# Quantization of spinor fields. III. Fermions on coherent (Bose) domains

Piotr Garbaczewski [a]
*Department of Physics, Brown University, Providence, Rhode Island 02912*

(Received 21 July 1981; accepted for publication 7 December 1981)

A formulation of the *c*-number classics-quanta correspondence rule for spinor systems requires all elements of the quantum field algebra to be expanded into power series with respect to the generators of the canonical commutation relation (CCR) algebra. On the other hand, the asymptotic completeness demand would result in the (Haag) expansions with respect to the canonical anticommutation relation (CAR) generators. We establish the conditions under which the above correspondence rule can be reconciled with the existence of Haag expansions in terms of asymptotic free Fermi fields. Then, the CAR become represented on the state space of the Bose (CCR) system.

PACS numbers: 11.10. – z

## 1. MOTIVATION

Our basic purpose is to deal with quantum field theory models (irrespective of the space-time dimensionality) whose elements of the field algebra admit a reconstruction in terms of one or more quantum free fields. By free we understand the field solutions of standard sourceless field equations like, e.g., the Klein–Gordon, Dirac, Maxwell, etc., ones. In addition, we require the equal-time canonical (anti)commutation relations to be satisfied on appropriate domains. The latter are, however, not required to belong to the Fock space.

For quantum fields with well defined asymptotics, the above reconstruction is realized in the form of the Haag series. In what follows, by Haag series we understand any power series in terms of the normal ordered products of the CCR or CAR algebra generators, denoted: $F(a^*, a)$:, $:F(b^*, b)$:, respectively.

As is well known, the asymptotic condition is not an obvious notion even for the simplest Fermi system; compare, e.g., Ref. 1 and references therein. In this connection we admit the Haag series reconstruction of quantum fields in terms of free fields which are not the asymptotic series in the usual sense of the word.[2]

In $1 + 1$ dimensions, for all models solvable via the Bethe *ansatz* technique, the construction of the eigenstates of the Hamiltonian explicitly involves the fundamental free fields; compare, e.g., Refs. 3–5. We know, for example,[6] that in case of the sine-Gordon system the underlying field is the massive neutral scalar. In case of the massive Thirring model the free massive Dirac field is used to construct the energy eigenstates. However, to relate this quantum model to its completely integrable *c*-number (semiclassical) relative, one is forced to adopt a "bosonization" in terms of the massive neutral vector boson.[1]

A quite analogous situation appears in the infrared QED, where a bosonization of the quantum Dirac field weakly coupled to the photon field is realized in terms of the Coulomb gauge free Maxwell field potential.[1]

A common property of both the Fermi and Bose models mentioned above is that to relate quantum and classical (*c*-number) levels of a given field theory model, one starts from the Haag-like expansions $\hat{F} = F(a^*, a)$ in terms of the fundamental CCR algebra generators. Then one makes a boson transformation $a \to a + \bar{\lambda}$, $a \to a + \lambda$, where $\lambda$ is a *c*-number function, and finally calculates the Fock vacuum expectation value in the tree approximation

$$(0|\hat{F}_\lambda|0) \to (0|:\hat{F}_\lambda:|0) = (0|:F(a^* + \bar{\lambda}, a + \lambda):|0) = F(\bar{\lambda}, \lambda). \tag{1.1}$$

The functional power series $F(\bar{\lambda}, \lambda)$ stand for classical, *c*-number relatives of the quantum objects $\hat{F} = F(a^*, a)$, to which $:F(a^*, a)$: corresponds in the tree approximation. One knows that the tree approximation prescription can be used to recover the classical Euler analogs of the quantum equations of motion.

It is of special importance to know these boson transformation parameters $\lambda$, which in the tree approximation give rise to the classical solitons. This problem was partially solved (for solitons) for the Korteweg–de Vries[7] and $\lambda\Phi^4$ models,[8,9] and more generally for the sine-Gordon system.[6,10,11] The latter case, using the Orfanidis' formulas,[12] allows an identification of at least some soliton solutions of the massive Thirring model. For a few other models in connection with a coherent state description of hadrons, see Ref. 13. The tree approximation procedure can be described as follows:

$$\hat{F} = F(a^*, a) \to F(a^* + \bar{\lambda}, a + \lambda) = \hat{F}_\lambda,$$

$$(0|F(a^* + \bar{\lambda}, a + \lambda)|0): = (\lambda|F(a^*, a)|\lambda) = (\lambda|\hat{F}|\lambda), \tag{1.2}$$

$$(\lambda|:F(a^*, a):|\lambda) = F(\bar{\lambda}, \lambda),$$

where $|\lambda)$ stands for a generalized coherent state for the field (CCR) algebra. In general $|\lambda)$ is not an element of the Fock space and hence gives rise to its own $|\lambda)$th Hilbert space irreducibility sector for the CCR algebra, incomplete direct product space IDPS $(|\lambda)) \subset H$ in the general Hilbert space $H$. For the particular case of Fermi models one can start from the Haag expansions in terms of the CAR generators: $F(b^*, b)$: but then the bosonization enters via $b = b(a^*, a)$,

[a] Permanent address: Institute of Theoretical Physics, University of Wroclaw, 50-205 Wroclaw, Poland.

