On quantum solitons and their classical relatives: Reducible quantum fields and infinite constituent "elementary" systems

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We demonstrate that the emergence of translation modes in the quantization of some at least nonlinear field theory models (like, e.g., ϕ^4 or the sine–Gordon systems) implies a specific structure of their state spaces namely this of the direct integral Hilbert space, which follows from the reducibility of the involved quantum field canonical commutation relations (CCR) algebras. As a special manifestation of this structure, one recovers infinite constituent "elementary" quantum systems living in the commutant of the CCR algebra, which appear as the Schrödinger or the two level ones. The corresponding Hamiltonians are derived. In addition, we propose a modification of the standard infrared Hilbert (photon field) space construction employed in quantum electrodynamics. We demonstrate that, in principle, Fermi (CAR) generators, carrying the spin–charge–momentum labels of Dirac particles, can be defined as operators in the electromagnetic (photon field) Hilbert space. The photon field (CCR) algebra is highly reducible, and in the present case fermions arise in the commutant of it, playing the role of intertwining operators.

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1. MOTIVATION

Usually one quantizes the classical field theory models under a tacit assumption that the field ϕ exhibits at most a space-time functional dependence: $\phi = \phi(\mathbf{x}, t)$, $\mathbf{x} \in R^3$ (or rather that the $\mathbf{x}, t$ dependence only is relevant for the quantization). An example of a classical field theory where the configuration space variable $\mathbf{x}$ is augmented by an auxiliary real variable w was considered in Ref. 1:

$$\mathscr{L} = \mathscr{L}(x, w) = \tfrac{1}{2}[\partial_\mu \phi(x, w)]^2 - \tfrac{1}{2}m^2\phi^2(x, w) - V[\phi, x, w] \tag{1.1}$$

with $x \in M^4$, ∂_μ being the space-time derivative, $x = (\mathbf{x}, t)$.

One insists there on a dynamical independence for all space and time of fields with distinct w value. This w-ultralocality of the field (1.1) has been exploited in Ref. 1 to construct a relatively simple quantized model. However, under an assumption that the quantum field CCR algebra is defined as follows:

$$[A_l(w), A^*_{l'}(w')]_- = \delta_{ll'}\delta(w - w'),$$
$$[A_l(w), A_l(w')]_- = 0,\ A_l(w)\,\Omega = 0 \quad \forall l, w. \tag{1.2}$$

Here $l = 1, 2, \cdots$ enumerates the oscillatorlike degrees of freedom of the wth field, and Ω is the Fock state. Let us recall that a conventional quantization procedure for the neutral scalar field theory model [omit w in (1.1)] would result in the commutation relations

$$[A_l, A^*_{l'}]_- = \delta_{ll'} \quad [A_l, A_{l'}]_- = 0 = [A^*_l, A^*_{l'}]_-$$
$$A_l\,\Omega = 0 \quad \forall l \tag{1.3}$$

and, as noticed, for example, in Refs. 2–4 still remains successful for models of the type (1.1). An oversimplified model with local fields which are at most bilinear in the generators (1.2) does not occur in case (1.3).

On the other hand, the quantization procedures developed during the 1970s for, for example, the sine–Gordon model, did ignore the fact that classical solutions of its field equation, at least in the soliton sector, exhibit a parametrization additional to x,t. In the simplest 1-soliton case it reduces to the real w-parametrization of (1.1), while for N-solitons the situation becomes more complex. The underlying parametrization enters soliton solutions $\phi = \phi(x, t)$ of the $(1+1)$-dimensional field equation:

$$\Box\phi(x, t) = m^2 \sin\phi(x, t) \quad m > 0, \tag{1.4}$$

via nonplane wave solutions $\varphi(x, t)$ of the free field equation

$$(\Box - m^2)\varphi(x, t) = 0,$$
$$\varphi(x, t) = \varphi_a(x, t) = \exp[m\gamma_a(x - v_a t) + \delta],$$
$$v_a = \frac{|a|^2 - 1}{|a|^2 + 1}, \quad \gamma_a = (\operatorname{sgn} a)(1 - v_a^2)^{-1/2}, \tag{1.5}$$
$$a \in R^1, \quad |a| \in (0, \infty)$$

so that (see Refs. 2–5)

$$\phi = \phi(x, t) = \phi[\varphi_a](x, t) = \phi_a(x, t) = \tfrac{1}{4}\tan^{-1}\varphi_a(x, t) \tag{1.6}$$

for one soliton, while

$$\phi(x, t) = \phi[\varphi_{a_1}, \ldots, \varphi_{a_N}](x, t) = \phi_{a_1 \cdots a_N}(x, t) \tag{1.7}$$

for N solitons, in the absence of "breathing" components. When the latter are present the parameters may become complex, and then are required to appear in complex conjugate pairs as, e.g., in the N-soliton (one breather) case of

$$\phi(x, t) = \phi_{aa_2a_3,\ldots,a^*,\ldots,a_N}(x, t), \tag{1.8}$$

where N-2 parameters are real and there is a complex conjugate pair.

Let us mention that in (1.5)–(1.8) one has still a freedom of choice of the phases $\delta_1, \ldots, \delta_N$. For 1-solitons, the sine–

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Gordon Lagrangian density has the form (1.1) with the parameter w replaced by a. In the N-soliton case the parameter w of (1.1) may be replaced by a sequence $(a_1,\dots,a_N)=(a)$.

As mentioned before, the (a) parametrization is insufficient for a unique characterization of the sine–Gordon soliton fields: the (δ) parametrization should also be taken into account. In the notation

$$\varphi_a(x,t)=\exp m\gamma_a(x+q)=\varphi_{aq}(x),$$
$$q=q_a=v_a(t_0-t)\Rightarrow\delta=\delta_a=m\gamma_a v_a t_0,$$

we have absorbed both δ and time t dependence of φ_a in a new parameter q. The a dependence of $q=q_a$ can be ignored due to the freedom of choice of the initial time instant t_0. Hence (1.7) can be rewritten as

$$\phi(x,t)=\phi_{(a,q)}(x)=\phi_{a_1\cdots a_N q_1\cdots q_N}(x), \tag{1.9}$$

thus leading to the $(2N+1)$-parameter family of N-solitons, where both (a) and (q) are viewed as independent parametric families. Notice that the time variable completely disappears from the formalism, which is obviously an effect of the translation freedom of (1.4).

Let $\hat{\varphi}_{\rm in}(x,0),\hat{\pi}_{\rm in}(x,0)$ be a canonical pair generating the CCR algebra of the mass m neutral scalar field in $1+1$ dimensions:

$$[\hat{\varphi}_{\rm in}(x,0),\hat{\pi}_{\rm in}(y,0)]_-=i\delta(x-y),$$
$$a(k)=\int dx\exp(-ikx)\left[\sqrt{k^2+m^2}\hat{\varphi}_{\rm in}(x,0)+i\hat{\pi}_{\rm in}(x,0)\right], \tag{1.10}$$
$$a(k)|0)=0\quad\forall k.$$

We introduce the coherent states (coherent soliton states of Refs. 2–4) as these states of the field (1.10) which satisfy

$$a(k)|\varphi)=\alpha(k)|\varphi),$$
$$(\varphi|\hat{\varphi}_{\rm in}(x)|\varphi)=(0)\{\hat{\varphi}_{\rm in}(x)+\varphi(x)\}|0)=\varphi(x), \tag{1.11}$$
$$(\Box-m^2)\varphi(x)=0.$$

Here $\alpha(k)$ is not a square-integrable function, since we make an identification of $\varphi(x)$ with either $\varphi_{aq}(x)$, (1.8) in the sine–Gordon soliton case, or with $\Sigma_{i=1}^N\varphi_{a_iq_i}(x)$ as required in the N-soliton case (without breathers).[2–4] Coherent states (2.11) should be viewed as continuous generalizations of the more familiar direct product coherent states

$$|\varphi)=\cup_\varphi^\otimes|0)=\prod_k^\otimes\{\exp(\alpha a^*-\overline{\alpha}a)f_0\}_k \tag{1.12}$$

with an ultraviolet cutoff implicit, f^0 being a Fock state of the Schrödinger representation of the CCR algebra. If one takes a coherent soliton state $|\varphi)$, then applies polynomials in creation and annihilation operators (1.10) to $|\varphi)$, and finally closes the resulting set of vectors, one arrives at the Hilbert space IDPS $(|\varphi))$, which is separable and carries an irreducible representation of the CCR algebra. IDPS $(|\varphi))$ is the one among infinitely many separable subspaces of the complete (von Neumann's) Hilbert space $\mathscr{H}$ of the quantum field (1.10).

Since we use coherent product states as the generating vectors for IDPS $(|\varphi))$, the theory of Ref.6 can be applied to classify the unitarily (in) equivalent representations of the CCR algebra affiliated with different coherent states. In Ref.2 we have done it for the soliton coherent states of the sine–Gordon fields; compare also Ref.7.

Because the above-introduced parametrization (a,δ) characterizes uniquely both the soliton field (1.9) and its free field constituents φ_{aq}, we arrive at the unambiguous labeling of quantum soliton (Hilbert space) sectors:

$$|\varphi)=|a_1,\dots,a_N,q_1,\dots,q_N)\Rightarrow\text{IDPS}(|\varphi)):=\mathscr{H}^N_{(a,q)}. \tag{1.13}$$

It happens because the free fields $\Sigma_{i=1}^N\varphi_{a_iq_i}(x)$ are the boson transformation parameters of (1.11). Suppose we have two distinct coherent soliton states $|\varphi)$ and $|\varphi')$ in $\mathscr{H}$. If the respective boson transformation parameters $\varphi(x)$ and $\varphi'(x)$ satisfy

$$\int_{R^1}dx\,[\varphi(x)-\varphi'(x)]^2\not<\infty, \tag{1.14}$$

then the representations of the CCR algebra in IDPS $(|\varphi))$ and IDPS$(|\varphi'))$ respectively, are unitarily inequivalent,[3,6] and the scalar product $(\varphi|\varphi')=0$ is conventionally introduced in $\mathscr{H}$ for $\varphi\neq\varphi'$ while $\|\varphi\|=\|\varphi'\|=1$ and $\|\varphi\|^2=(\varphi|\varphi)$. Obviously, the function $\varphi_{aq}(x)$ of (1.8) is not square integrable on the real line R^1 and the same concerns both $\Sigma_{i=1}^N\varphi_{a_iq_i}(x)$ and any $(\varphi-\varphi')(x)$ with $\varphi\neq\varphi'$.

For an example of the one-soliton boson transformation parameter $\varphi_{aq}(x)$, let us notice that even if $q=q'$ and $a=-a'$ (i.e., $|a|=|a'|$), we still have $(\varphi|\varphi')=0$. More generally, classical N-soliton sine–Gordon fields give rise to the rich (nondenumerable) family of normalized and pairwise-inequivalent vectors together with the related CCR algebra carrier spaces of (1.13) $\{\mathscr{H}^N_{(aq)}\}^{N=1,2,\dots}_{a,q\in R^1}$, where each $\mathscr{H}^N_{(a,q)}=$ IDPS $(|a_1,\dots,a_N,q_1,\dots,q_N))$ is by definition separable. We get in fact the field $\mathscr{H}_t$ of separable Hilbert spaces labeled by a continuous index set $T\ni t$. The present paper is the fourth one in the series of investigations devoted to the problem of quantum solitons and their classical relatives. We continue the discussion of different aspects of the quantization of nonlinear fields, with emphasis on the $(1+1)$-dimensional models (Refs. 2–4; see also Ref.5). The presence of the soliton solutions is known to complicate the traditional local quantization program by giving rise to zero-energy modes (related to translations) and then necessitating the "collective coordinates."

