big[\phi_1^*(k): \exp\Big(-\sum_{j=1}^2\phi_i^*(k)\phi_i(k)\Big) : \phi_1(k) \\
&\quad + \phi_2^*(k): \exp\Big(-\sum_{j=1}^2\phi_i^*(k)\phi_i(x)\Big) : \phi_2(k)\Big] \\
&\quad + \frac{2g}{\delta}\phi_1^*(k)\phi_2^*(k): \\
&\quad \times \exp\Big(-\sum_{j=1}^2\phi_j^*(k)\phi_j(k)\Big) : \phi_2(k)\phi_1(k).
\end{aligned} \tag{5.8}$$

An infinitesimal propagator for $H_F(k)$ reads

$$\hat{U}_F^k(\Delta t) = \exp(iH_F(k)\Delta t) \cong 1_F(k) - i\Delta t H_F(k), \tag{5.9}$$

and its functional kernel is

$$\begin{aligned}
U_F^k(\Delta t) &= \Big[\exp\sum_{j=1}^2\beta_j^*(k)\beta_j(k)\Big]\cdot\langle\beta|U_F^k(\Delta t)|\beta\rangle \\
&= \{-i[\beta_1^*(k)\partial_\alpha\beta_1(k,\alpha)|_{\alpha=0}\exp\beta_2^*(k)\beta_2(k) \\
&\quad - \beta_2^*(k)\partial_\alpha\beta_2(k,\alpha)|_{\alpha=0}\exp\beta_1^*(k)\beta_1(k)] \\
&\quad + m[\beta_1^*(k)\beta_2(k) + \beta_2^*(k)\beta_1(k)] \\
&\quad + (2g/\delta)|\beta_1(k)|^2|\beta_2(k)|^2\}(-i\Delta t) \\
&\quad + \Big\{1 + \sum_{i=1}^2\beta_i^*(k)\beta_i(k) + |\beta_1(k)|^2|\beta_2(k)|^2\Big\} \\
&:= (1 - i\Delta t H(k)/\Pi_{i=1,2}(1+|\beta_i(k)|^2)) \\
&\quad \times \exp\sum_{i+1}^2\ln(1+|\beta_i(k)|^2) \\
&\cong \exp\Big\{\sum_{i=1}^2\ln[1+\beta_i^*(k)\beta_i(k)] \\
&\quad - i\Delta t H(k)/\Pi_{i=1,2}(1+\beta_i^*(k)\beta_i(k)).
\end{aligned} \tag{5.10}$$

We compose a product of such kernels for all sites: $U_F = \Pi_k U_F^k$, and by repeating the Fermi oscillator arguments, we arrive at the following expression on a lattice (time is continuous):

$$\operatorname{tr}\exp(-iH_Ft) = \int[d\beta][d\beta^*]\exp\sum_k i\int_0^t dt \times\Big[\sum_{i=1}^2 i\frac{\beta_i^*(k)\dot{\beta}_i(k)}{1+\beta_i^*(k)\beta_i(k)} - \frac{H(k)}{\Pi_{i=1,2}(1+\beta_i^*(k)\beta_i(k))}\Big]. \tag{5.11}$$

If now to notice that the coherent state representation of $\phi_i(k)$ reads

$$\beta_i(k) = \frac{1}{\sqrt{\delta}}\int\chi_k(x)\beta_i(x)dx \cong \sqrt{\delta}\beta_i(x) \quad x\in\Delta_k, \tag{5.12}$$

we arrive at

$$\mathrm{tr}_\delta \exp(-iH_F t)$$

$$= \int [d\beta][d\beta^*] \exp \sum_k i \int_0^t dt \times \left\{ \sum_{j=1}^{2} i\delta \frac{\beta_j^*(x)\dot\beta_j(x)}{1+\delta\beta_j^*(x)\beta_j(x)} - [(-i\delta)(\beta_1^*(x) \times \partial\beta_1(x) \exp(\delta\beta_2^*(x)\beta_2(x)) - \beta_2^*(x)\partial\beta_2(x) \times \exp(\delta\beta_1^*(x)\beta_1(x))) + m\delta(\beta_1^*(x)\beta_2(x) + \beta_2^*(x)\beta_1(x)) + 2g\delta|\beta_1(x)|^2|\beta_2(x)|^2] \Big/ \prod_{i=1,2}(1+\delta\beta_i^*(x)\beta_i(x)) \right\}. \tag{5.13}$$