 © 1983 American Institute of Physics 

Downloaded 20 Apr 2007 to 156.17.88.174. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp

$b^* = b^*(a^*, a)$,[14–16] so that

$$F(b^*, b) = F[b^*, b](a^*, a) = G(a^*, a),$$
$$(0|G(a^* + \bar{\lambda}, a + \lambda)|0) = (\lambda|F[b^*, b](a^*, a)|\lambda), \quad (1.3)$$
$$(\lambda|:G(a^*, a):|\lambda) = G(\bar{\lambda}, \lambda).$$

In particular $(\lambda|b(a^*,a)|\lambda) = b(\bar{\lambda},\lambda)$, $(\lambda|b^*(a^*,a)|\lambda) = b(\bar{\lambda},\lambda) = b(\lambda,\bar{\lambda})$ correspond to $b, b^*$, respectively. Here the CAR generators are by construction acting on the Bose domain, hence we *are confronted with a serious problem of representations of the CAR algebra living in the non-Fock representations of the CCR algebra the latter being based on generalized coherent states.*

Let us recall that the case of Fock representation has been investigated and solved in Ref. 14, while the non-Fock case was not considered in full generality. We know only[17] that the CAR do allow a local representation in the Hilbert space of the Bose system, i.e., that the CAR hold true (while on a lattice) for a finite number of degrees of freedom, but may not hold true for almost all would-be Fermi degrees of freedom, upon bosonization.

As we show below, only a very special class of (Bose) coherent states allows the existence of fermions (representation of the CAR) on subspaces of IDPS($|\lambda)$) and that in general the CAR are prohibited. In the latter case, the interacting spinor field does not possess an asymptotic spinor partner ("confinement" property), and this role is played by the fundamental boson(s) affiliated with the underlying representation of the CCR algebra. More precisely, it means that in the von Neumann–Hilbert space $H$ of the Bose system we can find irreducibility domains for the CCR algebra such that the CAR can be irreducibly represented on a subspace. On these subspaces an asymptotic expansion of the interacting spinor field $\hat{\Psi} = \Psi(\hat{\psi}_{\text{in}})$ in terms of the free fermion $\hat{\psi}_{in}$ makes sense. Whenever the CCR algebra irreducibility sector in $H$ does not carry an irreducible CAR algebra representation, the underlying expansion makes no sense, and $\hat{\Psi}$ should be expanded with respect to the free boson: The free fermion is then "confined" and $\hat{\Psi}$ does not possess an asymptotic spinor partner.

## 2. MAIN THEOREM

For clarity, we shall abandon the explicitly continuous case and restrict considerations to the product representations of the CCR and CAR algebras.[17–20] We refer to Ref. 20 in connection with the role of coherent states in this case.

Let $H = \Pi_k^{\otimes} h_k$, $h_k = h \forall k = 1, 2, \ldots$ be the von Neumann infinite direct product Hilbert space. It is an infinitely reducible carrier space for the representation of the CCR algebra generated by a countable sequence of Schrödinger representations $\{a^*, a\}_j$:

$$[a_i, a_j]_-|\psi) = 0 = [a_i^*, a_j^*]_-|\psi),$$
$$[a_i, a_j^*]_-|\psi) = |\psi)\delta_{ij}, \quad (2.1)$$
$$\forall ij, \ |\psi) \in H.$$

Let $|\psi) = \Pi_k^{\otimes} f_k, f_k \in h_k$ be a product vector with the property $\|f_k\| = 1 \forall k$. With each $|\psi)$ we have associated a separable Hilbert space IDPS($|\psi)$) on which a representation of the CCR algebra acts irreducibly. Among all possible product vectors in $H$, we shall distinguish the coherent states, which can formally be obtained from the Fock vacuum $|0) \in H$, $a_i|0) = 0 \forall i$, $|0) = \Pi_k^{\otimes} f_k^0$, by applying a product mapping $U_\lambda^{\otimes}$:

$$U_\lambda^{\otimes}|0) = \Pi_k^{\otimes}(U_\lambda f^0)_k := \Pi_k^{\otimes}|\lambda)_k = |\lambda),$$
$$U_\lambda = \exp(\lambda a^* - \bar{\lambda} a), \quad \lambda_k \in C \forall k. \quad (2.2)$$

Here $|\lambda)$ is determined by fixing a denumerable sequence $(\lambda)$ of complex parameters. We have $a_i|\lambda) = \lambda_i|\lambda)$, $(\lambda|\lambda) = 1$. The incomplete direct product space based on $|\lambda)$ we denote IDPS ($|\lambda)$). Two coherent product states are equivalent: $|\lambda) \sim |\gamma)$ if and only if the series

$$\sum_k (\bar{\lambda}_k \gamma_k - \tfrac{1}{2}|\lambda_k|^2 - \tfrac{1}{2}|\gamma_k|^2) \quad (2.3)$$

converges.[20] When $\Sigma_k |1 - |(\lambda_k|\gamma_k)|| < \infty$, we talk about a weak equivalence $|\lambda) \underset{w}{\simeq} |\gamma)$. One knows[20] that the weakest condition for the CCR algebra representations acting in IDPS ($|\lambda)$), IDPS ($|\gamma)$), respectively, to be unitarily equivalent is that $|\lambda) \underset{w}{\simeq} |\gamma)$. In particular $|\gamma) \sim |\lambda) \Rightarrow |\gamma) \underset{w}{\simeq} |\lambda)$. Notice that if $\Sigma_j |\lambda_j|^2 \not< \infty$ then $|0) \underset{w}{\neq} |\gamma)$. If $\Sigma_j |\lambda_j - \gamma_j|^2 \not< \infty$ then $|\lambda) \underset{w}{\neq} |\gamma)$.