In the present paper we show that the translation freedom implies a very specific form of the state space of the quantized nonlinear system, namely, this of the direct integral Hilbert space, which we describe in Sec. 2 together with the derivation of the (inherent) infinite constituent "elementary" quantum systems. In Sec. 3 we introduce the "elementary" two-level systems, and investigate possible forms of their interaction. In Sec. 4 we propose a modification of the standard infrared Hilbert space construction employed in quantum electrodynamics, to allow the direct integral procedures of Sec. 2. Then, we construct Fermi (CAR) generators and represent them as operators in the electromagnetic field Hilbert space. The fermions carry the spin–charge–momentum labels of the Dirac particles and belong to the commutant of the photon field algebra (this is a consequence of the reducibility of the latter).

Remark: Our knowledge of soluble models in 1 + 1 dimensions is not broad enough to state that the above sine–Gordon structure (1.4)–(1.15) may always be found. Since, however, some of these features were observed to be valid for ϕ^4 and appeared also in the quantization of the c-number massive Thirring model, we feel that, even if not quite general, the sine–Gordon structure assumption may be of some use for the construction of the nontrivial quantum field theory models, preserving a correct classics–quanta relationship.

2. A LITTLE BIT OF THE OLD-FASHIONED MATHEMATICS: CONTINUOUS DIRECT SUMS OF HILBERT SPACES AND "ELEMENTARY" QUANTUM SYSTEMS

A. Let $\{h_k\}_{k=1,2,\ldots}$ be a countable sequence of separable Hilbert spaces. By h we denote the set of all sequences $\xi = \{\xi_k\}_{l=1,2,\ldots}, \xi_k \in h_k$, subject to the restriction $\Sigma_k \|\xi_k\|^2 < \infty$ and satisfying the linearity properties $\xi + \eta = \{\xi_k + \eta_k\} \in h, \alpha\xi = \{\alpha\xi_k\} \in h, \quad \alpha \in C$, h is a Hilbert space called a direct sum of separable Hilbert spaces h_k. To deal with a continuous generalization of this concept, we shall follow Refs. 8 and 9.

Let T be a μ-measurable set, where μ is a positive measure. We introduce a field of separable Hilbert spaces labeled by elements of the index set T: $\{h_t, t \in T\}$.

By h we denote the set of all vector valued functions: ξ: $t \to \xi_t, \xi \equiv \{\xi_t \in h_t, \quad t \in T\}$ such that

$$\left|\int_T (\xi_t, \eta_t)\, d\mu(t)\right| < \infty \tag{2.1}$$

for any two functions $\xi, \eta \in h$. In particular, from (2.1) there follows the requirement:

$$\int_T \|\xi_t\|^2\, d\mu(t) < \infty \quad \forall \xi \in h. \tag{2.2}$$

In addition to (2.1) the linearity is introduced via

$$\xi + \eta = \{\xi_t + \eta_t\}, \quad \alpha\xi = \{\alpha\xi_t\}. \tag{2.3}$$

The scalar product formula in h is given by (2.1):

$$(\xi, \eta) = \int_T (\xi_t, \eta_t)\, d\mu(t);$$

h is a Hilbert space called a direct integral of Hilbert spaces h_t with respect to the measure μ:

$$h = \int_T^{\oplus} h_t\, d\mu(t). \tag{2.4}$$

Let $\varphi(t)$ be a continuous (together with derivatives) function $T \to L(T)$ with the property

$$\int_T |\varphi(t)|^2 \|\xi_t\|^2\, d\mu(t) < \infty \tag{2.5}$$

for all $\xi \in h$. We shall define a linear operator L_φ in h:

$$L_\varphi: \quad \xi \to L_\varphi \xi = \{\varphi(t)\xi_t\}. \tag{2.6}$$

Here upon $\int_T |\varphi(t)|^2\, d\mu(t) < \infty$ we have

$$\begin{aligned} \|L_\varphi \xi\|^2 &= \int_T |\varphi(t)|^2 \|\xi_t\|^2\, d\mu(t) \\ &\leqslant \left[\int_T |\varphi(t)|^2\, d\mu(t)\right] \cdot \|\xi\|^2 \\ &= \|\varphi\|_\mu^2 \|\xi\|^2, \end{aligned} \tag{2.7}$$

and, consequently, L_φ is a bounded operator in h.

By taking a set of suitable functions $\{\varphi\}$ we get a corresponding set of bounded linear operators in h, which form a weakly closed commuting ring of bounded in h linear operators with unity. Notice that if $L_\varphi, L_\psi \in R$ then also $L_\varphi + L_\psi = L_\delta \in R$ and $L_\varphi \cdot L_\psi = L_\gamma \in R$.

Moreover, if φ is a real function, then it gives rise to a self-adjoint operator in h. If $|\varphi(t)| = 1$, the corresponding operator is a unitary one.

In this way to each direct integral of Hilbert spaces we have assigned a commuting ring of bounded operators. An inverse problem of the decomposition of a Hilbert space into a direct integral, with respect to a given commuting ring, is much more involved,[7–9] albeit useful for the solution of the reduction problem once an operator algebra is given in a Hilbert space. Let us mention that for an example of the sine–Gordon system we have a detailed knowledge about the state space of the system, but no clear understanding of the sine–Gordon field algebra.[2,3,10,11]

B. Let us choose a one-parameter index set $T \in R^1$ and a related one-parameter family $\{h_\lambda, \lambda \in R^1\}$ of Hilbert spaces. Let U_t be a unitary element of the commuting ring in $h = \int^\oplus h_\lambda\, d\mu(\lambda)$:

$$U_t \xi = U_t\{|\lambda)\} = \{\exp(i\lambda t) \cdot |\lambda)\} = \xi_t. \tag{2.8}$$

Each h_λ is separable Hilbert space; hence it can be equipped with an orthonormal complete basis system

$$\{|n, \lambda)\}_{n=0,1,\ldots} \quad (n, \lambda|m, \lambda) = \delta_{nm},$$
$$\sum_n |n, \lambda)(n, \lambda)| = 1_\lambda, \quad 1_\lambda h_\lambda = h_\lambda. \tag{2.9}$$

A definition of U_t equivalent to (2.8) can be given as follows:

$$U_t = \int_{R^1}^{\oplus} \exp(it\lambda) \sum_n |n, \lambda)(n, \lambda|\, d\mu(\lambda). \tag{2.10}$$

Let us introduce an operator V_s which is *not an element of the commuting ring*:

$$V_s = \int_{R^1}^{\oplus} \sum_n |n, \lambda - s) \exp\left(-s\frac{\partial}{\partial\lambda}\right)(n, \lambda|\, d\mu(\lambda), \tag{2.11}$$

$$\begin{aligned} V_s\xi &= V_s\{|\lambda)\} \\ &= \{\sum_n |n, \lambda - s)\exp\left(-s\frac{\partial}{\partial\lambda}\right)(n, \lambda| \sum_m |m, \lambda)(m, \lambda|\lambda)\} \\ &= \{\sum_n |n, \lambda - s)\exp\left(-s\frac{\partial}{\partial\lambda}\right)(n, \lambda|\lambda)\} \\ &= \{\sum_n |n, \lambda - s)(n, \lambda - s|\lambda - s)\} = \{|\lambda - s)\}. \end{aligned}$$

Consequently,

$$V_s U_t \xi = V_s\{\exp(it\lambda)|\lambda)\} = \{\exp(it\lambda)|\lambda - s)\}\exp(-its), \tag{2.12}$$
$$U_t V_s \xi = U_t\{|\lambda - s)\} = \{\exp(it\lambda)|\lambda - s)\},$$

which yields

$$V_s U_t = \exp(-its) U_t V_s, \tag{2.13}$$

i.e., a typical definition of the CCR algebra as given in the Weyl form.

Notice that

$$(V_s\xi, V_s\xi) = \int_{R^1} (\lambda - s|\lambda - s)\, d\mu(\lambda) = (\xi, \xi); \tag{2.14}$$

hence V_s is a unitary operator. Moreover, in addition to $U_t U_{t'} = U_{t+t'}$ we have a semigroup property for V_s as well:

$$V_s V_{s'} \xi = V_s\{|\lambda - s')\} = \{|\lambda - s - s')\} = V_{s+s'}\xi. \tag{2.15}$$

A strong continuity for both U_t and V_s is apparent; hence, by an application of the Stone theorem, the infinitesimal (self-adjoint) generators Q and P of U_t and V_s are recovered,

$$Q = \int_{R^1}^{\oplus} \lambda \sum_n |n, \lambda)(n, \lambda|\, d\mu(\lambda),$$
$$P = \int_{R^1}^{\oplus} \sum_n |n, \lambda)\Big(-i\frac{\partial}{\partial\lambda}\Big)(n, \lambda|\, d\mu(\lambda) \tag{2.16}$$

such that

$$QP\xi = \{\lambda \sum_n |n, \lambda)\Big(-i\frac{\partial}{\partial\lambda}\Big)(n, \lambda|\lambda)\},$$
$$PQ\xi = P\{\lambda|\lambda)\} = \{\sum_n |n, \lambda)(-i)(n, \lambda|\lambda) + \lambda \sum_n |n, \lambda)\Big(-i\frac{\partial}{\partial\lambda}\Big)(n, \lambda|\lambda)\} = -i\xi + QP\xi, \tag{2.17}$$

which implies

$$[Q, P]_- \xi = i\xi, \tag{2.18}$$

provided an appropriate domain is chosen to guarantee that both PQ and QP have a meaning in it (see Remark 1 below).

In this way we have demonstrated that, in addition to the commuting ring, the noncommuting pair can be introduced, which generates the CCR algebra representation in the direct integral $h = \int_{R^1}^{\oplus} h_\lambda\, d\mu(\lambda)$ of separable Hilbert spaces. The respective creation and annihilation operators read as follows:

$$A = \frac{1}{\sqrt{2}}(Q + iP) = \int_{R^1}^{\oplus} \sum_n |n, \lambda)\frac{1}{\sqrt{2}}\Big(\lambda + \frac{\partial}{\partial\lambda}\Big)(n, \lambda|\, d\mu(\lambda),$$
$$A^* = \frac{1}{\sqrt{2}}(Q - iP) = \int_{R^1}^{\oplus} \sum_n |n, \lambda)\frac{1}{\sqrt{2}}\Big(\lambda - \frac{\partial}{\partial\lambda}\Big)(n, \lambda|\, d\mu(\lambda), \tag{2.19}$$
$$[A, A^*]_- \xi = \xi$$

Let us recall that if one intends (as we do) to choose $h_\lambda = \mathrm{IDPS}(|\lambda))$ as determined by starting from (2.10) and (2.11), i.e., by the canonical generators $\{a^*(k), a(k)\}_{k\in R^1}$, then the received generators (2.19) are essentially *new* quantum objects, since we cannot reconstruct A and A^* *solely* in terms of $a^*(k)$ and $a(k)$. This last feature is rather common for these nontrivial field theory models which lead to reducible field algebras; see, for example, Refs. 12–16, but also Refs. 5, 17–20, where an infrared problem for charged particles involves reducible electromagnetic field algebras. The pair (2.19) gives rise to the Schrödinger representation of the CCR algebra, with the (direct integral) vacuum $\Omega \in h = \int_{R^1}^{\oplus} h_\lambda\, d\mu(\lambda)$ selected by the requirements

$$\Omega = \{|\Omega, \lambda)\}, \quad \int_{R^1} \sum_n |(n, \lambda|\Omega, \lambda)|^2\, d\mu(\lambda) < \infty,$$
$$\frac{1}{\sqrt{2}}\Big(\lambda + \frac{\partial}{\partial\lambda}\Big)(n, \lambda|\Omega, \lambda) = 0 \quad \forall n. \tag{2.20}$$

Notice that all h_λ can be inequivalent to the Fock space of the field algebra (2.10). Moreover, an index n in $(n, \lambda|\Omega, \lambda)$ identifies the nth basis vector in h which can be received from $|\lambda)$ by applying the nth function of generators $a^*(k)$ and $a(k)$. It has nothing in common with an index N of

$$\frac{1}{\sqrt{N!}} A^{*N}\Omega := \{|N, \lambda)\} = \{\sum_n |n, \lambda)(n, \lambda|N, \lambda)\} \tag{2.21}$$

which corresponds to the Nth excitation level, but in terms of the secondary quanta (2.19).