Under an assumption that we restrict path integrations to these c-number paths only for which $\sigma_i(x) = \beta_i^*(x)\beta_i(x) < \infty$ on the whole space axis, we can consistently achieve a continuum limit. Then contributions from factors $\exp(\delta\beta_i^*(x)\beta_i(x))$, $\Pi_{i=1}^2(1+\delta\beta_i^*(x)\beta_i(x))$ become negligible; if compared with 1; hence

$$\mathrm{tr}_A \exp(-iH_F t)$$

$$= \int_{\{\sigma_i \leqslant A < \infty\}} [d\beta][d\beta^*] \exp i \int_0^t dt \int dx \times \left\{ \sum_{j=1}^{2} i\beta_j^*(x)\dot\beta_j(x) - [(-i)(\beta_1^*\partial\beta_2 - \beta_2^*\partial\beta_1) + m(\beta_1^*\beta_2 + \beta_2^*\beta_1) + 2g|\beta_1|^2|\beta_2|^2](x) \right\}$$

$$= \int_{\{\sigma_i \leqslant A < \infty\}} [d\beta][d\beta^*] \exp i \int_0^t dt \int dx \times \left\{ \sum_{j=1}^{2} i\beta_j^*(x)\dot\beta_j(x) - H_{\text{classical}}(x) \right\}. \tag{5.14}$$

It does not overcome problems with the continuum limit if no restrictions on trajectories are imposed, but at the same time it selects a subset of trajectories on which these problems disappear. Then the Fermi and Bose formula for the trace has an identical contribution from this subset, and the solutions of the original classical field equations do make stationary the action for both cases. Because of the importance of such stationary points in the semiclassical physics, we conclude that it makes sense to talk about the quantum meaning of classical c-number spinor fields in the Fermi quantized case.

6. REMARKS ON THE CHIRAL INVARIANT GROSS–NEVEU MODEL

The Hamiltonian density of the model reads

$$H_F(x) = -i \sum_{a=1}^{N} (\psi_{a+}^* \partial\psi_{a+} - \psi_{a-}^*\partial\psi_{a-}) + \sum_{a,b}^{N} 4g\psi_{a+}^*\psi_{b-}^*\psi_{b+}\psi_{a-}. \tag{6.1}$$

The Hamiltonian H_F is diagonalizable in the Fock space of the Fermi fields $\psi_{a\alpha}(x)$, $a = 1,2,..., N$, $\alpha = \pm$,

$$[\psi_{\alpha a}(x), \psi_{\beta b}^*(y)]_+ = \delta_{\alpha\beta}\delta_{ab}\delta(x-y), \quad \psi_{\alpha a}(x)|0\rangle = 0, \quad \forall\ \alpha, a, x. \tag{6.2}$$

The respective eigenvectors have the general form[27]

$$|F,\xi\rangle = \int dx_1 \cdots \int dx_n \sum_{\{\alpha,a\}} F(x_1,..., x_n, \alpha_1,..., \alpha_n) \cdot \xi(a_1,..., a_n) \prod_{i=1}^{n} \psi_{a_i\alpha_i}^*(x_i)|0\rangle, \tag{6.3}$$

where $F(x,\alpha)$ is an eigenfunction of the n-particle Hamiltonian

$$h = -i \sum_{j=1}^{n} \alpha_j\partial_j - 4g \sum_{i,j} \delta(x_i - x_j)P^{ij}\left[\frac{1}{2}(1-\alpha_i\alpha_j)\right]. \tag{6.4}$$

P^{ij} is an operator which exchanges chiralities α_i and α_j. Let us assume that the CAR algebra representation (6.2) is embedded in the CCR algebra representation, as in Ref. 24. For this purpose we need an antisymmetric function

$$\sigma = \sigma(x_1,\alpha_1,a_1,..., x_n, \alpha_n, a_n) = \sigma^3 \tag{6.5}$$

which changes a sign if triplets (x, α, a) in $\{(x, \alpha, a)\}$ are interchanged and takes the values 0, ± 1. Then, we arrive at

$$\prod_{i=1}^{n} \psi_{a_i\alpha_i}^*(x_i)|0\rangle = \sigma(x_1, \alpha_1, a_1,..., x_n, \alpha_n, a_n)\phi_{a_1\alpha_1}^*(x_1)\cdots\phi_{a_n\alpha_n}^*(x_n)|0\rangle, \tag{6.6}$$

where ϕ, ϕ^* are the corresponding Bose operators (with the same iso-indices as the fermions).