Let us denote

$$P = :\exp(-a^*a): + a^* :\exp(-a^*a):a \quad (2.4)$$

a projection on a two-dimensional subspace $h_F$ of $h$ spanned by vectors $f^0$ and $a^* f^0 = f^1$. For a countable sequence $\{a^*, a\}_j$ we introduce a corresponding countable sequence $\{P_j\}$, and observe that the operators

$$\sigma_j^+ = a_j^* :\exp(-a_j^* a_j): \equiv P_j a_j^* P_j,$$
$$\sigma_j^- = :\exp(-a_j^* a_j): a_j \equiv P_j a_j P_j \quad (2.5)$$

satisfy the following commutation relations on the Hilbert space:

$$\text{IDPS}_F(|0)) = 1_F \text{IDPS}(|0)), \quad 1_F = \Pi_k^{\otimes} P_k:$$
$$[\sigma_i^+, \sigma_j^-]_- = 0 = [\sigma_i^+, \sigma_j^+]_- = [\sigma_i^-, \sigma_j^-]_-, \quad i \neq j \quad (2.6)$$
$$[\sigma_i^-, \sigma_i^+]_+ = P_i:, \quad P_i|\psi) = |\psi) \forall i, \quad \forall |\psi) \in \text{IDPS}_F(|0)).$$

By applying the Jordan–Wigner transformation to the set $\{\sigma^+, \sigma^-\}_j$ one can easily reproduce a sequence $\{b^+, b\}_j$ of the related CAR algebra generators. *We wish to emphasize that the condition*

$$[\sigma_i^-, \sigma_i^+]_+|\psi) = |\psi) \forall i \quad (2.7)$$

*is a crucial requirement, to have the CAR algebra represented on a domain to which a vector $|\psi)$ belongs.* Notice that the relations (2.7) are immediate if $|\psi)$ appears in the form of the product vector:

$$|\psi) = \Pi_k^{\otimes}(\alpha f^0 + \beta f^1)_k,$$
$$|\alpha_k|^2 + |\beta_k|^2 = 1 \forall_k, \quad (2.8)$$
$$\alpha f^0 = 0, \quad \alpha^* f^0 = f^1.$$

Vectors of this form are the conventional product ones used to investigate representations of the CAR algebra.[18,19] Notice that (2.7) does not hold true if applied to a coherent product state $|\lambda)$. We relate the above mentioned representation of the CAR algebra to that of the spin $\frac{1}{2}$ algebra (2.6) via



Downloaded 20 Apr 2007 to 156.17.88.174. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp

the Jordan–Wigner trick:

$$b_k^* = \exp\Big(i\pi \sum_{j=1}^{k-1} \sigma_j^+ \sigma_j^-\Big)\sigma_k^+, \qquad k = 1, 2, \ldots \qquad (2.9)$$

$$b_k = \exp\Big(i\pi \sum_{j=1}^{k-1} \sigma_j^+ \sigma_j^-\Big)\sigma_k^- .$$

It is easy to verify that (2.7) reads

$$[b_k, b_k^*]_+|\psi) = [\sigma_k^-, \sigma_k^+]_+|\psi)\forall k, \qquad (2.10)$$

and that (2.6) implies

$$[b_k, b_j]_+|\psi) = 0, \quad k \neq j$$
$$[b_k, b_j^+]_+|\psi) = 0, \qquad (2.11)$$
$$[b_k^+, b_j^+]_+|\psi) = 0.$$

Moreover, if $\sigma_i^+, b_i^*, a_i^*, \sigma_i^-, b_i, a_i$ are applied to the Fock state $|0)$ we find

$$\sigma_i^+|0) = b_i^*|0) = a_i^*|0)\forall i, \quad b_i|0) = 0 = a_i|0) = \sigma_i^-|0), \qquad (2.12)$$

i.e., the basic property of the Fock representation constructed in Ref. 14.

**Theorem**: Suppose we have given IDPS($|\lambda)$), where $|\lambda)$ is a coherent product state determined by a complex sequence $(\lambda) = \{\lambda_1, \lambda_2, \ldots\}$, where $\lambda_k = |\lambda_k|\exp(i\delta_k)$, $|\lambda_k|, \delta_k \in \mathbf{R}^1$. In addition to the sequences $(|\lambda|)$ and $(\delta)$ let us introduce the three additional real ones $(\phi)$, $(\psi)$, $(\alpha)$. Assume that

(1) $\sum_k |\lambda_k|^2 = \infty$, (2) $\sum_k |\lambda_k|^4 < \infty$,

(3) $\lim_{k\to\infty} \frac{\phi_k}{|\lambda_k|^4} = A \neq 0, \infty$,

(4) $\lim_{k\to\infty} \frac{\psi_k - \sigma_k}{|\lambda_k|^2} = B \neq 0, \infty$,

(5) $\lim_{k\to\infty} \frac{\alpha_k}{|\lambda_k|} = 1.$

Then a product vector $|\psi) = \Pi_k^\otimes (uf^0 + vf^1)_k$, with

$$u_k = \cos\alpha_k \exp(i\phi_k), \quad v_k = \sin\alpha_k \exp(i\psi_k) \qquad (2.13)$$

is an element of IDPS($|\lambda)$).