Remark 1: As is well known,[8] the direct integral of separable Hilbert spaces with respect to any standard measure is a separable Hilbert space again. Hence properties of the representation (2.13) and its generators (2.17) can be understood on the basis of general results described in Ref. (9).

It is not useless to mention that elements of the direct integral space $\mathscr{H}_\mu = \int^\oplus \mathrm{IDPS}(|\lambda))\, d\mu(\lambda)$ have the form

$$\psi(f) = \int^\oplus f(\lambda)|\psi, \lambda)\, d\mu(\lambda), \quad |\psi, \lambda) \in \mathrm{IDPS}(|\lambda)),$$
$$(\psi, \lambda|\psi, \lambda) = 1 \quad \forall\lambda, \quad (\psi, \lambda|\psi, \lambda') = 0, \ \lambda \neq \lambda',$$
$$(\psi(f), \psi(f)) = \|\psi(f)\|^2 = \int |f(\lambda)|^2\, d\mu(\lambda) < \infty.$$

Notice that we can formally represent the continuous set of orthonormal (in von Neumann's space) vectors $(\psi, \lambda|\psi, \lambda') = 0, \ \lambda \neq \lambda', \ (\psi, \lambda|\psi, \lambda) = 1$ as generalized vectors associated with $\mathscr{H}_\mu$:

$$\psi_\lambda(\delta) := \int^\oplus \delta(\lambda - \lambda')|\psi, \lambda')\, d\mu(\lambda')$$

so that if considered in the topology of $\mathscr{H}_\mu$ we arrive at

$$(\psi_\lambda(\delta), \psi_{\lambda'}(\delta)) = \delta(\lambda - \lambda').$$

It demonstrates how the continuity of the orthonormal set $\{|\lambda)\}$ is lost while passing to the direct integral Hilbert space $\mathscr{H}_\mu$: $\psi_\lambda(\delta)$ is not a Hilbert space vector.

Because of $\psi(f) = \int^\oplus f(\lambda)|\psi, \lambda)\, d\mu(\lambda)$, where $f(\lambda)$ is a function $R^1 \to C$ whose modulus is square μ-integrable, the problem of finding a dense in $\mathscr{H}_\mu$ domain $\mathscr{D} \subset \mathscr{D}(P) \cap \mathscr{D}(Q)$ reduces to this of finding an appropriate domain $\mathscr{D}$ in the set of functions $f(\lambda)$. Notice that if $|\psi,\lambda) = \Sigma_n \psi_n |n,\lambda)$, $\psi_n \in C$, $\psi(f) = \int^\oplus f(\lambda)|\psi,\lambda)\,d\mu(\lambda)$, then the action of P, Q, PQ, and QP on $\psi(f)$ reduces to

$$QP\psi(f) = \int^\oplus \lambda \sum_n |n,\lambda)\cdot\psi_n\left(-i\frac{\partial f(\lambda)}{\partial\lambda}\right)d\mu(\lambda) = \int^\oplus \lambda\left(-i\frac{\partial f}{\partial\lambda}\right)\cdot|\psi,\lambda)\,d\mu(\lambda)$$

$$PQ\psi(f) = \int^\oplus d\mu(\lambda) \times\left\{-if(\lambda)|\psi,\lambda) + \lambda\left(-i\frac{\partial f(\lambda)}{\partial\lambda}\right)|\psi,\lambda)\right\},$$

i.e., to the well-known Schrödinger representation problem in the Hilbert space of square μ- integrable functions.

Remark 2: The representation (2.10) of the CCR algebra in the separable Hilbert space is unitarily equivalent to the direct sum of Schrödinger representations. Consequently, the vector Ω of (2.20) would be unique in the irreducible case only. However, this is not the case in $\mathscr{H}_\mu$. Let us define

$$\Omega = \Omega_n := \int^\oplus f_0(\lambda)|n,\lambda)\,d\mu(\lambda),$$

where $|n,\lambda)$ is the nth basis vector in IDPS $(|\lambda))$ and $f_0(\lambda)$ satisfies: $(1/\sqrt{2})(\lambda + \partial/\partial\lambda)f_0(\lambda) = 0$. Then

$$A\Omega_n = \int^\oplus \left[\frac{1}{\sqrt{2}}\left(\lambda + \frac{\partial}{\partial\lambda}\right)f_0(\lambda)\right]|n,\lambda) \times d\mu(\lambda) = 0 \quad \forall n = 0, 1, \cdots$$

and

$$|N,n) = \frac{1}{\sqrt{N!}}A^{*n}\Omega_n = \int^\oplus \frac{1}{\sqrt{N!}}\left\{\left[\frac{1}{\sqrt{2}}\left(\lambda - \frac{\partial}{\partial\lambda}\right)\right]^N f_0(\lambda)\right\} \times |n,\lambda)\,d\mu(\lambda), (N,n|M,m) = \delta_{nm}\delta_{NM}.$$

Hence the representation (2.13) is not irreducible in $\mathscr{H}_\mu$.

C. Since we have in $h = \int^\oplus_{R^1} h_\lambda\, d\mu(\lambda)$ a canonical pair P, Q, it seems rather natural to follow a conventional quantum route, and to search for a Hamiltonian system in h. In an abstract scheme, there is no natural choice of the Hamiltonian. However, for a particular quantization procedure for the classical (e.g., sine–Gordon) model H can be determined once time-dependent trajectories are established in the set of parameters $\lambda \in R^1$. This is the case when the (underlying) classical dynamics is taken into account.

Let us consider a family of sine–Gordon 1-solitons, each one with a fixed a value, but differing in the choice of $q = q_a = \lambda$ of (1.8). In fact,

$$\lambda = \lambda(t) = v_a(t_0 - t) = \lambda - v_a t = \lambda + \dot\lambda t, \tag{2.22}$$

and $\lambda = v_a t_0 \in R^1$ is quite arbitrary. We look for the time development generator H such that

$$Q\xi = \{\lambda|\lambda)\} \to Q(t)\xi = \exp(iHt)Q\exp(-iHt)\xi = \{\lambda'(\lambda,t)|\lambda'(\lambda,t))\}, \quad \lambda'(\lambda,t) = \lambda + \dot\lambda t. \tag{2.23}$$

Notice that for the infinitesimal time variations we have

$$Q(\Delta t) \cong Q + \Delta t\dot Q, \quad \dot Q = -i[Q,H]_-, \qquad Q(\Delta t)\xi \cong \{(\lambda + \Delta t\dot\lambda)|\lambda + \dot\lambda\Delta t)\}, \tag{2.24}$$

and, consequently, because of (2.23),

$$\ddot Q = -i[\dot Q, H]_- = 0 \tag{2.25}$$

so that the form (2.24) persists for all $t \in R^1$ in (2.23). Notice that $H = H(\dot Q)$ solves (2.25).

We shall deduce an explicit form of $H = H(\dot Q)$ by following the idea of Ref. 21, which, if appropriately modified, can be applied to our case.

For the translation operator P we have the formula (2.17). Let us rewrite it in the form

$$P = \int^\oplus_{R^1} d\mu(\lambda)\,T_{10}(\lambda), \quad T_{10}(\lambda) = T_{01}(\lambda), \qquad = \int^\oplus_{R^1}\sum_n |n,\lambda)\left(i\frac{\partial}{\partial\lambda}\right)(n,\lambda|\,d\mu(\lambda). \tag{2.26}$$

Analogously for H

$$H = \int^\oplus_{R^1} d\mu(\lambda)\,T_{00}(\lambda) \tag{2.27}$$

with $T_{00}(\lambda)$ being still unspecified. Let us now introduce a symmetric stress energy tensor $T_{\mu\nu}(\lambda)$ so that the generator of Lorentz transformations in h can be introduced as follows:

$$M_{10} = \int^\oplus_{R^1} d\mu(\lambda)\{\lambda\,T_{00}(\lambda) - tT_{10}(\lambda)\} = HQ - tP. \tag{2.28}$$

Upon a standard requirement

$$0 = \frac{dM_{10}}{dt} = \frac{\partial M_{10}}{\partial t} - i[M_{10}, H]_- = -P + H\dot Q - t\dot P, \tag{2.29}$$

provided with a momentum conservation demand $\dot P = 0$, we arrive at the identity

$$P = H\dot Q, \tag{2.30}$$

which yields

$$i = [Q,P]_- = [Q,\dot Q]_- H + i\dot Q^2. \tag{2.31}$$

Recall now that $H = H(\dot Q)$; hence formally one can introduce the notion of $\partial H/\partial\dot Q$. Suppose that in the domain of H there exists at least one vector on which the following two operator identities hold true (the nature of the constraint will be investigated below):

$$\left[Q, \frac{\partial H}{\partial\dot Q}\right]_- = i, \qquad 2i\dot Q = [Q, 2H]_- = \left[Q, \dot Q\frac{\partial H}{\partial\dot Q}\right]_- \tag{2.32}$$

In (2.32) for some vectors the following holds :

$$2i\dot Q = [Q,\dot Q]_- \frac{\partial H}{\partial\dot Q} + i\dot Q, \tag{2.33}$$

which by taking into account (2.31) leads to the conclusion that, for such vectors,

$$\frac{\partial H}{\partial\dot Q} = \frac{\dot Q}{1 - \dot Q}H, \tag{2.34}$$

which is satisfied by

$$H = M/(1-\dot{Q}^2)^{1/2}, \tag{2.35}$$

with M being an integration constant, $M \in R^1$. Furthermore,

$$P = M\dot{Q}/(1-\dot{Q}^2)^{1/2} \tag{2.36}$$

and

$$[Q,\dot{Q}]_- = i(1-\dot{Q}^2)H^{-1} = (i/M)(1-\dot{Q}^2)^{3/2}. \tag{2.37}$$

Because of (2.36) a conventional relativistic formula for H (in $1+1$ dimensions) follows:

$$H = (P^2+M^2)^{1/2}. \tag{2.38}$$

Due to $\dot{Q} = \dot{Q}(P)$ and $H = H(P)$, the representation of P in $h = \int_{R^1}^{\oplus} h_\lambda\, d\mu(\lambda)$ as given by (2.17) and (2.26) leads to

$$\dot{Q} = \int_{R^1}^{\oplus} \sum_n |n,\lambda) \frac{i\partial}{(M^2-\partial^2)^{1/2}} (n,\lambda|\, d\mu(\lambda),$$

$$H = \int_{R^1}^{\oplus} \sum_n |n,\lambda)(M^2-\partial^2)^{1/2}(n,\lambda|\, d\mu(\lambda). \tag{2.39}$$