If $F(x, \alpha)$ is an eigenfunction of h, then

$$F^a(x, \alpha) = F(x, \alpha)\cdot\sigma(x, \alpha, a) \tag{6.7}$$

is an eigenfunction again, because formally

$$\partial_i\sigma = \partial_i\sigma^{2n+1} = (2n+1)\sigma^{2n}\partial_i\sigma = (2n+1)\sigma^2\partial_i\sigma \Rightarrow \partial_i\sigma \equiv 0, \tag{6.8}$$

which yields (up to a set of measure zero)

$$-i\sum_j \alpha_j\partial_j F^a(x,\alpha) \equiv \left(-i\sum_j \alpha_j\partial_j F\right)\cdot\sigma. \tag{6.9}$$

On the other hand

$$\sum_{ij} \delta(x_i - x_j)P^{ij}\left[\frac{1}{2}(1-\alpha_i\alpha_j)\right]F^a(x_1\alpha) = \sum_{ij}\delta(x_i - x_j)P^{ij}\left[\frac{1}{2}(1-\alpha_i\alpha_j)\right] \times F(\cdots x_i\cdots x_j,...,..., \alpha_i\cdots\alpha_j\cdots) \times\sigma(\cdots x_i\cdots x_j\cdots\alpha_i\cdots\alpha_j\cdots) = \left\{\sum_{ij}\delta(x_i - x_j)P^{ij}\left[\frac{1}{2}(1-\alpha_i\alpha_j)\right]F\right\}\cdot\sigma \tag{6.10}$$

provided σ is symmetric under an interchange of α_i and α_j at $x_i = x_j$ and $(a_1,..., a_n)$ fixed. As an example of such σ we propose

$$\sigma(x_1, a_1, \alpha_1,..., x_n, a_n, \alpha_n) = \prod_{1\leqslant j<k\leqslant n} p_{jk},$$

$$p_{jk} = \delta_{\alpha_j\alpha_k}\delta_{a_ja_k}[\Theta(x_j - x_k) - \Theta(x_k - x_j)] + \delta_{a_ja_k}\Theta(|\alpha_j - a_k|)(-1)^{1+\Theta(x_j - x_k)} + \delta_{\alpha_j\alpha_k}\Theta(|a_j - a_k|)(-1)^{\Theta(x_j - x_k)} + (1-\delta_{\alpha_j\alpha_k})(1-\delta_{a_ja_k})(-1)^{\Theta(x_j - x_k)}. \tag{6.11}$$

Notice that at $x_i = x_j$ we have

$$p_{ik}(x_j = x_k) = \delta_{a_j a_k}\Theta(|\alpha_j - \alpha_k|) - \delta_{\alpha_j \alpha_k}\Theta(|a_j - a_k|) + (1 - \delta_{\alpha_j \alpha_k})(1 - \delta_{a_j a_k})(-1)\,. \quad (6.12)$$

One should notice also that an eigenvalue problem for h arises from an eigenvalue problem for H_F after commuting the annihilation operators to $|0\rangle$ through the product of ψ^*'s and then integrating the gradients by parts. If now to use the Bose CGN Hamiltonian $H_B = H_F(\psi^* \to \phi^*, \psi \to \phi)$ the procedure is exactly the same and the eigenvalue problem for h arises again, but with a wave function $F^a(x, \alpha)$ instead of $F(x, \alpha)$.

Consequently the Bethe ansatz eigenvectors of H_F are also eigenvectors of H_B, and satisfy the property $1_F|F, \xi\rangle = |F, \xi\rangle$, where 1_F is the operator unit of the *bosonized* Fermi algebra. All the arguments applied before to the massive Thirring model apparently apply in the CGN case, and for example it is not difficult to check that the coherent state expectation value of the *bosonized* Fermi Hamiltonian reads

$$\langle\beta|1_F H_B 1_F|\beta\rangle \cong \sum_k \Big\{\delta(-i)\sum_a [\beta^*_{a+}\exp(-\delta\beta^*_{a+}\beta_{a+})\partial\beta_{a+} - \beta^*_{a-}\exp(-\delta\beta^*_{a-}\beta_{a-})\partial\beta_{a-}] + 4g\delta\sum_{ab}\beta^*_{a+}\beta^*_{b-}\exp\Big(-\delta\sum_\alpha [\beta^*_{a\alpha}\beta_{a\alpha} + \beta^*_{b\alpha}\beta_{b\alpha}]\Big)\beta_{b+}\beta_{b-} \underset{\delta\to 0}{\to} \int dx \times \Big\{\sum_a (-i)(\beta^*_{a+}\partial\beta_{a+} - \beta^*_{a-}\partial\beta_{a-} + 4g\sum_{ab}\beta^*_{a+}\beta^*_{b-}\beta_{b+}\beta_{a-}\Big\} \quad (6.13)$$

for all c-number spinor functions which are regular enough, i.e., satisfy $\sigma_{\alpha a}(x) = \beta^*_{\alpha a}(x)\beta_{\alpha a}(x) \leqslant A < \infty$. The respective solutions of the classical CGN model field equations thus have a quantum meaning both in the Bose and Fermi quantization cases (see in this connection also Ref. 28).

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