*Proof*: It suffices to prove that vectors $|\lambda)$ and $|\psi)$ are equivalent. The equivalence cirterion is $\Sigma_k |z_k| < \infty$, where

$$z_k = 1 - [\cos\alpha_k \exp(i\phi_k) + |\lambda_k| \times \exp i(\psi_k - \delta_k)\sin\alpha_k]\exp(-|\lambda_k|^2/2). \qquad (2.14)$$

Let us consider $k \gg k_0 \gg 1$, when all the parameters are close to 0. Then, upon expanding $z_k$ into a Taylor series about 0, we have

$$\mathrm{Re}\, z_k \simeq 1 - \Big(1 - \frac{|\lambda_k|^2}{2}\Big)\Big[\Big(1 - \frac{\alpha_k^2}{2}\Big)\Big(1 - \frac{\phi_k^2}{2}\Big) + |\lambda_k|\Big(1 - \frac{(\psi_k - \delta_k)^2}{2}\Big)\alpha_k\Big],$$

$$\mathrm{Im}\, z_k \simeq \Big(1 - \frac{|\lambda_k|^2}{2}\Big)\Big[\Big(1 - \frac{\alpha_k^2}{2}\Big)\phi_k + |\lambda_k|(\psi_k - \delta_k)\alpha_k\Big], \qquad (2.15)$$

i.e., by virtue of (1)–(5),

$$\mathrm{Re}\, z_k \simeq |\lambda_k|^4, \quad \mathrm{Im}\, z_k \simeq (A + B)|\lambda_k|^4. \qquad (2.16)$$

Consequently,

$$\lim_{k\to\infty} \frac{|z_k|}{|\lambda_k|^4} = [1 + (A + B)^2]^{1/2} \neq 0, \infty. \qquad (2.17)$$

Because of (2) the equivalence criterion holds true, and $|\psi) \sim |\lambda)$. Consequently,

$|\psi) \in$ IDPS($|\lambda)$).

It is worth emphasizing that we must have here $\lim_{k\to\infty} |z_k| = 0$. It leads to $\mathrm{Re}\, z_k \underset{k\to\infty}{\to} 0$, i.e.,

$$[\cos\alpha_k \cos\phi_k + |\lambda_k|\sin\alpha_k \cos(\psi_k - \delta_k)] \to \exp(|\lambda_k|^2/2),$$

which holds true if and only if $|\lambda_k| \underset{k\to\infty}{\to} 0$.

*Remark 1*: Notice that in the above, at a fixed choice of parameters $|\lambda_k| \in R^+$, we still have a freedom in the choice of phases $(\delta)$ in the complex sequence $(\lambda)$, which is furthermore reflected in the appropriate freedom of choice of the phases $(\psi)$ in the product vector $|\psi)$. The latter is obviously regulated by

$$\lim_{k\to\infty} \frac{\psi_k - \delta_k}{|\lambda_k|^2} = B. \qquad (2.18)$$

A consequence of this is that if we have two sequences $(\lambda)$, $(\lambda')$,

$$\lambda_k = |\lambda_k|\exp(i\delta_k), \quad \lambda'_k = |\lambda_k|\exp(i\delta'_k)\forall k,$$

then the condition

$$\sum_k |\lambda_k|^2[\cos(\delta_k - \delta'_k) - 1] < \infty \qquad (2.19)$$

is a sufficient and necessary condition for the product vectors $|\psi) \in$ IDPS($|\lambda)$), $|\psi') \in$ IDPS($|\lambda')$) to be weakly equivalent. To see this, it is enough to notice that product vectors $|\lambda)$, $|\lambda')$ are weakly equivalent if and only if the real part of (2.3) converges. In fact

$$(\lambda|\lambda') = \exp\Big\{-\frac{1}{2}\sum_k |\lambda_k - \lambda'_k|^2 + i\sum_k \mathrm{Im}(\bar{\lambda}_k \lambda'_k)\Big\} \qquad (2.20)$$

and $\Sigma_k |\lambda_k - \lambda'_k|^2 < \infty$ is just the same as (2.19). Obviously, if $\Sigma_k |\lambda_k - \lambda'_k|^2 = \infty$, then $|\lambda) \underset{w}{\neq} |\lambda')$.

*Remark 2*: The above theorem can also be deduced as a special case of a more general theory of Ref. 19. Namely, if $h$ is a Hilbert space with an orthonormal basis $(e_k)_0^\infty$, and $p$ a projection on a linear span of $e_0, \ldots, e_N$ so that $P_N = p_1 \cdots p_N$ is a projection in IDPS($|\lambda)$), then

(1) there exists a limiting projection $P = \lim_{N\to\infty} P_N$ in IDPS($|\lambda)$);

(2) by expanding $|\lambda)_i = \Sigma_k \gamma_i^k e_k = \Sigma_k \lambda_i^k/(k!)^{1/2} e_k$, we arrive at the following conclusion:

$$P \neq 0 \text{ if and only if } \sum_i \Big[1 - \Big(\sum_{k=0}^N |\gamma_i^k|^2\Big)^{1/2}\Big] < \infty;$$

(3) the vector $|\psi)$, $P|\psi) \neq 0$ can be constructed as follows:

$$|\lambda) = \Pi_i^\otimes |\psi_i), \quad |\psi_i) = \Big(\sum_{k=0}^N |\bar{\gamma}_i^k|^2\Big)^{-1/2} \sum_{k=0}^N \bar{\gamma}_i^k e_k. \qquad (2.21)$$



Downloaded 20 Apr 2007 to 156.17.88.174. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp

In the special case of $N = 1$, we have $\gamma_i^0 = \exp(-|\lambda_i|^2/2)$ and $\lambda_i^1 = \lambda_i \exp(-|\lambda|^2/2)$, and $\Sigma_{k=1}^{\infty}|\lambda|_k^4 < \infty$, $\Sigma_{k=1}^{\infty}|\lambda_k|^2 = \infty$, is a necessary and sufficient condition for a projection $P$ to exist in IDPS($|\lambda\rangle$).