D. Let us now analyze the constraints (2.32) which diminish the arbitrariness in the choice of H, by demanding the existence of suitable vectors in the domain. By making use of the first constraint (2.32) we get formally

$$\frac{\partial H}{\partial \dot{Q}} = \frac{M\dot{Q}}{(1-\dot{Q}^2)^{1/2}} = \frac{PH^2}{M^2} \tag{2.40}$$

with $[P,H]_- = i\dot{P} = 0$. Hence the second one reads

$$\left[Q, \frac{\partial H}{\partial \dot{Q}}\right]_- |\psi) = \frac{1}{M^2}[Q, PH^2]_-|\psi)$$
$$= \left\{\frac{2P^2}{M^2} + i\frac{H^2}{M^2}\right\}|\psi) = i|\psi), \tag{2.41}$$

where $[\dot{Q},H]_- = 0$ is taken into account. From (2.41) it follows that $|\psi)$ is a common eigenvector of both P and H:

$$P|\psi) = 0, \quad H|\psi) = \sqrt{P^2+M^2}|\psi) = M|\psi). \tag{2.42}$$

Then we find

$$2i\dot{Q} = \left[Q, \dot{Q}\frac{\partial H}{\partial \dot{Q}}\right]_- = \frac{1}{M^2}[Q, P^2H]_-, \tag{2.43}$$

and hence

$$(P^2+2H^2)/2M^2\,\dot{Q}\,|\psi) = \dot{Q}\,|\psi)$$
$$\Rightarrow \dot{Q}|\psi) = 0 \ \text{ or } \ [\dot{Q},H]_- = 0 = [\dot{Q},P]_-, \tag{2.44}$$

which may impose a restriction on $\dot{Q}$ if applied to $|\psi)$. Consequently, (2.32) is equivalent to

$$H|\psi) = M|\psi), \quad P|\psi) = 0, \tag{2.45}$$

and thus the parameter M corresponds to the rest mass of the elementary quantum system associated with the direct integral $h = \int_{R^1}^{\oplus} h_\lambda\, d\mu(\lambda)$ of separable Hilbert spaces. The main problem now is to find M while maintaining consistency with the classical field equations (e.g., the sine-Gordon one) which underlies the whole derivation of (2.42). Recall that M appears in (2.35) as an integration constant, but is not at all constrained to be a c-number. The more natural requirement is that M belongs to the commutant of the $\{P, Q\}$ C^*-algebra. Since the $\{P,Q\}$ pair arises in the reducible representation of the primary scalar field, (2.10) algebra, both P and Q do commute with the CCR algebra generators $\{a^*(k), a(k)\}_{k\in R^1}$. If M is a c-number then H does commute also, as being solely constructed in terms of P and Q. However, an integration procedure leading to (2.35) does not exclude the fact that $M = M(a^*, a)$ is an operator element of the neutral scalar field algebra. This route has been followed in Ref. (22); however, the authors start from another assumption: (1) about the Hilbert space structure, which is a direct product of the Fock space for collective modes and the Hilbert space for the primary quantum field; (2) about the existence of the particle number operator for the neutral scalar field in its soliton sectors.

Let us assume to have a rest frame vector $|\psi)$ of (2.45), and let us further assume that $H|\psi) = M_0|\psi)$, $M_0 \in R^1$. By applying to $|\psi)$ polynomials in $a^*(k)$, $a(k)$ and then making a Hilbert space closure of the set obtained, we arrive at the previously introduced notion of IDPS $(|\psi))$. Consequently, if one has any Hamiltonian operator $H_0(a^*,a) = H_0$ generating a unitary in time evolution *in* IDPS$(|\psi))$, it can be safely added to M_0, thus giving rise to the following modification of $H = \sqrt{P^2+M_0^2}$:

$$H = \sqrt{P^2+(M_0+H_0)^2}, \quad H_0 = H_0(a^*,a). \tag{2.46}$$

Recall that $\hbar = c = 1$. In Ref. (22) H_0 is supposed to be a free neutral scalar field Hamiltonian, which, however, has no eigenvectors outside of the Fock space IDPS $(|0))$. Our IDPS$(|\psi))$ is inequivalent to IDPS $(|0))$. However, H_0 can be regarded to be (if specialized to our example) the quantum sine-Gordon Hamiltonian, constrained to the particular soliton sector. Then we can expect[2,3,10,11] that the conventional sine-Gordon spectrum and eigenvectors can be produced in IDPS $(|\psi))$.

Remark: A construction of coherent soliton states for the sine-Gordon system is motivated by the following assumptions: If $\hat{\phi}(x,t)$ is an interacting sine-Gordon field, then it admits the Haag type expansion in terms of (asymptotic-like but *not* asymptotic at all) free mass m neutral scalar field generators $a^*(k)$ and $a(k)$, i.e., $\hat{\phi}(x,t) = \phi(a^*,a,x,t)$. Then, a coherent state expectation value of $\hat{\phi}$ in the tree (zero loop) approximation,

$$(\alpha|{:}\hat{\phi}(a^*,a,x,t){:}|\alpha) = (0|{:}\hat{\phi}(a^*+\overline{\alpha}, a+\alpha, x, t){:}|0)$$
$$= \phi(\overline{\alpha},\alpha,x,t) = \phi_{cl}(x,t), \tag{2.47}$$

should allow us to restore both the classical sine-Gordon field, its equations of motion, and the Hamiltonian

$$(\alpha|{:}H(\hat{\phi}){:}|\alpha) = (\alpha|{:}H(\hat{\phi})(a^*,a){:}|\alpha)$$
$$= (0|{:}H(\phi)(a^*+\overline{\alpha}, a+\alpha){:}|0)$$
$$= H(\phi)(\overline{\alpha},\alpha). \tag{2.48}$$

In the above, $|0)$ is the Fock state for the $\{a^*,a\}$ field algebra.

Consequently, the quantum sine-Gordon Hamiltonian, consistent with the above tree approximation mappings, should implement an evolution unitary in time in each of the irreducibility sectors for the $\{a^*,a\}$ field algebra, i.e., in each IDPS $(|\alpha))$. Hence it is a rather natural choice to identify $H_0 = H_0(a^*,a)$ of (2.46) with the sine-Gordon Hamiltonian, while constructed *solely* in terms of $a^*(k)$ and $a(k)$.

Let us add that the original Haag expansions in terms of free asymptotic fields of the model are of the perturbative nature. In the above we admit expansions which formally have the Haag form (infinite power series), but (1) are nonperturbative and (2) can be defined with respect to the free fields which are not asymptotic ones for the model under consideration ("confinement").

The choice of the c-number constant M_0 is to some extent arbitrary, but we are motivated by the fact that the sine-Gordon parameter m of (2.4) gives rise to the classical 1-soliton mass equal $8m$. The 1-soliton momentum $k_a = 8mv_a/\sqrt{1-v_a^2}$ is in a relativistic relationship with $8m$: $E=\sqrt{(8m)^2+k^2}$. Hence an identification $M_0 = 8m$ is quite natural.

Suppose now that we have solved the spectral problem for H_0 in IDPS $(|\psi))$. If $|E, \psi)$ is an eigenvector of H_0 in IDPS$(|\psi))$, then it is also an eigenvector of H,

$$H|E,\psi) = \sqrt{P^2 + (8m+E)^2}\,|E,\psi). \tag{2.49}$$

Hence we are in principle capable of producing quantum corrections (due to the primary field excitations built over the extended particle state) to the 1-soliton mass $8m$. If we adopt the discrete (bound state, WKB) spectrum of the sine-Gordon Hamiltonian: $\{E_i\}_{i=1,2,\ldots}$ then the secondary Hamiltonian $H = H(H_0, P, Q) = H(a^*, a, P, Q)$ has a discrete mass spectrum with $H_0|0,\psi) = 0$ corresponding to the $H|\psi) = 8m|\psi)$ equation.

Let us emphasize that in contrast to Ref. (22), but in agreement with the observations of Ref. (6), the asymptotic problem for the neutral scalar (sine-Gordon) field cannot be solved in IDPS $(|\psi))$. There is no unitary mappings (*solely* in the scalar field algebra) of the soliton Hilbert space IDPS $(|\psi))$ into a Fock space IDPS $(|0))$. *Consequently, we have a typical "confinement" of the quantum scalar field constituents of the extended particle state (soliton) generating IDPS* $(|\psi))$.

On the other hand, a folk-lore statement is to attribute the notion of a quantum particle to an elementary quantum system (i.e., an irreducible representation of the CCR algebra, which the pair P, Q does indeed generate). Consequently, for each fixed a value, the 1-soliton Hilbert space $h_a = \int_{R^1}^{\oplus} h_{a\lambda}\, d\mu(\lambda)$ can be interpreted as the carrier state space for a quantum particle, which though "elementary" is still an infinite constituent object, see, e.g., in this connection Ref. (23). The constituents can never be seen in the conventional (Fock space) sense, due to the above-mentioned "confinement" property.

Let us emphasize a paradoxical situation: a folk-lore understanding of P, Q is that an elementary particle is structureless. Quite the contrary, our derivation of P, Q is based on the rich infinite constituent structure underlying the construction of the state space for the sine-Gordon field, while soliton sectors are taken into account.

3. "ELEMENTARY" TWO LEVEL SYSTEMS AND THEIR INTERACTION IN 1 + 1 DIMENSIONS

A. The modulus $|a|$ of the 1-soliton parameter a is relevant to the velocity v_a and $v_a = v_{-a}$. Here sgn a is relevant to the topological invariant value:

$$R = \frac{1}{4\pi}\int_{R^1} \frac{\partial \phi}{\partial x}\,dx, \tag{3.1}$$

which for the ath 1-soliton reads $R = R(a) = R(\pm|a|) = \pm\frac{1}{2}$. We assume $a \neq 0$ and then compose a direct sum of the $\pm a$th soliton Hilbert spaces:

$$h(a) = h_a \oplus h_{-a}, \quad a \in R^+. \tag{3.2}$$

One can observe (see, e.g., Ref. 3) that a classical 1-soliton momentum value reads $k = 8m(a^2-1)/2|a|$ and hence the a label can be replaced by a joint (R, k) label: $h_{\pm a} \to h_{Rk}$ so that

$$h(a) = h_k = \oplus_{R=\pm 1/2}\, h_{Rk}. \tag{3.3}$$

It means that now we are able to give a quantum meaning to the topological invariant R. For this purpose, we shall define the spin-$\frac{1}{2}$ SU(2) group raising and lowering operators, as has been previously done in Ref. (3), in a slightly different context:

$$\sigma_k^+ = \int_{R^1}^{\oplus} \sum_n |n, k, +, \lambda)(n, -, k, \lambda|\, d\mu(\lambda),$$
$$\sigma_k^- = \int_{R^1}^{\oplus} \sum_n |n, k, -, \lambda)(n, +, k, \lambda|\, d\mu(\lambda). \tag{3.4}$$

Because of $(n, a, \lambda|m, -a, \lambda) = 0 \quad \forall n, m, \lambda$, we find immediately that

$$\sigma_k^+ \xi_{\mp k} = \xi_{\pm k}, \quad \xi_{+k} \in h_{+k}, \xi_{-k} \in h_{-k},$$
$$(\sigma_k^{\pm})^2 = 0, \tag{3.5}$$

i.e.,

$$\sigma_k^3 \xi_{\pm k} = (\pm\tfrac{1}{2})\xi_{\pm k}, \quad \sigma_k^3 = (-\tfrac{1}{2})1_k + \sigma_k^+\sigma_k^-,$$
$$1_k = [\sigma_k^+, \sigma_k^-]_+ \tag{3.6}$$

In accordance with Ref. 2 and 3 the N-soliton coherent state (without breathers) exhibits the following parametrization:

$$|k_1, R_1, \lambda_1, \ldots, k_N, R_N, \lambda_N) = |\underline{k}, \underline{R}, \underline{\lambda})_N,$$
$$k_1 < k_2 < \cdots < k_N \tag{3.7}$$

where, due to the classical momentum ordering, R_i is a topological invariant of the k_ith (asymptotic) 1-soliton.