*Remark 3*: Notice that states $|\psi\rangle = \Pi_i^{\otimes}|\psi_i\rangle$ in (2.21), (2.13) have exactly the structure required by the spin 1/2 approximation procedure of Ref. 4 for quantum Bose systems. The above (Remark 2) statement is more general, however, and allows a construction of quantum spin chain states (with a fixed finite spin) in the Hilbert space of an interacting (non-Fock) Bose system; see in this connection also Ref. 15. The Holstein–Primakoff SU(2) generators

$$\begin{aligned} S_i^+ &= (2s)^{1/2} a_i^*(1 - a_i^* a_i/(2s))^{1/2}, \\ S_i^- &(2s)^{1/4}(1 - a_i^* a_i(2s))^{1/2} a, \\ S_i^3 &= s - a_i^* a_i, \end{aligned} \tag{2.22}$$

provide us with an irreducible (at each $i$th site) representation of the SU(2) group Lie algebra corresponding to spin $s = N/2$, given by

$$\mathbf{S}_P = P\mathbf{S}P, \tag{2.23}$$

where $P$ is a limiting projection of Remark 1.

## 3. DISCUSSION

Let us notice that the existence of $|\psi\rangle$ in IDPS($|\lambda\rangle$) guarantees that all vectors equivalent to $|\psi\rangle$, of the form $\Pi_k^{\otimes}(\alpha f^0 + \beta f^1)_k$, $|\alpha_k|^2 + |\beta_k|^2 = 1 \forall_K$, are elements of IDPS($|\lambda\rangle$). A Hilbert space closure of the set of all linear combinations of such equivalent product vectors, $\text{IDPS}_F(|\psi\rangle)$ is a subspace of IDPS($|\lambda\rangle$). The CAR are irreducibly represented on $\text{IDPS}_F(|\psi\rangle)$ provided $\{b^*, b\}_j$ are constructed from $\{a^*, a\}_j$ according to Ref. 14. Let us also observe[19] that once we have any product vector $|\gamma\rangle \in \text{IDPS}(|\lambda\rangle)$ with the basic property $[\sigma_j^-, \sigma_j^+]_+|\gamma\rangle = |\gamma\rangle \forall_j$ then the following two properties cannot be simultaneously satisfied: (1) $\sigma_i^-|\gamma\rangle = 0\ \forall_i$, (2) $|\gamma\rangle \neq 0$ under an additional restriction (3) $|\lambda\rangle \underset{w}{\neq} |0\rangle$, where $|0\rangle$ is a Fock state in $H$, and $|\lambda\rangle$ is a coherent product state. Consequently, there exists a unitary inequivalence of the CCR algebra representations associated with IDPS($|\lambda\rangle$), IDPS($|\lambda'\rangle$), where $|\lambda\rangle \underset{w}{\neq} |\lambda'\rangle$ implies a unitary inequivalence of the related CAR algebra representations in $\text{IDPS}_F(|\psi\rangle)$, $\text{IDPS}_F(|\psi'\rangle)$, respectively. Let us here emphasize that a particular form of the boson transformation parameter for a concrete field theory model follows from its equations of motion. This severe restriction may violate, and in general it does, the condition (2) of the Theorem of Section 2. In this case the bosonic semiclassic (i.e., the CCR representation based on the coherent product state) prevents us from having represented the CAR on the appropriate domain. *The "semiclassical Hilbert space" allows at most a local representation of the* CAR *on a subspace*,[17] i.e., with a property $[b_i, b_i^*]_+|\gamma\rangle = |\gamma\rangle$ for a *finite*, though arbitrarily large, number of modes, $|\gamma\rangle$ belonging to this subspace. Notice that by defining an arbitrary polynomial $W_{(j)}(b^*, b)$ in terms of "bosonized" Fermi generators $\{b^*, b\}_{j\in(j)}$, $(j)$ being a finite set of indices, we arrive at the following definition of *locally Fermi, but globally coherent (Bose) quantum states*:

$$|\lambda\rangle_{BF} = |\lambda\rangle_{(j)} = W_{(j)}(b^*, b)|\lambda\rangle. \tag{3.1}$$

One can easily verify that on $|\gamma\rangle_{BF}$ the CAR hold true for all $j\in(j)$, but not for $j\notin(j)$, albeit $[b_i, b_j^*]_+ = 0$ for all $i\neq j$; compare, e.g., Ref. 17. Suppose now that the coherent product state $|\lambda\rangle$ obeys the restrictions of the theorem of Sec. 2. Then, the *semiclassical Hilbert space IDPS* ($|\lambda\rangle$) *does carry a Fermi system on a subspace*: the CCR algebra possesses the manifestly Fermi states in IDPS($|\lambda\rangle$); compare, e.g., also Ref. 21. In this case, we can say that *both fundamental free bosons and fermions can exist in the same state space on an equal footing*. However, if the restrictions of the theorem are not satisfied by $|\lambda\rangle$, then the only fundamental free field that remains is the Bose one. No fundamental free fermions are allowed. In the case of interacting Fermi systems such a phenomenon would correspond to a "confinement" of their fundamental free excitations (absence of asymptotic free fermions).