A generalization of the previous direct integral procedure reads now as follows:

$$|\underline{k}, \underline{R}, \underline{\lambda})_N \to \mathrm{IDPS}(|\underline{k}, \underline{R}, \underline{\lambda})_N),$$
$$h_{k_1R_1\cdots k_NR_N} = h^N_{\underline{k}\,\underline{R}} = \int_{R^1}^{\oplus} d\mu(\lambda_1)\cdots\int_{R^1}^{\oplus} d\mu(\lambda_N) \times \mathrm{IDPS}(|\underline{k}, \underline{R}, \underline{\lambda})_N). \tag{3.8}$$

By our choice of the sharply ordered momentum sequence $k_1 < k_2 < \cdots < k_N$ we have guaranteed a fulfillment of the "classical Pauli exclusion principle" regarding that neither a_i in the parametric sequence $\{a_1, \ldots, a_N\}$ can appear more than once. By composing a direct sum with respect to all possible configurations of $\pm$'s in the sequence $\{R_1, \ldots, R_N\}$ we finally arrive at the N-soliton analog of the two-level Hilbert space h_k of (3.3):

$$h_{k_1\cdots k_N} = \oplus_{\mathrm{conf}(\underline{R})}\, h_{k_1R_1\cdots k_NR_N}. \tag{3.9}$$

The corresponding set of spin-$\frac{1}{2}$SU(2)^N generators can be constructed by adopting the idea of Ref. (3) to our (direct integral) case:

$$\sigma_i^+ = \sigma_{ik_1\ldots k_N}^+ = \sum_{\mathrm{conf}(\underline{R})}^{(i)} \int_{R^1}^{\oplus} \cdots \int_{R^1}^{\oplus} d\mu(\lambda_1)\cdots d\mu(\lambda_N) \times|k_1, R_1, \lambda_1, \ldots, R_i = +\tfrac{1}{2}, \ldots, k_N, R_N, \lambda_N)(k_1 R_1, \lambda_1, \ldots, R_i = -\tfrac{1}{2}, \ldots, k_N, R_N, \lambda_N|,$$

$$\sigma_i^- = \sum_{\mathrm{conf}(\underline{R})}^{(i)} \int_{R^1}^{\oplus} \cdots \int_{R^1}^{\oplus} d\mu(\lambda_1)\cdots d\mu(\lambda_N) \times|k_1, R_1, \lambda_1, \ldots, R_i = -\tfrac{1}{2}, \ldots, k_N, R_N, \lambda_N) \times(k_1, R_1, \lambda_1, \ldots, R_i = +\tfrac{1}{2}, \ldots, k_N, R_N, \lambda_N|, \quad (3.10)$$

where $\Sigma^{(i)}_{\mathrm{conf}(\underline{R})}$ means that we perform summations over all admissible configurations of $\{R_1,\ldots,R_N\}$ under an assumption that R_i is left untouched. Obviously,

$$[\sigma_i^+, \sigma_j^+]_- = 0 = [\sigma_i^-, \sigma_j^-]_-,$$
$$[\sigma_i^-, \sigma_j^+]_- 0, \quad i\neq j,$$
$$[\sigma_i^+, \sigma_i^-]_+ = 1_{k_1\cdots k_N} = 1_k^N, \quad (3.11)$$
$$1_k^N h_k^N = h_k^N.$$

All mappings changing a configuration of $\{R_1,\ldots,R_N\}$ at a fixed choice of $k_1 < \ldots < k_N$ leave the Hilbert space $h_{k_1k_2\ldots k_N}$ invariant. Since with such configuration-to-configuration mappings we have automatically associated the spin-$\frac{1}{2}$ *xyz* Heisenberg model Hamiltonian (see Refs. 2 and 3), $h_{k_1k_2\cdots k_N}$ can be viewed as the carrier Hilbert space for a system of N interacting spins $\frac{1}{2}$ on a linear lattice with sites labeled by $k_1,\ldots,k_N$:

$$H_{xyz} = -\sum_{a=1}^{3}\sum_{i=1}^{N} \mathscr{J}_a \sigma_i^a \sigma_{i+1}^a, \quad (3.12)$$

$$\sigma_i^1 = \frac{1}{\sqrt{2}}(\sigma_i^+ + \sigma_i^-), \quad \sigma_i^2 = \frac{i}{\sqrt{2}}(\sigma_i^+ - \sigma_i^-),$$
$$\sigma_i^3 = -\tfrac{1}{2} + \sigma_i^+\sigma_i^-.$$

Needless to say, the spectrum of the weakly anisotropic *xyz* model, while going to continuum is mapped into the WKB sine-Gordon/massive Thirring model spectrum. Hence a close relationship with the quantum sine-Gordon model still persists despite the direct integrals involved. Be aware, however, that (3.12) is not a continuum but an (irregular) lattice Hamiltonian.

An explicit construction of the eigenvectors of (3.12), which can be easily reproduced in each $h_{k_1k_2\cdots k_N}$, was given together with the derivation of the eigenvalues in Ref. 24; see also Ref. 11. Let us emphasize that the spin-$\frac{1}{2}$ notion arising in the above is of the purely topological (classical topological invariant) origin; hence, with H_{xyz} in mind, we should say rather about an interacting system of topological spins $\frac{1}{2}$.

B. Before, we have introduced the Hamiltonian system in $h_{k_1\cdots k_N}$ by allowing mappings among configurations $\{R_1,\ldots,R_N\}$ at a fixed choice of $k_1 < \cdots < k_N$, which resulted in the spin $\frac{1}{2}$ H_{xyz} operator as the appropriate generator of time translations.

We can, however, define a more restrictive set of mappings, by following the route of Refs. 25 and 26, which aims at the description of particle scattering in 1 + 1 dimensions, for systems subject to the infinite set of conservation laws. Namely, let us adopt the following scattering principles: (1) absence of particle production; (2) equality of the sets of initial and final momenta $\{p_1,\ldots,p_N\} = \{p'_1,\ldots,p'_N\}$; (3) if there is more than one type of particles involved, then the numbers n_i of particles of the same type are unchanged in the scattering process.

Now let us make an assumption that the classical soliton momentum indices appearing in $h_{k_1\cdots k_N}$ are the "particle momenta" of the would-be scattered extended quantum systems. Then $h_{k_1\cdots k_N}$ appears as a substitute (albeit completely different in its structure) of the conventional N-particle Fock space sector: $\otimes_{i=1}{}^N(h_i)$. Instead of speaking about different (topological $R = \pm\frac{1}{2}$) spin projections for a given "particle," we can view them as completely distinct species, having thus introduced a soliton and antisoliton as the two-"particle" types. Due to requirement (3), the number of soliton or antisoliton labels remains unchanged in any scattering, and, moreover, by (1)–(3) we find that[25] the S operator acts on the states in $h_{k_1\cdots k_N}$ by a possible permutation of momentum labels. In terms of soliton states

$$S:\ |k_1,R_1\ldots k_N,R_N)\to|k_{\pi(1)}, R_1,\ldots,k_{\pi(N)},R_N) = |k_1, R_{\pi(1)},\ldots,k_N, R_{\pi(N)}), \quad (3.13)$$

which means that the scattering process is described by an "exchange" of topological charges at a fixed $k_1 < \cdots < k_N$ sequence, which is subject to our requirements (3). Here

$$h_{k_1\cdots k_N} \ni |f,k_1,\ldots,k_N) = |f):$$
$$= \sum_{\{R_1\cdots R_N\}} f_{R_1\cdots R_N}|k_1,R_1,\ldots,k_N, R_N) \quad (3.14)$$

so that

$$S:|f)\to \sum_{\{\underline{R}\}} f_{R_1\cdots R_N}|k_1, R_{\pi(1)},\ldots,k_N, R_{\pi(N)}) = \sum_{\{\underline{R}\}} f_{R_{\pi(1)}\cdots R_{\pi(N)}}|k_1,R_1,\ldots,k_N, R_N) = |f') \quad (3.15)$$

with the normalization

$$(k_1, R_1,\ldots,k_N, R_N|k_1, R'_1,\ldots,k_N,R'_1) = \delta_{R_1R'_1}\cdots\delta_{R_NR'_N}. \quad (3.16)$$

Because of (1)–(3) the S operator should be factorized if defined on states in $h_{k_1\cdots k_N}$ (see Refs. 25 and 26):

$$(g'|f) = (g, k_1\ldots,k_N|S|f, k_1\ldots,k_N) = (g, k_1,\ldots,k_N|\prod_{1\leqslant i<j\leqslant N} S(k_i,k_j)|f, k_1\ldots,k_N), \quad (3.17)$$

where, since the two-particle S operators $S(k_i, k_j)$ in general fail to commute, it is necessary to specify the order of factors occuring in (3.17). A possible choice which corresponds to the ordering of momenta $k_1 < \ldots < k_N$ (see Ref. 25) is

$$S_{N-2,N}\cdots(S_{2N}\cdots S_{23})(S_{1N}\cdots S_{12}) = \prod_{1\leqslant i<j\leqslant N} S_{ij}, \quad (3.18)$$

where

$$S_{ij}S_{kl} = S_{kl}S_{ij},$$
$$S_{ij}S_{ik}S_{jk} = S_{jk}S_{ik}S_{ij}, \tag{3.19}$$

i,j,k,l all unequal. Let us once more recall the sharp momentum ordering we use in the above.

Suppose for a while that $h_{k_1 \ldots k_N}$ is formally constructed without any reference to the $k_1 < \ldots < k_N$ demand. Then let us notice that the antisymmetrizing symbol

$$\sigma(k_1,\ldots,k_N) = \prod_{1 < i < j \leqslant N} [\Theta(k_i - k_j) - \Theta(k_j - k_i)] \tag{3.20}$$

with $\Theta(p) = 1$ for $p > 0$ and $\Theta(p) = 0$ for $p \leqslant 0$ equals 1 except for the case of an odd permutation when it equals -1, and for the case of coinciding momenta ($k_i = k_j$ for some choice of i, j) when it equals 0.

Therefore, σ^2 is nonnegative and equals 1 for any choice of noncoinciding k's in the momentum sequence. It implies that

$$\sigma^2(1-\sigma^2) = 0, \quad \sigma^3 = \sigma,$$
$$h_{k_1 \ldots k_N} = \sigma^2 h_{k_1 \ldots k_N} \oplus (1-\sigma^2) h_{k_1 \ldots k_N} = h^1_{k_1 \ldots k_N} \oplus h^2_{k_1 \ldots k_N} \tag{3.21}$$

and $h^1_{k_1 \ldots k_N}$ can always be written in the form (3.14) with $k_1 < \cdots < k_N$. The analysis of Ref. 27 (see also Ref. 11) shows that $h^1_{k_1 \ldots k_N}$ includes Fermi states of the Bose system defined in $h_{k_1 \ldots k_N}$ with k's unrestricted by (3.20).