*Example 1*: *Sine-Gordon versus massive Thirring model.*

(1) Both the Mandelstam[22] construction and the Orfanidis[12] observations allow a bosonization of the massive Thirring field in terms of the interacting sine-Gordon field under appropriate constraints. Namely, we can symbolically write an operator identity:

$$\hat{\Psi} = \Psi(\hat{\Phi}), \quad \hat{\Phi} = \Phi(\hat{\phi}_{\text{in}}), \quad (\Box - m^2)\hat{\phi}_{\text{in}} = 0, \tag{3.2}$$

so that according to the tree approximation scheme, we should have calculated a coherent state expectation value:

$$\langle\lambda|: \Psi(\Phi)[\hat{\phi}_{\text{in}}]: |\lambda\rangle = \Psi(\Phi)[\phi] = \Psi(\phi), \tag{3.3}$$

where $\phi$ is a free classical field (the scalar neutral one) of Ref. 6, $\hat{\phi}_{\text{in}}$ in the above is the plane-wave solution of the Klein–Gordon equation in 1 + 1 dimensions, and the normal ordering refers to its (plane-wave solution) creation–annihilation generators. Classically,[12] one knows that if $\Phi = \Phi(\phi)$ is the sine-Gordon 1-soliton, then $\Psi(\phi)$ introduced according to

$$\begin{aligned} \Psi_1 &= \Psi_1^a = ia^{-1/2}\left(\tfrac{1}{2}\sin\frac{\Phi_a}{2}\right)^{1/2}\exp(-i\Phi_a/r), \\ \Psi_2 &= \Psi_2^a = a^{1/2}\left(\tfrac{1}{2}\sin\frac{\Phi_a}{2}\right)^{1/2}\exp(i\Phi_a/4), \\ \Phi_a &= \Phi(\phi_a), \end{aligned} \tag{3.4}$$

satisfies the massive (mass 1) Thirring model equations of motions, which are the classical ($c$-number) ones:

$$\begin{aligned} -i\partial_x\Psi_1 &= \tfrac{1}{2}\Psi_2 - 2\Psi_2^+\Psi_2\Psi_1, \\ i\partial_t\Psi_2 &= \tfrac{1}{2}\Psi_1 - 2\Psi_1^+\Psi_1\Psi_2. \end{aligned} \tag{3.5}$$

The underlying coherent 1-soliton states were constructed in Ref. 6, and their boson transformation parameters satisfy

$$\frac{1}{2\pi}\int_{R^1}\frac{dk}{(k^2+m^2)^{1/2}}\bar{\lambda}(k)\lambda(k) = \int dx[\phi(x)]^2 = \infty, \tag{3.6}$$

where $\phi(x) = \phi_a(x) = \exp m\gamma_a x$, $\gamma_a = (a^2+1)/2a$; hence Condition (2) of the theorem of Sec. 2 is manifestly violated. As a consequence *no free fermion is allowed in the 1-soliton Hilbert space IDPS* ($|\lambda\rangle$) *for the sine-Gordon system*.

(2) On the other hand, the spectral solution of the mas-



Downloaded 20 Apr 2007 to 156.17.88.174. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp

sive Thirring model given in Ref. 23 proves that the fundamental free field, to be used in the Haag expansions of the model, is the massive Dirac one in 1 + 1 dimensions. Its creation and annihilation operators are required to satisfy the CAR:

$$[b_i(p), b_j^*(q)]_+ = \delta_{ij}\delta(p-q),$$
$$[b_i(p), b_j(q)]_+ = 0 = [b_i^*(p), b_j^*(q)]_+, \quad ij = 1, 2 \tag{3.7}$$

and $\hat{\Psi} = \Psi(\hat{\psi}_{\text{in}}) = \Psi(b^*, b)$. Notice that in 1 + 1 dimensions one can introduce both Bose and Fermi fields on the common Hilbert space domain, without bothering about any spin-statistics problems (this is not the case in 1 + 3 dimensions). A bosonization of $\{b^*, b\}_{i=1,2}$ involves the corresponding Bose degrees of freedom $\{a^*, a\}_{i=1,2}$ (see Refs. 1 and 14) so that

$$b^* = b^*(a^*, a), \quad b = b(a^*, a),$$
$$\hat{\Psi} = \Psi(\hat{\psi}_{\text{in}}) = \Psi(b^*, b) \equiv \Psi(a^*, a) = \Psi(\hat{U}_\mu), \tag{3.8}$$

where $\hat{U}_\mu$ is the massive vector field in 1 + 1 dimensions with no Proca condition imposed.[1] If the construction of semiclassical domains IDPS($|\lambda\rangle$), i.e., of coherent states $|\lambda\rangle$, respects the coexistence of fermions and bosons on a common domain, both $\hat{\psi}_{\text{in}}$ and $\hat{U}_\mu$ are *equally fundamental* and give rise to equivalent Haag series expansions of the quantum fields on the subspace of IDPS($|\lambda\rangle$).

(3) The above picture breaks down if the coherent state $|\lambda\rangle$ does not respect restrictions of the theorem. Then the CAR are no longer satisfied by $\hat{\psi}_{\text{in}}$, and an appropriate (and then unique) fundamental free field is $\hat{U}_\mu$, i.e., the Bose one. In particular, if we impose a Proca condition we arrive at Case (1), where the fundamental free field is a massive neutral scalar $\hat{\phi}_{\text{in}}$, i.e., a boson again.

To summarize: The massive Thirring model always admits a bosonization in terms of $\hat{U}_\mu$. Nevertheless, the notion of a free fundamental fermion can still be saved if coherent states $|\lambda\rangle$ obey the theorem. Otherwise, either $\hat{U}_\mu$ or $\hat{\phi}_{\text{in}}$ plays the role of fundamental field in the model. Consequently, this special Fermi model admits in principle the three different types of Haag expansions—in terms of $\hat{\psi}_{\text{in}}$, $\hat{\phi}_{\text{in}}$, or $\hat{U}_\mu$, *depending on the choice of the state space in H*. Let us once more emphasize that an expansion in terms of $\hat{\psi}_{\text{in}}$ can always be rewritten as an equivalent expansion in terms of $\hat{U}_\mu$. This is obviously a peculiarity of the 1 + 1 dimensional spacetime, where the spin-statistics theorem does not apply. The inverse statement in general is not true, because once having specified a domain for $\Psi(\hat{U}_\mu) = \hat{\Psi}$ in *H*, we may have prohibited the existence of the CAR on it. Then, even having started from an expansion $\hat{\Psi} = \Psi(\hat{\psi}_{\text{in}})$ one must realize that $\hat{\psi}_{\text{in}}$ is no longer a free Fermi field in the conventional sense of the word. It is worth mentioning at this point that quite a variety of spinor models in 1 + 1 dimensions do not meet the requirement of asymptotic completeness; the asymptotic spinor field related to a given interacting spinor field does not exist on the state space of the latter, see, e.g., Ref. 24, but also Refs. 1 and 25–27, where the spinor field asymptotic in 1 + 3 dimensions is considered.

*Example 2*: QED *in the infrared domain, or the gauge field transcription of the Dirac-photon system.*

The statement of Ref. 1, that the correspondence principle allowing us to relate the classical (*c*-number) and quantum levels of spinor systems in 1 + 1 and 1 + 3 dimensions, involves free Bose systems with unbounded-from-below Hamiltonians. With any element of the spinor field algebra in hand, upon bosonization we can calculate its coherent state expectation value in the tree approximation, thus arriving at the corresponding semiclassical entity.

In 1 + 1 dimensions, the free (asymptotic) fermion can in prinicple coexist with the subsidiary (background) boson on the same state space in *H*. Then an *interacting fermion can have its free asymptotic Fermi partner.* However, in 1 + 3 dimensions, the spin-statistics theorem must be taken into account. By using a chain of heuristic arguments, we demonstrated in Ref. 1 that a Dirac field, if weakly coupled to the photon field (a nonlinear system of coupled Maxwell and Dirac equations), allows a bosonization in terms of the pure gauge field itself. We use here the Maxwell potential in the Coulomb gauge

$$\hat{\psi} = \psi(\hat{A}_\mu) \leftrightarrow \psi(A_\mu) = (\lambda\,|:\psi(\hat{A}_\mu):|\,\lambda), \tag{3.9}$$

where $|\lambda\rangle$ is an appropriate coherent photon state, $A_\mu$ being a solution of the sourceless Maxwell equations. A really striking peculiarity of (3.9) is that an interacting spin $\frac{1}{2}$ field appears as a nonlinear and nonlocal excitation in the spin 1 free field algebra. This observation can hardly be reconciled with the traditional wisdom about the (perturbative) QED, and its asymptotic problem solution.[25–27] Namely, in the latter case the interacting fields, both Bose and Fermi, have expansions in terms of free Bose and Fermi fields via the Haag series. The Haag series is written in terms of free Fermi and Bose fields commuting among themselves, which is distinct from the bosonization recipe, as discussed in (2.4)–(2.17). The asymptotic infraparticle states of QED found in Ref. 27 require both free bosons and fermions to commute among each other.

In the bosonized case, while using (2.5) and (2.9), we find that, for example,

$$[b_k, a_k^*]_\pm = \exp i\pi \sum_{j=1}^{k-1} \sigma_i^+ \sigma_j^- [\sigma_k^-, a_k^*]_\pm, \tag{3.10}$$

hence neither commutation nor anticommutation occurs.

On the other hand, the observation (3.9) is fully consistent with the attempts of Righi and Venturi[28–30] to construct charged fermion fields from extended particlelike solutions in their nonlinear approach to quantum electrodynamics. An example of the fully bosonized interacting spinor field which satisfies the CAR, and does not at all commute with the electromagnetic field, is given in Ref. 29. An analogy with the previously considered sine-Gordon/Thirring case appears to be striking.