Because the soliton Hilbert space of (3.14) respects the sharp ordering of momenta, we have $\sigma^2 h_{k_1 \ldots k_N} = h^1_{k_1 \ldots k_N} = h_{k_1 \ldots k_N}$. Consequently, the soliton "particle" scattering is the same as the scattering of particles subject to the Pauli exclusion principle.

Since $R = +\frac{1}{2} \rightarrow R = -\frac{1}{2}$ implies the soliton $\rightarrow$ antisoliton, i.e., the "particle" $\rightarrow$ "antiparticle" mapping, the general features of the scattering (3.15) should coincide with these observed for the conventional Fermi massive Thirring model, irrespective of the fact that we make the whole of the construction for the Bose system only.

The state $|f, k_1,\ldots,k_N)$ is a normalized vector for any N; hence the two-particle S matrices S_{ij} can be studied in more detail:

$$S_{ij} = T_{ij} + R_{ij},$$
$$(f, k_1, k_2 | T_{ij} | f, k_1, k_2) = t_{ij}, \tag{3.22}$$
$$(f, k_2, k_4 | R_{ij} | f, k_2, k_1) = r_{ij}.$$

In particular, if we consider a particle–antiparticle (i.e., soliton–antisoliton) pair, we have only four functions occurring in (3.22): $t_{\bar{f}\bar{f}}, t_{f\bar{f}}, t_{\bar{f}\bar{f}}, r_{f\bar{f}}$, which are still related by crossing, symmetry and unitarity. If to introduce the rapidity Θ,

$$k^0_i = m_i \cosh \Theta_i, \quad k^1_i = m_i \sinh \Theta_i,$$
$$(k_i + k_j)^2 = m_i^2 + m_j^2 + 2m_i m_j \cosh \Theta_{ij}, \; \Theta_{ij} = \Theta_i - \Theta_j, \tag{3.23}$$
$$(k_i - k_j)^2 = m_j^2 + m_i^2 + 2m_i m_j \cosh(i\pi - \Theta_{ij}),$$

then, by making use of the crossing and unitarity relations, one arrives[25] at

$$t = t_{\bar{f}\bar{f}}, \quad r = r_{f\bar{f}}, \quad u = t_{ff} = t_{\bar{f}\bar{f}}$$
$$t(\Theta) = u(i\pi - \Theta), \quad r(\Theta) = r(i\pi - \Theta), \tag{3.24}$$

and

$$r^2(\Theta) = t^2(\Theta)\{1 - [t(-\Theta)t(\Theta)^{-1}]\}, \tag{3.25}$$

which proves that the scattering is described by only one independent function, $t(\Theta)$, say.

A particular choice of $t(\Theta)$ determines the quantum field theory model which governs the scattering process. If we, for example, decide to follow Refs. 25 and 28, then the choice of the transmission amplitude:

$$t(\Theta) = (-1)^\lambda \prod_{k=1}^{\lambda} \frac{e^{\Theta - i\pi k/\lambda} + 1}{e^{\Theta} + e^{-i\pi k/\lambda}}, \; \lambda \text{ integer}, \tag{3.26}$$

corresponds to the so called soliton–antisoliton scattering, and $t(\Theta)$ has poles at $\Theta_k = i\pi(1 - k/\lambda)$ which is in relation with the WKB sine-Gordon spectrum: $m_k = 2m \sin \pi k/2\lambda$ provided that $\lambda = 8\pi/\gamma = 8\pi/\beta^2(1-\beta^2/8\pi)$ and the coupling term of the sine-Gordon equation reads $(m^2/\beta)\sin \beta\phi$. Due to the fact that λ is chosen to be an integer, the reflection amplitude $r(\Theta)$ vanishes.

Consequently, we have found it possible to introduce in the soliton Hilbert space $h_{k_1 \ldots k_N}$, a concept of what is usually known as the "soliton-antisoliton scattering" for the quantized sine-Gordon system. If we make one more identification, $\lambda = 1 + 2g/\pi$, the amplitude (3.26) describes the transmission phenomena in the massive Thirring model with the coupling constant g.

4. ON REDUCIBLE FIELD ALGEBRAS IN QUANTUM ELECTRODYNAMICS

A. One says that a representation $\pi(\mathscr{U})$ of some (field) algebra $\mathscr{U}$ is reducible in a Hilbert space $\mathscr{H}$, if there is at least one nontrivial (i.e., different from zero and unity) operator R, which (1) commutes in $\mathscr{H}$ with the whole of $\pi(\mathscr{U})$ and (2) is not an element of $\pi(\mathscr{U})$

In terms of quantum fields, one may assume that $\mathscr{U}$ is generated by (say) the CCR algebra creation–annihilation operators $\{a^*, a\}$ of some Bose field. Then the weakest statement about R is that it cannot be *solely* constructed in terms of $\{a^*, a\}$[12–14]: Obviously, for an irreducible representation, any element of $\pi(\mathscr{U})$ can be given as a function of $\{a^*, a\}$. A particular example of such a situation was considered before in Sec. 2. One of the observations of Ref. 29 was that a quantized solution of the free massive Dirac equation, satisfying the usual CAR algebra commutation relations can be reconstructed in terms of the two potential Maxwell (Bose) operators. Then, upon imposing suitable constraints, one can arrive at the free electromagnetic field (the Bose quantized Coulomb gauge potential $\hat{A}_\mu$) reconstruction of $\hat{\psi}(x)$, so that $\hat{\psi}(x) = \psi(\hat{A}_\mu)(x)$.

If by $\mathscr{U}(\hat{A})$ we denote the electromagnetic free field algebra, then $\hat{\psi}$ is necessarily an element of it, as all its spinor components can be *solely* expressed in terms of $\hat{A}_\mu$. However, then $\hat{\psi}$ does not in any case commute with $\hat{A}_\mu$: $[\hat{\psi}(x), \hat{A}_\mu(y)]_- \neq 0$,[5] while the mutual commutativity requirement $[\hat{\psi}(x), \hat{A}_\mu(y)]_- = 0$ lies at the foundations of the conventional (perturbative) investigations in the infrared quantum electrodynamics.[17–19] On the other hand, this assumption is not at all necessary for the understanding of the QED Hilbert space structure, as described in Ref. 20. At this point, it is worthwhile to recall the fundamental conjecture of Ref. 20 that the asymptotic electron (Dirac) field must not locally commute with the electromagnetic field. Consequently, even if one follows the traditional route by starting from the mutually commuting free fields $[\hat{\psi}(x), \hat{A}_\mu(y)]_- = 0$, the final goal must be a construction of the asymptotic free fields (arising as weak limits of the interpolating ones) which do fail to mutually commute.

In the Kulish–Faddeev–Zwanziger approach,[17,19] one incorporates the radiation/Coulomb phase operator so as to arrive at the asymptotic electron field $\hat{\psi}_{\rm as} = \psi_{\rm as}(\hat{\psi}, \hat{\psi}, \hat{A}_\mu)$, which is a nonlocal function of the mutually commuting plane wave solutions to respective free field equations: nonasympotic electron $\hat{\psi}(x)$ and photon $\hat{A}_\mu$ ones (see also Ref. 30).

While constructing $\hat{\psi}_{\rm as}$, one believes[19] that an appropriate Hilbert space is the direct product of the free field representation space for photons and the traditional Fock representation space for electrons. Let us mention that this kind of assumption has been made by Matsumoto, *et al.*[21,22] to give account of the collective degrees of freedom for soliton fields.

On the other hand, it is well known that the free electromagnetic field algebra has a highly reducible representation in the Hilbert space of infrared states,[6,18,20,30] hence quite a natural way of getting the mutually commuting free fields $\{\hat{\psi}, \hat{A}_\mu\}$ would be to follow the construction of Sec. 2. *Then the nonasymptotic free Fermi field would be in principle identifiable in the commutant of the (reducible) free photon field algebra.*

For this purpose we need, however, an appropriate set of electromagnetic field (coherent) states. In the traditional perturbative framework[6,31,17,19,30] one encounters the coherent photon states, which describe the soft photon clouds accompanying one or more Dirac particles. Like the soliton states $|p_1, R_1, \lambda_1, ..., p_N, R_N, \lambda_N)$ of (3.7) the photon coherent states $|p_1, e_1, ..., p_N, e_N)$ can be used to generate the respective incomplete direct product spaces, which carry pairwise unitarily inequivalent irreducible representations of the CCR algebra. However, in the photon case the parametrization is not rich enough, and the translation freedom, which is so crucial for the procedures of Sec. 2, is lacking.

B. Let $\{b_s^*(\mathbf{k}), d_s^*(\mathbf{k}), b_s(\mathbf{k}), d_s(\mathbf{k})\}_{=1,2}$ be the CAR algebra generators associated with the free Dirac field,

$$\hat{\psi}(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \left(\frac{m}{p_0}\right)^{1/2} \sum_s [b_s(\mathbf{p}) w_s(\mathbf{p}) e^{i\mathbf{px}} + d_s^*(\mathbf{p}) v_s(\mathbf{p}) e^{-i\mathbf{px}}]\, d\mathbf{p},$$

$$\hat{\psi}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int \left(\frac{m}{p_0}\right) \sum_s [b_s^*(\mathbf{p}) \overline{w}_s(\mathbf{p}) e^{-i\mathbf{px}} + d_s(\mathbf{p}) \overline{v}_s(\mathbf{p}) e^{i\mathbf{px}}]\, d\mathbf{p}, \tag{4.1}$$

$$[b_i(\mathbf{p}), b_j^*(\mathbf{q})]_+ = \delta(\mathbf{p}-\mathbf{q})\delta_{ij} = [d_i(\mathbf{p}), d_j^*(\mathbf{p})]_+,$$

and let $\{a_\mu^*(\mathbf{p}), a_\mu(\mathbf{p})\}$ be the CCR algebra ones for the free Maxwell field:

$$\hat{A}_\mu(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int [a_\mu^*(\mathbf{k}) e^{-i\mathbf{kx}} + a_\mu(\mathbf{k}) e^{i\mathbf{kx}}] \frac{d\mathbf{k}}{(2k_0)^{1/2}} \tag{4.2}$$

$$[a\mu(\mathbf{k}), a_\nu^*(\mathbf{p})]_- = -g_{\mu\nu}\delta(\mathbf{k}-\mathbf{p})$$

with $\operatorname{diag} g_{\mu\nu} = (1, -1, -1, -1)$, $p = ((\mathbf{p}^2 + m^2)^{1/2}, \mathbf{p})$, $k = (|\mathbf{k}|, \mathbf{k})$. The mutual commutativity condition

$$[a_\mu^\#(\mathbf{k}), b_s^\#(\mathbf{p})]_- = 0 = [a_\mu^\#(\mathbf{k}), d_s^\#(\mathbf{p})]_- \tag{4.3}$$

is imposed according to convention.

The electric charge operator for the Dirac particles reads

$$Q = -e \int d^4p\, \delta(p^2 - m^2)\Theta(p^0) \sum_s [b_s^*(p) b_s(p) - d_s^*(p) d_s(p)]. \tag{4.4}$$

One knows that the n-particle–antiparticle Dirac state vector

$$b_{s_1}^*(\mathbf{p}_1) \dots b_{s_n}^*(\mathbf{p}_n) d_{t_1}^*(\mathbf{q}_1) \cdots d_{t_m}^*(\mathbf{q}_m)|0)_F \tag{4.5}$$

induces the infrared coherent photon state describing the associated radiation

$$|\mathbf{p}_1, e_1, ..., \mathbf{p}_{n+m}, e_{n+m}) = \exp\left\{\frac{e}{(2\pi)^{3/2}} \int [f_{n,m}^\mu(\cdots|\mathbf{k}) a_\mu^*(\mathbf{k}) - \bar{f}_{n,m}^\mu(\cdots|\mathbf{k}) a_\mu(\mathbf{k})] \frac{d\mathbf{k}}{(2k_0)^{1/2}}\right\}|0)_B. \tag{4.6}$$

Here $|0)_F$ and $|0)_B$ are the respective Fock vacua and one should realize that $|\mathbf{p}_1, e_1, ..., \mathbf{p}_{n+m}, e_{n+m})$ is not an element of the Fock space, since

$$f_{n,m}^\mu(\cdots|\mathbf{k}) = \left(\sum_{i=1}^n \frac{p_i^\mu}{p_i \cdot k} - \sum_{i=1}^m \frac{q_i^\mu}{q_i \cdot k}\right) \times \varphi(\mathbf{k}, \mathbf{p}_1, ..., \mathbf{p}_n, \mathbf{q}_1, ..., \mathbf{q}_m), \tag{4.7}$$

where $\varphi(\mathbf{k}, \mathbf{p}_1 ..., \mathbf{p}_n, \mathbf{q}_1, ..., \mathbf{q}_m) = 1$ for $|\mathbf{k}| \leqslant \delta \ll 1$ while φ rapidly vanishes for $|\mathbf{k}| > \delta$. Notice that because of (4.5) no coinciding $(\mathbf{p}, e)$ pairs can appear in (4.6). By introducing the orthonormal transverse polarization vectors $\epsilon_s^\mu(\mathbf{k})$, $s = 1, 2$, $\epsilon_s^0(\mathbf{k}) = 0$, $\boldsymbol{\epsilon}_s \cdot \mathbf{k} = 0$ we can rewrite (4.6) in the (noncovariant) Chung form,[31]

$$|\mathbf{p}_1, e_1, ..., \mathbf{p}_{n+m}, e_{n+m}) = \exp\left\{\frac{e}{(2\pi)^{3/2}} \times \int \sum_{s=1}^2 [f_{n,m}^s(\cdots|\mathbf{k}) a_s^*(\mathbf{k}) - \bar{f}_{n,m}^s(\cdots|\mathbf{k}) a_s(\mathbf{k})] \frac{d\mathbf{k}}{(2k_0)^{1/2}}\right\}|0)_B := W_{nm}(\underline{p}, \underline{e})|0)_B, \tag{4.8}$$

where

$$f_{n,m}^s(\cdots|\mathbf{k}) = f_{nm}(\cdots|\mathbf{k}) \cdot \epsilon_s(\mathbf{k}) \tag{4.9}$$

so that

$$[a_i(\mathbf{k}), a_j^*(\mathbf{p})]_- = \sum_{\mu\nu} \epsilon_i^\mu(\mathbf{k})\,\epsilon_j^\nu(\mathbf{p})[a_\mu(\mathbf{k}), a_\nu^*(\mathbf{p})]_- = \delta_{ij}\delta(\mathbf{k}-\mathbf{p}) \tag{4.10}$$

provided $\Sigma_{\mu\nu}(-g_{\mu\nu})\epsilon_i^\mu(\mathbf{k})\epsilon_j^\nu(\mathbf{k}) = \delta_{ij}$.

C. In the above discussion we have distinguished a ball Ω with the radius smaller than δ in the momentum space. Let δ be the photon momentum detectability threshold. Then photons with $|\mathbf{k}|\leqslant\delta$ may be called "soft" while those with $|\mathbf{k}|>\delta$ the "hard" ones.[31] The soft photons are described by the previously introduced coherent states, while, for the hard ones, the conventional occupation number representation is adopted in the literature.

Let Ω_s be the momentum space region with $s\cdot\delta\leqslant|\mathbf{k}|<(s+1)\delta$. The previous Ω is Ω_0 in the present notation. Let $\chi_s(\mathbf{p})$ be the characteristic function of the set Ω_s: $\chi_s(\mathbf{p})=1$ for $\mathbf{p}\in\Omega_s$, 0 otherwise, and let V_s be the respective momentum space volume of Ω_s. Then we introduce

$$a_i^\#(s) = \frac{1}{\sqrt{V_s}}\int d\mathbf{p}\; a_i^\#(\mathbf{p})\chi_s(\mathbf{p}),$$
$$[a_i(s), a_j^*(t)]_- = \delta_{ij}\delta_{st},\; i,j=1,2 \;\; s,t=0,1,2,\cdots, \tag{4.11}$$

and quite analogously

$$[b_i(s), b_j^*(t)]_+ = \delta_{ij}\delta_{st} = [d_i(s), d_j^*(t)]_+ \tag{4.12}$$

for Fermi operators (4.1).

At this point let us consider a single hard $(s>0)$ photon mode $\{a_i^*(s), a_i(s)\}_{i=1,2}$. By $W_s(a^*,a)$ we denote a polynomial in $a_i^*(s)$, $a_i(s)$, $i=1,2$, s fixed. Let us consider the set of all such polynomials $\{W_s(a^*,a)\}$, s fixed. Then the Hilbert space closure of the *set of vectors* $\{W_s(a^*,a)|0)_B\}$ is a Hilbert space $h_s = \overline{\{W_s(a^*,a)|0)_B\}}$ of the sth photon mode.

The lattice index s will be, for simplicity, omitted in below. While in $h_s = h$ we have the two Bose degrees of freedom $\{a_i^*, a_i\}_{i=1,2}$. By using them we can easily construct the infinitesimal generators of the $E(2)$ group Lie algebra in h, namely, the obvious formulas:

$$a = q+ip,\quad a^* = q-ip \tag{4.13}$$

allows us to introduce in h the two translation generators

$$N_j = (1/2i)(a_j - a_j^*) = p_j,\; q_j = \tfrac{1}{2}(a_j + a_j^*),\; j=1,2, \tag{4.14}$$

which together with the rotation generator

$$\mathscr{J}_3 = q_1p_2 - q_2p_1 \tag{4.15}$$

form the $E(2)$ group Lie algebra

$$[\mathscr{J}_3, N_1]_- = iN_2,$$
$$[\mathscr{J}_3, N_2]_- = -iN_1,\; [N_1,N_2]_- = 0. \tag{4.16}$$

In the most obvious Schrödinger representation (4.14)–(4.15) read

$$N_1 = -i\frac{\partial}{\partial u},\; q_1 = u,\; N_2 = -i\frac{\partial}{\partial v},\; q_2 = v,$$
$$\mathscr{J}_3 = -i\left(u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}\right),\; \hbar = c = 1. \tag{4.17}$$

The unitary in h translation operators have the form

$$T_\lambda = \exp i\boldsymbol{\lambda}\mathbf{N} = \exp\sum_{j=1}^{2}\frac{\lambda_j}{2}(a_j - a_j^*); \tag{4.18}$$

hence, if applied to the Fock vacuum, they generate the coherent states for the sth (hard photon) mode, parametrized by real labels λ_1, λ_2:

$$T_\lambda|0) = |\lambda_1,\lambda_2),\;\; a_j|\lambda_1,\lambda_2) = \tfrac{1}{2}\lambda_j|\lambda_1,\lambda_2). \tag{4.19}$$

If to recall that the coherent soft photon states (4.8) were constructed under an assumption that all entering photon modes belong to Ω_0, we find that each soft photon state may be considered as a substitute for the (Fock) vacuum for all non-Ω_0 (i.e., hard) photon modes:

$$a_j(s)|\mathbf{p}_1,e_1,\ldots,\mathbf{p}_{n+m},e_{n+m}) = 0$$
$$\forall j=1,2,\;\; \forall s>0. \tag{4.20}$$

On the other hand, due to the sufficiently bad $\mathbf{k}=0$ behavior of the boson transformation parameters f^μ,

$$a_\mu(\mathbf{k}) \rightarrow a_\mu(\mathbf{k}) + f^\mu_{nm}(\cdots|\mathbf{k}) \tag{4.21}$$

the hard photon (but soft) vacua (4.7) generate the rich family of unitarily inequivalent representations of the photon field (CCR) algebra, despite the fact that one exploits only the $\mathbf{k}\in\Omega_0$ modes for the construction of generating vectors. It is rather clear that the concentration on the infrared aspect of the radiation field made people[6,17–20,30,31] not even notice that theory does not seem to forbid such boson transformation parameters for the radiation field, which (1) behave like $f^\mu_{n,m}(\cdots|\mathbf{k})$ of (4.7) for $\mathbf{k}\in\Omega_0$ i.e., $|\mathbf{k}|\ll 1$, but (2) behave as badly at $|\mathbf{k}|\rightarrow\infty$, as, say, the sine-Gordon parameters (1.8). Property (1) would allow for the standard infrared construction, while property (2), upon appropriately varying the boson transformation parameters with respect to their (bad) $|\mathbf{k}|\rightarrow\infty$ behavior, allows in principle for the construction of the set of unitarily inequivalent families of infrared representations: Within each family the unitarily inequivalent infrared representations would still persist.

Below we shall not enter into the problem of determining the boson transformation parameters which in addition to (1) and (2) would exhibit a consistency with the equations of motion for the radiation field [as $f^\mu_{n,m}$ of (4.8) do]. Instead we shall assume the appropriate parameters to be granted, and analyze some consequences of this ansatz.

D. Let us notice that $f^\mu_{n,m}(\cdots|\mathbf{k})$ may be written as a sum of contributions following from single charged particles:

$$f^\mu_{n,m}(\cdots|\mathbf{k}) = \sum_{i=1}^{n}\frac{p_i^\mu}{p_i\cdot k}\varphi(\mathbf{k},p_i) - \sum_{j=1}^{m}\frac{q_i^\mu}{q_i\cdot k}\varphi(\mathbf{k},q_i) = \sum_{i=1}^{n+m} f^\mu(e_{ij},\mathbf{r}_i|\mathbf{k}), \tag{4.22}$$
$$e_i = \pm 1,\; \mathbf{r}_i = \mathbf{p}_i,\; i\leqslant n,\; \mathbf{r}_i = \mathbf{q}_i,\; i>n,$$
$$\varphi(\mathbf{k},r_i) = 1 \;\text{ for }\; \mathbf{k}\in\Omega_0\;\; \forall i=1,2,\ldots,n+m.$$

Our idea is to introduce a modified boson transformation parameter in the place of this in (4.7)–(4.8):

$$f^\mu(e,\mathbf{r},\lambda,\lambda'|\mathbf{k}) = \chi_0(\mathbf{k})f^\mu(e,\mathbf{r}|\mathbf{k}) + [1-\chi_0(\mathbf{k})]g^\mu(e,\mathbf{r},\lambda,\lambda'|\mathbf{k}) \tag{4.23}$$

with $\chi_0(\mathbf{k})=1$ for $\mathbf{k}\in\Omega_0$, 0 otherwise. We demand further that the $|\mathbf{k}|\rightarrow\infty$ dependence be regulated by the choice of two real parameters λ,λ'. An analogy with the sine-Gordon considerations of Sec. 2 suggests that the modified coherent photon states

$$|\mathbf{p}_1, e_1, \lambda_1, \lambda_1', ..., \mathbf{p}_N, e_N, \lambda_N, \lambda_N') = \exp\left\{ \frac{e}{(2\pi)^{3/2}} \int \sum_{s=1}^{2} \sum_{j=1}^{N} [f^s(\mathbf{p}_j, e_j, \lambda_j, \lambda_j'|\mathbf{k}) a_s^*(\mathbf{k}) - f^s(\mathbf{p}_j, e_j, \lambda_j \lambda_j'|\mathbf{k}) a_s(\mathbf{k})] \frac{d\mathbf{k}}{(2k_0)^{1/2}} \right\} |0)_B \tag{4.24}$$

should form the set of vectors, which are pairwise neither equivalent nor weakly equivalent, unless the respective parametric sets $\mathscr{P}$ and $\mathscr{P}'$ with $\mathscr{P} = \{\mathbf{p}_1, e_1, \lambda_1, \lambda_1', ..., \mathbf{p}_N, e_N, \lambda_N, \lambda_N'\}$ do coincide: $\mathscr{P} \cap \mathscr{P}' = \mathscr{P} = \mathscr{P}'$. Then at a fixed choice of N, and $\{\lambda_i, \lambda_i'\}_{1<i<N}$ we remain on the level of standard infrared recipes. The infrared photon Hilbert spaces received from the generating vectors $|\mathbf{p}_1, e_1, ..., \mathbf{p}_N, e_N)_{\underline{\lambda}\underline{\lambda}'}$ by varying $\mathbf{p}_i$, e_i carry pairwise inequivalent representations of the photon field algebra. In addition, the infrared families arising for different choices of $\{\lambda_i, \lambda_i'\}_{1<i<N}$ are unitarily inequivalent, albeit for each fixed choice of $\{\lambda_i, \lambda_i'\}_{1<i<N}$ the same infrared physics is described.

At this point let us fix N and the other charged particles data $\{e_i, \mathbf{p}_i\}_{1<i<N}$. For each sequence $\{\lambda_i, \lambda_i'\}_{1<i<N}$ we deal with a separable Hilbert space IDPS $(|e, p, \underline{\lambda}, \underline{\lambda}')_N)$ [whose orthonormal basis system we distinguish by an additional index $n = 1, 2,...$; compare, (2.9)]. We shall adopt both the direct integral procedures of Sec. 2 [see (2.20)] and the spin-$\frac{1}{2}$ SU(2) operator construction of (3.10) to arrive at operators in the Hilbert space:

$$\mathscr{H}_f^N = \sum_{\{e\}} \int_{R^1}^{\oplus} \text{IDPS}(|e, p, \underline{\lambda}, \underline{\lambda}')_N) \, d\mu(\underline{\lambda}, \underline{\lambda}'). \tag{4.25}$$

Let us introduce

$$A_j^{\alpha\pm} = \frac{1}{\sqrt{2}}(Q_j^{\alpha\pm} + iP_j^{\alpha\pm} = \sum_{\text{conf}\{e\}}^{(j)} \int_{R^1}^{\oplus} \cdots \int_{R^1}^{\oplus} d\mu(\underline{\lambda}, \underline{\lambda}') \times \sum_n |n, ..., \mathbf{p}_j, \pm, \lambda_j^1, \lambda_j^2, \cdots) \times \frac{1}{\sqrt{2}} \left(\lambda_j^\alpha + \frac{\partial}{\partial \lambda_j^\alpha} \right) (n, ..., \mathbf{p}_j, \pm \lambda_j^1, \lambda_j^2, \cdots | \tag{4.26}$$

with $(A_j^{\alpha\pm})^* = A_j^{*\alpha\pm}$, $\alpha = 1, 2, \lambda_j = \lambda_j^1$, $\lambda_j' = \lambda_j^2$, $j = 1, 2..., N$, the sum $\Sigma_{\text{conf}\{e\}}^{(j)}$ having exactly the same meaning as the one in (3.10). The object $A_j^{*\alpha\pm}$ differs from $A_j^{\alpha\pm}$ in the replacement of $(1/\sqrt{2})(\lambda_j^\alpha + \partial/\partial\lambda_j^\alpha)$ by $1/\sqrt{2}(\lambda_j^\alpha - \partial/\partial\lambda_j^\alpha)$.

Due to the orthogonality of inequivalent coherent states constituting objects (4.26) we have

$$A_j^{\alpha\pm} A_j^{\alpha\mp} = 0 = A_j^{\alpha\pm} A_j^{*\alpha\mp}, \quad [A_j^{\alpha+}, A_j^{*\alpha+}]_- + [A_j^{\alpha-}, A_j^{*\alpha-}]_- = 1 \tag{4.27}$$

so that the operators

$$B_j^{\alpha q} = A_j^{\alpha+} + i(-1)^{q-1} A_j^{\alpha-}, \quad \alpha = 1, 2, \; q = 1, \; 2, \; j = 1, 2, ..., N \tag{4.28}$$

together with their Hermitian adjoints satisfy the (infinitesimal) CCR algebra commutation relations

$$[B_j^{\alpha q}, B_{j'}^{*\alpha' q'}]_- = \delta_{\alpha\alpha'} \delta_{qq'} \delta_{jj'}, \quad [B_j^{\alpha q}, B_{j'}^{\alpha' q'}]_{-1} = 0 = [B_j^{*\alpha q}, B_{j'}^{*\alpha' q'}]_{-1}. \tag{4.29}$$

The operators (4.28) do carry the charge–spin–momentum labels of the Dirac particles (fermions), though acting explicitly in the (direct integral) Hilbert space of the (reducible) electromagnetic field algebra and satisfying the canonical commutation relations (the CCR). Notice that by construction, operators (4.28) belong to the commutant of the electromagnetic field algebra and the Fock-ness property for the representation (4.29) can be introduced analogously to this of (2.21).

Once the CCR algebra generators are given, one can exploit the study of isomorphisms between Hilbert spaces of symmetric and antisymmetric functions,[27] to construct the CAR algebra generators and then to represent them in the Hilbert space of the Bose system. For Dirac fermions, this construction has been accomplished in Refs. 32, 29, and 5 (see also Ref. 33, and the number of integral degrees of freedom $\{\alpha, q\}$ is preserved in this CCR $\to$ CAR = CAR(CCR) mapping.

Operators (4.28) completely suffice for the construction of the Fermi set $\{b_\alpha(s), b_\alpha^*(s), d_\alpha(s), b_\alpha^*(s)\}$ of (4.12), where, however, in contrast to the case of Ref. 29, the Fermi operators do belong to the commutant of the photon field algebra, and obviously (this time like in the case of Ref. 29) do not need any separate Fermi Hilbert space to have them represented.

Let us once more emphasize that the present construction differs essentially from the one given in Ref. 29 and Ref. 5, example 2, where a possible electromagnetic field content of the massive Dirac field was analyzed. Namely, in the latter case fermions were essentially arising in the photon field algebra, while in the present case they arise in the commutant of this algebra, thus not belonging to it (but acting in the photon field Hilbert space).

It seems to be a quite appealing idea that the asymptotic solution for the quantized coupled Dirac–Maxwell system given for example, by Zwanziger[19] (on the basis of the ansatz concerning the asymptotic limit of the renormalized Heisenberg electric current operator) can be given in terms of the objects we have introduced above. However, for this purpose, we find it unavoidable to have solved the existence problem for boson transformation parameters (4.23). By existence we mean a consistency with the classical equations of motion for the coupled Dirac–Maxwell system.

Let us recall[29] our statement of belief: *A necessary condition for the existence of any physically meaningful quantum field theory model is that the corresponding classical model exists and is soluble.*

For the classical Dirac–Maxwell system both the existence and (local in time) solubility were proved,[34,35] but in contrast to (1 + 1)-dimensional models the explicit solutions are painfully lacking.

Let us mention that, despite the latter problem, the relationship between the quantized Fermi–Dirac–Maxwell system and the corresponding classical (c-number) one has been investigated in the series of papers of the present author[32,29,5,36] on the quantization of spinor fields, see also Refs. 37, 38.

[1]J. R. Klauder, Phys. Rev. Lett. **20**, 763 (1972).
[2]P. Garbaczewski, J. Math. Phys. **22**, 574 (1981).
[3]P. Garbaczewski, J. Math. Phys. **22**, 1272 (1981).
[4]P. Garbaczewski, Ann. Phys. (N.Y.) **139**, 293 (1982).
[5]P. Garbaczewski, J. Math. Phys. **24**, 341 (1983).
[6] T. W. B. Kibble, J. Math. Phys. **9**, 315 (1968).
[7]L. Mercaldo, I. Rabuffo, and G. Vitiello, Nucl. Phys. B **188**, 193 (1981).
[8]I. M. Gel'fand and N. Ja. Vilenkin, *Generalized Functions* (Moscow, 1961) (in Russian), Vol. 4.
[9]M. A. Neumark, *Normed Rings* (Nauka, Moscow, 1968) (in Russian).
[10]A. E. Nussbaum, Duke Math. J. **31**, 33 (1964).
[11]P. Garbaczewski, J. Math. Phys. **24**, 651 (1983).
[12]R. Haag, Nuovo Cimento **25**, 287 (1962).
[13]H. Araki and E. J. Woods, J. Math. Phys. **4**, 637 (1963).
[14]J. R. Klauder, J. Math. Phys. **8**, 1666 (1965).
[15]J. Lopuszański and H. Reeh, J. Math. Phys. **7**, 148, 1821 (1966).
[16]R. T. Powers, Commun. Math. Phys. **4**, 3 (1967).
[17]P. P. Kulish and L. D. Faddeev, Teor. Mat. Fiz. **4**, 153 (1970).
[18]J. Fröhlich, Ann. Inst. H. Poincaré, **19**, 1 (1973).
[19]D. Zwanziger, Phys. Rev. **D 11**, 3481 (1975).
[20]J. Fröhlich, G. Morchio, and F. Strocchi, Ann. Phys. (N.Y.) **119**, 241 (1979).
[21]H. Matsumoto, N. J. Papastamatiou, G. Semenoff, and H. Umezawa, Phys. Rev. **D 23**, 1339 (1981).
[22]H. Matsumoto, N. J. Papastamatiou, G. Semenoff, and H. Umezawa, Phys. Rev. **D 24**, 406 (1981).
[23]J. G. Taylor, Ann. Phys. (N.Y.) **115**, 153 (1978).
[24]L. A. Tahtajan and L. D. Faddeev, Usp. Mat. Nauk **34**, 13 (1979).
[25]M. Karowski and H. J. Thun, Nucl. Phys. B **130**, 295 (1977).
[26]A.B. Zamolodchikov and A. B. Zamolodchikov, Ann. Phys. (N.Y.) **120**, 253 (1979).
[27]P. Garbaczewski, Commun. Math. Phys. **43**, 131 (1975).
[28]V. E. Korepin, L. D. Faddeev, Theor. Math. Phys. **25**, 1039 (1975).
[29]P. Garbaczewski, J. Math. Phys. **23**, 442 (1982).
[30]J. M. Jauch and F. Rohrlich, *Quantum Theory of Photons and Electrons* (Springer-Verlag, Berlin, 1976).
[31]V. Chung, Phys. Rev. **140**, B110 (1965).
[32]P. Garbaczewski, J. Math. Phys. **19**, 642 (1978).
[33]P. Garbaczewski, Phys. Rep. C. **36**, 65 (1978).
[34]L. Gross, Comm. Pure Appl. Math. **19**, 1 (1966).
[35]J. M. Chadam, J. Math. Phys. **13**, 597 (1972).
[36]P. Garbaczewski, "Quantization of spinor fields IV. Joint Bose-Fermi spectral problems," J. Math. Phys. (to appear).
[37]P. Garbaczewski, Nucl. Phys. B **218**, 321 (1983).
[38]P. Garbaczewski, Ann. Phys. (NY) **149** (2) (1983).