Obviously the field $\hat{A}_\mu$ is not free, but its Haag series do apparently fit in our framework. Hence a construction of the appropriate coherent photon states is quite in order. In the case of the relativistic field theory, we expect that the presence of free fermions should be forbidden in the fully bosonized Fermi system. Hence one should look for coherent states which do not conflict with this theorem. We still cannot propose a final solution to this problem; let us, however, indicate that the coherent photon states invented by



Downloaded 20 Apr 2007 to 156.17.88.174. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp

Chung[25] in the conventional approach to the QED do not allow the existence of free fermions on any subspace of the semiclassical (photon) Hilbert space. The coherent states of interest read (a single electron case)

$$|\lambda\rangle_p = U_\lambda^* |0\rangle:$$
$$= \exp\left\{ \frac{e}{(2\pi)^{3/2}} \int \sum_{i=1,2} \left[ F^i(k,p) a_i^*(\mathbf{k}) - \overset{*}{F}{}^i(k,p) a_i(\mathbf{k}) \right] \frac{d^3k}{(2k_0)^{1/2}} \right\} |0\rangle, \tag{3.11}$$

where

$$F^i(k,p) = \frac{p\cdot\epsilon^i}{p\cdot k} \phi(k,p) \tag{3.12}$$

and $p$, $k$, $\epsilon^i$ are the four-vectors, $p\cdot k$ being the corresponding scalar product formula. Here $p$ stands for the four-momentum of the electron to which the state $|\lambda\rangle_p$ is assigned. The function $\phi(k,p)$ equals 1 in the vicinity of $k=0$. By also taking into account a factor $1/(2k_0)^{1/2}$, $k_0 = |\mathbf{k}|$, one easily verifies that the coherent photon state $|\lambda\rangle_p$ violates Condition (2) of the main theorem due to the singularity of $|F^i(k,p)|$ at $k=0$. Let us mention that in analogy to $|\lambda\rangle_p$, the soliton states of the massive Thirring–sine-Gordon example did exhibit a manifest parametrization $|\lambda\rangle = |\lambda\rangle_a$ in terms of the 1-soliton parameter $a$; compare, e.g., (3.6). Because the 1-soliton total momentum reads $k = 8m(|a|^2 - 1)/2|a|$, $|\lambda\rangle_a$ provides us with a momentum parametrization as well.


## ACKNOWLEDGMENTS

I would like to express my warm thanks to Professors H. M. Fried and A. Jevicki for arranging my visit to Brown University, where the present paper was completed.



[1]P. Garbaczewski, "Quantization of spinor fields II. Meaning of bosonization in 1 + 1 and 1 + 3 dimensions," J. Math. Phys. **23**, 442 (1982).
[2]N. N. Bogolubov, A. A. Logunov, and I. I. Todorov, *Axiomatic quantum field theory* (Nauka, Moscow, 1969), in Russian.
[3]H. B. Thacker, Rev. Mod. Phys. **53**, 253 (1981).
[4]J. Honerkamp, "Quantization of exactly integrable field theoretic models," in *Les Houches Colloquim Proceedings* (Springer, Berlin, 1980).
[5]P. Garbaczewski, J. Math. Phys. **22**, 574 (1981); see also **21**, 2670 (1980).
[6]P. Garbaczewski, "On quantum solitons...II. Fermion-boson reciprocity and the classical versus quantum problem for the sine-Gordon system," J. Math. Phys. **22**, 1272 (1981).
[7]Y. H. Ichikawa, N. Yajima, and K. Takano, Prog. Theor. Phys. **55**, 1723 (1976).
[8]H. Matsumoto, P. Sodano, and H. Umezawa, Phys. Rev. D **19**, 511 (1979).
[9]M. S. Swanson, Phys. Rev. D **22**, 1497 (1980).
[10]G. Oberlechner, M. Umezawa, and Ch. Zenses, Lett. Nuovo Cimento **23**, 697 (1978).
[11]H. Matsumoto, N. J. Papastamatiou, H. Umezawa, and M. Umezawa, Phys. Rev. D **23**, 1339 (1981).
[12]S. J. Orfanidis, Phys. Rev. D **14**, 472 (1976).
[13]K. Ishikawa, Prog. Theor. Phys. **55**, 588 (1976).
[14]P. Garbaczewski, Commun. Math. Phys. **43**, 131 (1975).
[15]P. Garbaczewski, Phys. Rep. **36C**, 65 (1978).
[16]P. Garbaczewski, J. Math. Phys. **19**, 642 (1978).
[17]P. Garbaczewski and Z. Popowicz, Rep. Math. Phys. **11**, 69 (1977).
[18]G. G. Emch, *Algebraic methods in statistical mechanics and quantum field theory* (Wiley, New York, 1972).
[19]P. Garbaczewski, "Representations of the CAR generated by representations of the CCR. Non-Fock case," Wroclaw Univ. prepring, 1981.
[20]T. W. B. Kibble, J. Math. Phys. **9**, 315 (1968).
[21]R. F. Streater and I. F. Wilde, Nucl. Phys. B **24**, 561 (1970).
[22]S. Mandelstam, Phys. Rev. D **11**, 3026 (1975).
[23]H. Bergknoff and H. B. Thacker, Phys. Rev. D **19**, 3666 (1979).
[24]N. Nakanishi, Prog. Theor. Phys. **59**, 607 (1978).
[25]V. Chung, Phys. Rev. **140**, B1110 (1965).
[26]J. K. Storrow, Nuovo Cimento **54A**, 15 (1978).
[27]P. P. Kulish and L. D. Faddeev, Teor. Math. Fiz. **4**, 153 (1970).
[28]R. Righi and G. Venturi, Nuovo Cimento A **52**, 166 (1979).
[29]R. Righi and G. Venturi, Nuovo Cimento A **52**, 511 (1979).
[30]R. Righi and G. Venturi, Lett. Nuovo Cimento **31**, 487 (1981).





Downloaded 20 Apr 2007 to 156.17.88.174. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp