Mechanisms of the "fermion–boson reciprocity"

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We analyze two completely integrable quantized systems: the nonlinear Schrödinger and sine–Gordon ones, with the aim explicitly to recover the (spin-$\frac{1}{2}$ approximation) mechanisms responsible for their being allowed in the continuum limit equivalence with Fermi systems.

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I. MOTIVATION

Since Refs. 1–3, a relationship between the (lattice) spin *xyz* Heisenberg and the (continuum) massive Thirring models has become well known. On the other hand, there is a correspondence[4] between the massive Thirring model and the sine–Gordon model, in which, upon adjustments of suitable parameters, the Schwinger functions of both models are identical, and hence the spectra are the same. In fact, the latter coincide with the semiclassical spectrum of the sine–Gordon system. In an *indirect* way we thus have a relationship between the spin-$\frac{1}{2}$ *xyz* and the sine–Gordon model.

There is an isolated attempt[5] to establish the correspondence *directly*, via the so-called spin-$\frac{1}{2}$ approximation of the sine–Gordon system on a lattice. One finds that in a Hilbert space (state space) of the sine–Gordon model there exists a proper subspace for which the reduction of the sine–Gordon Hamiltonian to this subspace makes it identical to the appropriate spin-$\frac{1}{2}$ *xyz* type Hamiltonian. The spin-$\frac{1}{2}$ approximation idea is then to recover these properties of the original Bose system which allow it to approximate, at least weakly, the related spin-$\frac{1}{2}$ or Fermi system (see also Ref. 6). One knows that if the spectra of both are identical, we can expect that then $H_{\rm sG}$ commutes with the underlying projection P: $[H_{\rm sG},P]=0$ and then either $H_{\rm sG}=PH_{\rm sG}P$ or $H_{\rm sG}=PH_{\rm sG}P+(1-P)H_{\rm sG}(1-P)$. *However this simple property does not hold true while on a lattice, in general, but it can be recovered in the continuum limit.*

In the present paper we exploit the results of the quantum inverse scattering analysis for the nonlinear Schrödinger and sine–Gordon models, to demonstrate explicitly the mechanisms of the "fermion–boson reciprocity" known to occur for both in the continuum limit, with the emphasis on the involved spin-$\frac{1}{2}$ approximation formalism.

In case of the nonlinear Schrödinger model, it is important to have available explicit formulas for Bethe-type eigenfunctions of the Hamiltonian. In the case of the sine–Gordon system such functions are not explicitly known, but nevertheless we are able to recover the spin-$\frac{1}{2}$ approximation scheme by analyzing the algebraized form of the Bethe ansatz and relating it to this of the spin-$\frac{1}{2}$ *xyz* Heisenberg model.

II. NONLINEAR SCHRÖDINGER MODEL IN THE REPULSIVE CASE

(A) Let us consider the famous Hamiltonian Bose system in 1 + 1 dimensions:

$$H=\frac{1}{2}\int\nabla\phi^*\nabla\phi\,dx+\frac{1}{2}\int\int\phi^*(x)\phi^*(y)V(x-y)\phi(x)\phi(y)\,dxdy,$$

$$[\phi(x),\phi^*(y)]_-=\delta(x-y),\quad[\phi(x),\phi(y)]_-=0, \tag{2.1}$$

$$i\dot\phi=[\phi,H]_-=-\frac{1}{2}\nabla\phi^2+\int V(x-y)\phi^*(y)\phi(y)\,dy\,\phi(x). \tag{2.2}$$

The latter equation of motion, in case of $V(x-y)=c\delta(x-y)$ is known as the nonlinear Schrödinger equation.

Let us choose a countable set of square integrable functions in $\mathscr{L}^2(R^1)$ subject to the restrictions: $\{\varphi_s\}$, $\operatorname{supp}\varphi_s=\Delta_s$, $\Delta_s\cap\Delta_{s+1}\neq\varnothing$, $\Delta_s\cap\Delta_t=\varnothing$ otherwise, Δ_s being a closed interval in R^1. Now let us approximate $\phi(x)$ by

$$\phi(x)\cong\sum_s a_s\varphi_s(x),\quad[a_s,a_t^*]_-=\delta_{st},\quad[a_s,a_t]_-=0. \tag{2.3}$$

Then (2.1) reads

$$H\cong\sum T_{mn}a_m^*a_n+\sum G_{mnjk}a_m^*a_n^*a_ja_k, \tag{2.4}$$

where

$$T_{mn}=\frac{1}{2}\int\nabla\bar\varphi_n\nabla\varphi_m\,dx,$$

$$G_{mnjk}=\frac{1}{2}\int\int\bar\varphi_m(x)\bar\varphi_n(y)V(x-y)\varphi_j(x)\varphi_k(y)\,dxdy \tag{2.5}$$

and because of (2.2) only the nearest neighbor exchange integrals remain (provided they exist):

$$T_{mm}=\frac{1}{2}\int|\nabla\varphi_n|^2dx=\alpha_m,$$

$$G_{mmmm}=\frac{c}{2}\int|\varphi_m|^4dx=\beta_m,$$

$$T_{mn}=\delta_{n\,m-1}\alpha_{m-1}+\delta_{n\,m+1}\alpha_{m+1}, \tag{2.6}$$

$$G_{mnjk}=\delta_{n\,m-1}\delta_{jm}\delta_{k\,m-1}\beta_{m-1}+\delta_{n\,m+1}\delta_{jm}\delta_{k\,m+1}\beta_{m+1}$$

with

$$\alpha_{m\pm1}=\frac{1}{2}\int\nabla\bar\varphi_m\nabla\varphi_{m\pm1}dx,$$

$$\beta_{m\pm1}=\frac{c}{2}\int|\varphi_m|^2|\varphi_{m\pm1}|^2dx. \tag{2.7}$$

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Hence, using (2.6), the formula (2.4) becomes

$$H \cong \sum_m [a_m^* (\alpha_{m+1} a_{m+1} + \alpha_{m-1} a_{m-1} + \alpha_m a_m) + a_m^{*2} (\beta_{m+1} a_{m+1}^2 + \beta_{m-1} a_{m-1}^2 + \beta_m a_m^2)]. \quad (2.8)$$

(B) Let $|0)$ be a Fock vacuum for the representation of the CCR algebra generated by (2.3): $a_k|0) = 0 \ \forall k$. We denote

$$p_k = :\exp(-a_k^* a_k): + a_k^* :\exp(-a_k^* a_k): a_k \quad (2.9)$$

and observe that

$$p_k^* = p_k, \quad p_k^2 = p_k, \quad [p_k, p_l] = 0, \quad k \neq l; \quad (2.10)$$

and, moreover,

$$p_k a_k^* p_k \equiv \sigma_k^+, \quad p_k a_k p_k \equiv \sigma_k^-, \quad (2.11)$$

i.e.,

$$\begin{aligned}(\sigma_k^+)^2 &= 0 \\ &= (\sigma_k^-)^2 \\ &= [\sigma_k^-, \sigma_l^+]_- \\ &= [\sigma_k^-, \sigma_l^-]_- \\ &= [\sigma_k^+, \sigma_l^+]_-, \quad k \neq l, \\ & [\sigma_k^-, \sigma_k^+]_+ = p_k, \end{aligned} \quad (2.12)$$

thus giving rise to spin-$\frac{1}{2}$ Pauli operators for the linear chain. An operator

$$P = \prod_k p_K \quad (2.13)$$

is a well-defined projection operator in the Fock space of our Bose system,[7] and allows us to consider a projected Hamiltonian $H_F = PHP$ of the (spin-$\frac{1}{2}$ approximation) type of Ref. 6:

$$H = H_F + (1-P)HP + PH(1-P) + (1-P)H(1-P). \quad (2.14)$$

Obviously (2.8) reads

$$H_F = PHP = \sum_m [\sigma_m^+ (\alpha_{m+1} \sigma_{m+1}^- + \alpha_{m-1} \sigma_{m-1}^- + \alpha_m \sigma_m^-)]. \quad (2.15)$$

If, for any vector $|\lambda)$, we have $P|\lambda) = |\lambda)$, then

$$H|\lambda) = H_F|\lambda) + (1-P)H|\lambda),$$

i.e.,

$$(1-P)H|\lambda) = 0 \Rightarrow H|\lambda) = H_F|\lambda). \quad (2.16)$$

Suppose that H is diagonalized, and $|\lambda)$ is an eigenvector of H: $H|\lambda) = E|\lambda)$. Then,

$$P|\lambda) = |\lambda) \Rightarrow H|\lambda) = H_F|\lambda) = E|\lambda), \quad (2.17)$$

and because of $[P, H] = 0$ we can equivalently write

$$H = H_F + (1-P)H(1-P)]. \quad (2.18)$$

(C) Notice that (2.15) is a familiar form of Schultz's Hamiltonian[8] for the impenetrable Bose lattice gas in one space dimension, provided one adopts

$$\alpha_{m+1} = \alpha_{m-1} = -1/2M\delta^2, \quad \alpha_m = 1/M\delta^2 \quad (2.19)$$

with δ being the lattice spacing, M the single-particle mass, and the finite volume restriction (to N sites) being conventionally imposed. One knows that after making the Jordan Wigner transformation from spins-$\frac{1}{2}$ to anticommuting (CAR) variables, followed by the appropriate canonical transformation, a resulting Hamiltonian describes the free Fermi gas. In the continuum limit the Girardeau eigenvalues and wavefunctions arise.[9] The free Bose gas Hamiltonian should then be used instead of H:

$$H = H_0 = -\frac{1}{2M} \int_0^L \phi^*(x) \frac{\partial^2}{\partial x^2} \phi(x)\, dx \quad (2.20)$$

provided one imposes the domain restriction:

$$\left[\int dx' \phi^*(x) \phi^*(x') \delta(x-x') \phi(x') \phi(x)\right] |\phi) = 0, \quad (2.21)$$

which eliminates the coupling term

$$\tfrac{1}{2} c \int dx \int dx' \phi^*(x) \phi^*(x') \delta(x-x') \phi(x') \phi(x)$$

of the original nonlinear Schrödinger problem.

In Girardeau's model it is obvious that $[H, P]_- = 0$ and hence all eigenvectors of H satisfying $P|\lambda) = |\lambda)$ are just those obseying (2.21). Here P projects on a subspace of the (Hilbert) eigenspace of H, on which (2.21) holds true. Let us, however, mention that the role of the coupling constant c, by virtue of (2.21) is missing, while from Refs. 10 and 11 one knows that at $c = 0$ the Bose (free) gas appears while at $c = \infty$ H converts into $H_F = PHP$, i.e., the equivalent free Fermi gas appears.

(D) Let H be a nonlinear Schrödinger model Hamiltonian. For any value of $c > 0$, the normalized in-eigenstates of H read as follows[12,13]:

$$\begin{aligned}|\phi(k_1, \cdots, k_n))_{\text{in}} &= R(k_1) \cdots R(k_n) |0) \\ &= \int \left[\prod_{i=1}^n dx_i \exp(ik_i x_i)\right] \\ &\times \left\{\prod_{1<j<i<n} \left[\Theta(x_j - x_i) + \Theta(x_i - x_j) \frac{k_i - k_j - ic}{k_i - k_j + ic}\right]\right\} \\ &\times \phi^*(x_1) \cdots \phi^*(x_n)|0), \quad k_1 < k_2 < \cdots < k_n. \end{aligned} \quad (2.22)$$

Here the Zamolodchikov operators $R(k)$, $R^*(k)$ can be introduced to algebraize the Bethe ansatz for the eigenvectors of H, and one has

$$R(k_i) R(k_j) = \frac{k_i - k_j - ic}{k_i + k_j + ic} R(k_j) R(k_i) \quad (2.23)$$

and

$$R(k_i) R^*(k_j) = \frac{k_i - k_j - ic}{k_i - k_j + ic} R^*(k_j) R(k_i) + 2\pi\delta(k_i - k_j) \quad (2.24)$$

together with

$$[H, R^*(k)]_- = k^2 R^*(k), \quad (2.25)$$

which implies that $|\phi(k_1, \cdots, k_n))$ is an eigenvector of H. By making use of (2.22), one finds easily that at $c = 0$

$$|\phi(k_1,\cdots,k_n)_B = \int\Big\{\prod_{i=1}^{n} dx_i\Big[\exp\Big(i\sum_i k_i x_i\Big)\Big]\Big\}\phi^*(x_1|\cdots\phi^*(x_n|0) = a^*(k_1)\cdots a^*(k_n)|0), \quad (2.26)$$

which corresponds to the free Bose gas problem, while the free Fermi gas problem appears at $c=\infty$, when

$$|\phi(k_1,\cdots,k_n))_F = \int dx_1\cdots\int dx_n\Big[\exp\Big(i\sum_i k_i x_i\Big)\Big] \times\sigma(x_1,\cdots,x_n]\phi^*(x_1)\cdots\phi^*(x_n)|0) \quad (2.27)$$

and

$$\sigma(x_1,\cdots,x_n) = \prod_{1<j<i\leq n}[\Theta(x_i-x_j)-\Theta(x_j-x_i)] \quad (2.28)$$

satisfies

$$\sigma_n^3=\sigma_n,\quad \sigma_n^2(1-\sigma_n^2)=0,$$
$$\sigma(\cdots x_i\cdots x_j\cdots) = -\sigma(\cdots x_j\cdots x_i\cdots). \quad (2.29)$$

Obviously $R=R(\phi^*,\phi)$ follows from (2.22); see Refs 12, 13. Girardeau's version of the free Bose gas wavefunctions arriving at the free Fermi gas is realized here by means of σ_n. This is a special case of the general isomorphisms between linear spaces of symmetric functions and antisymmetric functions[14] see also Ref. 7.

Let us emphasize that the general theory of Ref. 14 allows us to recover Fermi states of Bose systems in more than one space dimension. Obviously the simple multiplicative alternation σ_n is insufficient for such a purpose. Because of (2.29) for an n-point function f_n, one has

$$f_n = \sigma_n^2 f_n + (1-\sigma_n^2)f_n = \overset{1}{f_n} + \overset{2}{f_n} \quad (2.30)$$

so that if $f_n = f(x_1,\dots,x_n)$ is a symmetric function, its antisymmetric image á la Girardeau[9] is

$$\sigma_n \overset{1}{f_n} = \sigma(x_1,\dots,x_n)\overset{1}{f}(x_1,\dots,x_n).$$

In our case, the vector

$$\sigma(k_1,\dots,k_n)|\phi(k_1,\dots,k_n)\rangle_F = \int dx_1\cdots\int dx_n\Big[\exp\Big(i\sum_j k_j x_j\Big)\Big] \times\sigma^2(x_1,\dots,x_n)\phi^*(x_1)\cdots\phi^*(x_n|0) \quad (2.31)$$

is a typical element of the range of a projection operator $P=1_F$ of Ref. 7, i.e., $P|\overset{1}{\phi}\rangle_B = |\overset{1}{\phi}\rangle_B$. If adopted to our notation, 1_F reads

$$1_F = \sum_n \frac{1}{n!}\int dx_1\cdots\int dx_n\,\sigma^2(x_1,\dots,x_n)\phi^*(x_1)\cdots\phi^*(x_n) \times:\exp\Big[-\int dy\,\phi^*(y)\phi(y)\Big]:\phi(x_1)\cdots\phi(x_n). \quad (2.32)$$

It is an operator unit of the CAR algebra (a Fock representation):

$$[b(x),b^*(y)]_+ = \delta(x-y)1_F,\quad [b(x),b(y)]_+=0$$
$$b(x) = \sum_m(1+n)^{1/2}\int dy_1\cdots\int dy_n \times\sigma(y_1,\dots,y_n)\sigma(x,y_1,\dots,y_n) \times\phi^*(y_1)\cdots\phi^*(y_n):\exp\Big[-\int dz\,\phi^*(z)\phi(z)\Big]: \times\phi(x)\phi(y_1)\cdots\phi(y_n). \quad (2.33)$$

By inspection one easily verifies that

$$|\overset{1}{\phi})_B = \sigma(k_1,\dots,k_n)|\phi(k_1,\dots,k_n))_F = \sigma(k_1,\dots,k_n)\int dx_1\cdots\int dx_m\Big[\exp\Big(i\sum_j k_j x_j\Big)\Big] \times b^*(x_1|\cdots b^*(x_n)|0) = \sigma(k_1,\dots,k_n)b^*(k_1)\cdots b^*(k_n)|0). \quad (2.34)$$

Obviously at $c=0$, the $\{R,R^+\}$ algebra generates the CCR algebra, while at $c=\infty$ the CAR's arise; see (2.22)–(2.24).

By using the Jordan Wigner transformation, (2.34) can be replaced by

$$|\overset{1}{\phi})_B = \sigma^+(k_1)\cdots\sigma^+(k_n|0\rangle, \quad (2.35)$$

where $|\overset{1}{\phi})_B$ vanishes if any two k's coincide, and is permutation invariant. In this connection compare also Ref. 15, where the quantum Gel'fand–Levitan transform problem has been solved for the nonlinear Schrödinger field: $\phi(x)=\phi[R^*,R](x)$ in the limit of the infinite repulsion reduces to the Jordan Wigner mapping from the CAR generators $\{R,R^*\}$ to Pauli operators $\{\phi,\phi^*\}$.

(E) From Ref. 6 one knows that a family $\{|\phi(k_1,\dots,k_n))\}_{n=0,1,2,\dots}$ of vectors forms a complete eigenfunction system for the nonlinear Schrödinger model Hamiltonian if $c\geqslant 0$. However, one should realize that a continuous transition from $c=0$ to $c=\infty$ results in the contraction of the dynamically accessible state space for our Bose system from the whole of the Fock space $\mathscr{F}$ $(c=0)$ to its proper subspace $P\mathscr{F}=\mathscr{F}_F$.

This peculiar property is precisely a (very special) realization of the Bose→Fermi metamorphosis phenomenon via the spin-$\frac{1}{2}$ approximation procedure, as described in Refs. 16, 6.

Recall that $P=1_F$, (2.32), is a continuous version of the lattice projection P, which replaces each single Bose degree of freedom by the two-level (spin-$\frac{1}{2}$) degree in the system. The formulas (2.14)–(2.18) apply both to the continuous and lattice cases.

Generally upon the parametric dependence $H=H(\lambda)$ we expect to arrive at $[H(\infty),P]_-=0$ so that $H(\infty)=PH(\infty)P+(1-P)H(\infty)(1-P)$ may hold true either on the lattice level or, if impossible, on the continuum level. If all eigenstates of $H(\infty)$ belong to the range of P, then $H(\infty)=H_F=PH(\infty)P$, which is the case for the nonlinear Schrödinger model at $c=\infty$. Then a "fermion–boson reciprocity" idea apparently applies. Otherwise, the diagonalization of H_F does not resolve the spectral problem for H, and one is forced to diagonalize H itself to recover a complete set of eigenvectors. Obviously, it is frequently much more favorable to diagonalize the Bose Hamiltonian than the related Fermi one. If one succeeds with H, then a solution for H_F is immediate, provided $[H,P]=0$. This idea lies at the foundations of the boson expansion methods as applied in the many-body (Fermi or finite spin) systems in the diagonalization-before-the-projection procedure of Refs. 17,18.

III. THE QUANTUM SINE–GORDON SYSTEM VS SPIN-½ *xyz* HEISENBERG MODEL: PRELIMINARIES

(A) As far as the nonlinear Schrödinger model is concerned, we are fortunate to have available an exact spectrum of the problem with an explicit form of the eigenvectors. This fact enables one to control *how the Bose system is becoming equivalent to Fermi one as c varies from 0 to* ∞, and to notice that $[H,P]\neq 0$ except for $c=0$ and $c=\infty$.

For the sine–Gordon system, despite the spectacular equivalence with the massive Thirring model[4] and the demonstration of complete integrability via the quantum inverse scattering transform method,[19] even the knowledge of the Bethe ansatz states for the massive Thirring model did not enable one to construct explicitly Bethe wavefunctions. On the other hand, we know from Ref. 5 that the sine–Gordon Hamiltonian, while put on a lattice with the nearest-neighbor coupling gradient term, in the spin-½ approximation (at $T>0$) reduces to the spin-½ *xyz* problem. The latter under the weak anisotropy assumption in the continuum limit is known to reproduce the WKB spectrum of the sine–Gordon system.[1]

In addition, our lattice analysis of Ref. 5 suggests that in the continuum limit the ("fermion–boson reciprocity") statement:

$$H_{sG}=PH_{sG}P\equiv H_{xyz} \tag{3.1}$$

is not realized if at all, but rather

$$H_{sG}\equiv H_{xyz}+(1-P)H_{sG}(1-P). \tag{3.2}$$

Consequently, a selection of the appropriate subspace of the sine–Gordon state space is necessary to arrive at the Coleman's equivalence on the level of irreducible fields. To support this conjecture, we shall make an analysis of the available inverse scattering results for the sine–Gordon[19] and the spin-½ *xyz*[20] models with the emphasis on some limiting properties of both. Both systems are considered on the finite lattice of length L and spacing δ. This restriction is essential because the continuum limit of the sine–Gordon model does not exist for all coupling constant values.

Within the inverse scattering formalism, the basic object is the one-parameter family of the local transition matrices $L_n(\lambda)$, $n=1,\dots,N=L/\delta$ which give rise to the matrix:

$$T(\lambda)=L_N(\lambda)\cdots L_1(\lambda)=\begin{pmatrix}A(\lambda) & B(\lambda)\\ C(\lambda) & D(\lambda)\end{pmatrix}, \tag{3.3}$$

where the following commutation relations hold true:

$$\begin{aligned}&[A(\lambda),A(\mu)]_-=[B(\lambda),B(\mu)]_-=0,\\ &B(\lambda)A(\mu)=b(\lambda,\mu)B(\mu)A(\lambda)+c(\lambda,\mu)A(\mu)B(\lambda),\\ &B(\mu)D(\lambda)=b(\lambda,\mu)B(\lambda)D(\mu)+c(\lambda,\mu)D(\lambda)B(\mu),\\ &c(\lambda,\mu)[C(\lambda),B(\mu)]_-=b(\lambda,\mu)[A(\mu)D(\lambda)-A(\lambda)D(\mu)],\\ &c(\lambda,\mu)[D(x),A(\mu)]_-=b(\lambda,\mu)[B(\mu)C(\lambda)-B(\lambda)C(\mu)].\end{aligned} \tag{3.4}$$

As a consequence of (3.4), one has, for example,

$$[\mathcal{T}(\lambda),\mathcal{T}(\mu)]_-=0,\mathcal{T}(\lambda)=A(\lambda)+D(\lambda). \tag{3.5}$$

We wish to have represented relations (3.4) in terms of the operator algebra in the N-particle Hilbert space

$$\mathcal{H}=\prod_{i=1}^{N}{}^{\otimes} h_i,\quad h_i=h\quad\forall i,$$

including a reference state Ω such that

$$\begin{aligned}C(\lambda)\Omega=0,\quad A(\lambda)\Omega&=\exp[a(\lambda)N]\Omega,\\ D(\lambda)\Omega&=\exp[d(\lambda)N]\Omega.\end{aligned} \tag{3.6}$$

The particular choice of functions $b(\lambda,\mu),c(\lambda,\mu),a(\lambda),d(\lambda)$ determines the model of interest.

The eigenstates of $\mathcal{T}(\lambda)$ are constructed from Ω as follows:

$$|\lambda_1,\dots,\lambda_n)=\prod_{i=1}^{n}B(\lambda_i)\Omega \tag{3.7}$$

provided the following (periodicity) condition holds true:

$$\exp\{[a(\lambda_k)-d(\lambda_k)]N\}=\prod_{\substack{i=1\\ j\neq k}}^{n}\frac{c(\lambda_j,\lambda_k)}{c(\lambda_k,\lambda_j)},\quad k=1,2,\dots,n. \tag{3.8}$$

Then (3.7) is an eigenvector of the operator $\mathcal{T}(\lambda)$ with the eigenvalue

$$A(\lambda,\lambda_1,\dots,\lambda_n)=e^{a(\lambda)N}\prod_{j=1}^{n}\frac{1}{c(\lambda_j,\lambda)}+e^{d(\lambda)N}\prod_{j=1}^{n}\frac{1}{c(\lambda,\lambda_j)}. \tag{3.9}$$

(B) The sine–Gordon model in the above framework is defined[19] as follows,

$$\begin{aligned}&b(\lambda,\mu)=\frac{\sinh(v-v')}{\sinh(v-v'+i\gamma)},\quad c(\lambda,\mu)=\frac{i\sin\gamma}{\sinh(v-v'+i\gamma)},\\ &v=\ln\lambda,\ v'=\ln\mu,\\ &a(\lambda)=d(\bar\lambda)=\tfrac{1}{8}m^2\delta^2\cosh(2v-i\delta)\end{aligned} \tag{3.10}$$

provided one starts from the field equation

$$\varphi_{tt}-\varphi_{xx}=(m^2/\beta)\sin\beta\varphi,\quad \gamma=\tfrac{1}{8}\beta^2, \tag{3.11}$$

and then discretizes the problem: $L=N\delta$, δ being the lattice spacing. The local transition operator $L_n(\lambda)$ appears then in the form

$$L_n(\lambda)=\begin{pmatrix}\mu_n, & (m/4)[\lambda v_n^*-(1/\lambda)v_n]\\ (m/4)[(1/\lambda)v_n^*-\lambda v_n], & \mu_n^*\end{pmatrix}, \tag{3.12}$$

where the Weyl commutation relations

$$\begin{aligned}&v_nv_m=v_mv_n,\quad u_nu_m=u_mu_n,\quad v_n^*v_m=v_mv_n^*,\\ &u_n^*u_m=u_mu_n^*,\quad u_nv_n=e^{-i\gamma}v_nu_n,\quad u_nv_n^*=e^{i\gamma}v_n^*u_n\end{aligned} \tag{3.13}$$

are satisfied by operators

$$\begin{aligned}&v_n=\delta\exp\left[(i\beta/2\delta)\int_{x_n}^{(x_n+\delta)}\varphi(x)dx\right],\\ &p_n=\tfrac{1}{4}\beta\int_{x_n}^{x_n+\delta}\dot\varphi(x)dx,\quad u_n=\exp(-ip_n),\\ &[\varphi(x),\varphi(y)]_-=0=[\dot\varphi(x),\dot\varphi(y)]_-,\\ &[\varphi(x),\dot\varphi(y)]_-=i\delta(x-y).\end{aligned} \tag{3.14}$$

Let us emphasize that the defining commutation relations (3.9) can be rewritten in the compact tensor product form:

$$R(\lambda,\mu)[T(x)\otimes T(\mu)] = [T(\mu)\otimes T(\lambda)]R(\lambda,\mu) \quad (3.15)$$

where $R = R(\lambda,\mu)$ is a 4×4 matrix with c-valued elements:

$$R_{\rm sG} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = b(\lambda,\mu),\ c = c(\lambda,\mu). \quad (3.16)$$

To specify a concrete model in the inverse scattering formalism, one must thus know R and solutions of the eigenvalue equations (3.6). This amount of knowledge suffices to reconstruct the whole model.

(C) Now we shall recall the basic results for the spin-$\frac{1}{2}$ *xyz* Heisenberg model, with a final goal of identifying these eigenvectors for the sine–Gordon system, which belong to the range of the projection P of Sec. I, in

$$\mathscr{H} = \prod_{i=1}^{N}{}^{\otimes} h_i, \quad h_i = \mathscr{L}^2(R^1) \quad \forall i,$$

after making a transition to the continuum. Without entering into the detailed (Baxter's) parametrization, let us notice that the form of the local transition matrix[20]

$$L_n(\lambda) = \begin{pmatrix} w_4\sigma_n^4 + w_3\sigma_n^3, & w_1\sigma_n^1 - iw_2\sigma_n^2 \\ w_1\sigma_n^1 + iw_2\sigma_n^2, & w_4\sigma_n^4 - w_3\sigma_n^4 \end{pmatrix}, \quad (3.17)$$

where $[\sigma_n^i,\sigma_m^j]_- = 0,\ n\neq m$, and operators $\{\sigma_n^i\}^{i=1,2,3,4}$ are equivalent to 2×2 Pauli matrices assigned to the nth site, allows a representation of $L_n(\lambda)$ in

$$P\mathscr{H} = \prod_{i=1}^{N}{}^{\otimes}(ph)_i \equiv \prod_{i=1}^{N}{}^{\otimes} C_i^2$$

provided we introduce

$$\begin{aligned} \varphi_k &= \delta^{-1/2}\int_{R^1}\varphi(x)\chi_k(x)\,dx = 2^{-1/2}(a_k^* + a_k), \\ \dot{\varphi}_k &= (1/i2^{1/2})(a_k^* - a_k), \\ \chi_k(x) &= 1 \quad \text{for } x\in(x_k,x_k+\delta) \\ &= 0 \quad \text{otherwise.} \end{aligned} \quad (3.18)$$

Then define $\{\sigma_n^i\}_{n=1,2,\dots}^{i=1,2,3}$ and $\sigma_n^4 = p_n$ via formulas (1.10)–(1.13). However, we are still far from any relationship with the sine–Gordon system, especially because the R matrix reads

$$R_{xyz} = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}, \quad a\neq 1, \quad d\neq 0,$$

$$\begin{aligned} a &= \Theta(2\eta)\Theta(\lambda-\mu)H(\lambda-\mu+2\eta), \\ b &= H(2\eta)\Theta(\lambda-\mu)\Theta(\lambda-\mu+2\eta), \\ c &= \Theta(2\eta)H(\lambda-\mu)\Theta(\lambda-\mu+2\eta), \\ d &= H(2\eta)H(\lambda-\mu)H(\lambda-\mu+2\eta) \end{aligned} \quad (3.19)$$

with H, Θ being Jacobi's eta and theta elliptic functions.

(D) One knows that, for each family $\{L_n(\lambda)\}_{n=1,\dots,N}$ of local transition matrices for the *xyz* model, there exists a sequence $\{M_n^l(\lambda) = M_n^l(\lambda,s,t)\}$ of matrices with complex-valued matrix elements, being functions of two integers l and n, $n,l = 0,\pm 1\pm 2,\dots$, and two arbitrary complex parameters s and t (to be omitted below for simplicity) such that the operators

$$\begin{aligned} L_n^l(\lambda) &= L_n^l(\lambda,s,t) \\ &= M_{n+l}^{-1}L_n(\lambda)M_{n+l-1} \\ &= \begin{pmatrix} \alpha_n^l(\lambda) & \beta_n^l(\lambda) \\ \gamma_n^l(\lambda) & \delta_n^l(\lambda) \end{pmatrix} \end{aligned} \quad (3.20)$$

with $M_n^l := M_{n+(l-1)}$ serve as the new local transition operators for the *xyz* model, with a resulting one-parameter family of transition operators:

$$T^l(\lambda) = \prod_{n=1}^{N} L_n^l(\lambda) = \begin{pmatrix} A^l(\lambda) & B^l(\lambda) \\ C^l(\lambda) & D^l(\lambda) \end{pmatrix}. \quad (3.21)$$

Let us consider a single-site Hilbert space $h = \mathscr{L}^2(R^1)$ and let $\{e_i\}_{i=0,1,\dots}$ be a complete orthonormal system in it with $a^*e_0 = e_1$, $ae_0 = 0$, $e_n = (1/\sqrt{n!})a^{*n}e_0$. We denote $e_0 = e^-$, $e_1 = e^+$. In this notation, we introduce a one-parameter family of vectors[20]

$$\omega^l = H(s+2(\eta+l)\eta-\eta)e^+ + \Theta(s+2(\eta+l)\eta-\eta)e^-, \quad (3.22)$$

given as linear combinations of e^+, e^- in terms of Jacobi elliptic functions of the modulus k, η being one more (real) parameter. Each vector ω^l, (3.22) is annihilated by a respective γ^l of (3.20), and, moreover, we have

$$\begin{aligned} \gamma^l(\lambda)\omega^l &= 0, \\ \alpha^l(\lambda)\omega^l &= h(\lambda+\eta)\omega^{l-1}, \\ \delta^l(\lambda)\omega^l &= h(\lambda-\eta)\omega^{l-1}, \end{aligned} \quad (3.23)$$

with

$$h(u) = \Theta(0)H(u)\Theta(u). \quad (3.24)$$

The above local formula allows a construction of the one-parameter family of N-particle vectors in

$$P\mathscr{H} = \prod_{i=1}^{N}{}^{\otimes}(p\mathscr{L}^2(R^1))_i:$$

$$\Omega^l = \omega_1^l\otimes\omega_2^l\otimes\cdots\otimes\omega_N^l, \quad \omega_i^l = \omega^l \quad \forall i \quad (3.25)$$

such that operators $A^l(\lambda), B^l(\lambda), C^l(\lambda), D^l(\lambda)$ of (3.21) satisfy

$$\begin{aligned} A^l(\lambda)\Omega^l &= h^N(\lambda+\eta)\Omega^{l-1}, \\ D^l(\lambda)\Omega^l &= h^N(\lambda-\eta)\Omega^{l-1}, \\ C^l(\lambda)\Omega^l &= 0 \end{aligned} \quad (3.26)$$

for each l. The formula (3.21) can be rewritten as follows:

$$T^l(\lambda) = M_{N+l}^{-1}(\lambda)T(\lambda)M_l(\lambda) := T_{N+l,l}(\lambda), \quad (3.27)$$

where the notation

$$T_{k,l}(\lambda) = M_k^{-1}(\lambda)T(\lambda)M_l(\lambda) \quad (3.28)$$

is introduced, and the corresponding (operator-valued) matrix elements read $A_{kl}(\lambda)$, $B_{kl}(\lambda)$, $C_{kl}(\lambda)$, $D_{kl}(\lambda)$. By using these operators, one constructs a $1+n$ operator family of vectors:

$$\begin{aligned} \Psi_l(\lambda_1,\dots,\lambda_n) \\ = B_{l+1,l-1}(\lambda_1)\cdots B_{l+n,l-n}(\lambda_n)\Omega^{l-n}, \quad n = N/2. \end{aligned} \quad (3.29)$$

The transfer operator of the model,[20]

$$\mathrm{Tr}[T(\lambda)] = \mathscr{T}(\lambda)$$
$$= A(\lambda) + D(\lambda) = \mathscr{T}_{ll}(\lambda) = A_{ll}(\lambda) + D_{ll}(\lambda) \quad (3.30)$$

satisfies the following eigenvalue equation:

$$\mathscr{T}(\lambda)\Psi_\Theta(\lambda_1,...,\lambda_n) = \Lambda(\Theta,\lambda,\lambda_1,...,\lambda_n)\Psi_\Theta(\lambda_1,...,\lambda_n), \quad (3.31)$$

where

$$\Psi_\Theta(\lambda_1,...,\lambda_n) = \sum_{l=-\infty}^{+\infty} e^{2\pi i l\Theta}\Psi_l(\lambda_1,...,\lambda_n),$$
$$\Lambda(\Theta,\lambda,\lambda_1,...,\lambda_n) = e^{2\pi i\Theta}\Lambda_1 + e^{-2\pi i\Theta}\Lambda_2,$$
$$\Lambda_1(\lambda,\lambda_1,...,\lambda_n) = h^N(\lambda+\eta)\prod_{k=1}^{n}\alpha(\lambda,\lambda_k), \quad (3.32)$$
$$\Lambda_2(\lambda,\lambda_1,...,\lambda_n) = h^N(\lambda-\eta)\prod_{k=1}^{n}\alpha(\lambda_k,\lambda),$$
$$\alpha(\lambda,\mu) = h(\lambda-\mu-2\mu)/h(\lambda-\mu), \quad n = N/2,$$

and by construction $\Psi_\Theta(\lambda_1,...,\lambda_n)\in P\mathscr{H}$.

The periodicity condition reads

$$\frac{h^N(\lambda_j+\eta)}{h^N(\lambda_j-\eta)} = e^{-4\pi i\Theta}\sum_{\substack{k=1\\k\neq j}}^{n}\frac{\alpha(\lambda_k,\lambda_j)}{\alpha(\lambda_j,\lambda_k)},$$
$$j = 1,2,...,n = N/2. \quad (3.33)$$

By exploiting a property $H(u) = -H(-u)$, $\Theta(u) = \Theta(-u)$, i.e., $h(u) = -h(-u)$, we can write this condition as follows:

$$\left[\frac{h(\lambda_j+\eta)}{h(\lambda_j-\eta)}\right]^N e^{4\pi i\Theta} = \sum_{\substack{k=1\\k\neq j}}^{n}\frac{h(\lambda_j-\lambda_k+2\eta)}{h(\lambda_j-\lambda_k-2\eta)}. \quad (3.34)$$

IV. FROM THE *xyz* MODEL TO THE SINE–GORDON MODEL: SPIN-$\frac{1}{2}$ APPROXIMATION IDEA REVIVED

(A) In the above the elliptic Jacobi functions $H(u)$ and $\Theta(u)$ are of the modulus k with $k = (1-l'')/(1+l'')$ $0 < l'' < 1$. We are interested in the properties of the *xyz* Heisenberg system while approaching the limit $(1-k')/(1+k') = l'\to 0$, which implies $k = (1-k'^2)^{1/2}\to 0$. The transition from the modulus k to l', under an assumption $l'\to 0$, can be equivalently rewritten as

$$h(u) = H(u,k)\Theta(u,k)\Theta(0,k) = (\mathrm{const})H(u,l') \quad (4.1)$$

(see Ref. 3). It allows us to evaluate the limiting properties of (3.34) upon the following change of variables (formerly introduced in Ref. 3)

$$(\lambda_i-\lambda_j)\to -\tfrac{1}{2}i(\beta_i-\beta_j),$$
$$\eta\to\tfrac{1}{2}(\pi-\mu) \quad (4.2)$$

caused by

$$\lambda_i\to -\tfrac{1}{2}i\beta_i + iK_l, \quad \eta\to\tfrac{1}{2}(\pi-\mu),$$
$$K_l = \int_0^{\pi/2}\frac{d\varphi}{(1+l^2\sin^2\varphi)^{1/2}}, \quad K_l \underset{l'\to 0}{\to} \ln(4/l')\to\infty. \quad (4.3)$$

The following limits were investigated in Ref. 3:

$$\ln\left[\frac{h(\lambda+\eta)}{h(\lambda-\eta)}\right] \underset{l'\to 0}{\to} -i(\pi-\mu) - \tfrac{1}{16}il'^2\sin\mu\sinh\beta_i,$$
$$\ln\left[\frac{h(\lambda_i-\lambda_j+2\eta)}{h(\lambda_i-\lambda_j-2\mu)}\right] \underset{l'\to 0}{\to} \ln\left[\frac{\sinh(\frac{1}{2}(\beta_i-\beta_j)-2i\mu)}{\sinh(\frac{1}{2}(\beta_i-\beta_j+2i\mu)}\right]. \quad (4.4)$$

Consequently, we arrive at the following form of the periodicity condition,

$$\exp\{i[4\pi\Theta-(\pi-\mu)N]\}\cdot\exp(-ik_iL)$$
$$= \prod_{\substack{k=1\\k\neq i}}^{n}\frac{\sinh[\frac{1}{2}(\beta_i-\beta_j-2i\mu)]}{\sinh[\frac{1}{2}(\beta_i-\beta_j+2i\mu)]}$$
$$k_i = m_0\sinh\beta_i, \quad m_0 = \tfrac{1}{16}l'^2/\delta\sin\mu, \quad N = L/\delta, \quad (4.5)$$

and, upon the additional demands

$$\Theta = (\pi-\mu)N/4\pi, \quad \beta_i = 2v_i, \quad \mu\to -\mu, \quad (4.6)$$

we get finally

$$\exp(ik_iL) = \prod_{\substack{k=1\\k\neq l}}^{n}\frac{\sinh(v_i-v_j+i\mu)}{\sinh(v_i-v_j-i\mu)}, \quad i = 1,2,...,n = N/2. \quad (4.7)$$

On the other hand, by recalling (3.9) and (3.10), we realize that the original sine–Gordon periodicity condition reads

$$\exp\{[a(\lambda_K)-\overline{a(\lambda_k)}]N\} = \exp[\tfrac{1}{4}m^2\delta\sin\gamma\sinh 2v_k\cdot L]$$
$$= \prod_{\substack{j=1\\j\neq k}}^{n}\frac{\sinh(v_k-v_j+i\gamma)}{\sinh(v_k-v_j-i\gamma)}, \quad \gamma\to\mu, \quad (4.8)$$

i.e., upon an identification

$$\frac{l'^2}{16\delta} = \frac{m^2\delta}{4} \quad (4.9)$$

we can identify (3.34) with the sine–Gordon periodicity condition.

(B) We are interested in the $l'\to 0$ limit, upon the replacements (4.2), (4.6), and (4.9) of all the basic formulas for the *xyz* Heisenberg model. For this purpose let us make use of Ref. 21 and make transparent the l' dependence of the Jacobi functions. Namely we have

$$H(u,k) = \theta_1(u/2K_k,q), \quad \Theta(u,k) = \theta_4(u/2K_k,q),$$
$$k^2 + k'^2 = 1, \quad k' = (1-l')/(1+l'), \quad (4.10)$$

where

$$q = \exp(-\pi K_l/K_{l'}), \quad K_l = \int_0^{\pi/2} d\varphi/(1+l^2\sin^2\varphi)^{1/2},$$
$$K_l \underset{l'\to 0}{\to} \ln(4/l')\to\infty, \quad K_{l'} \underset{l'\to 0}{\to} \pi/2, \quad (4.11)$$
$$q \underset{l'\to 0}{\to} \exp(-2K_l)\to l'^2/16,$$

and the following q expansions hold true[21]:

$$\theta_1(v,q) = 2q^{1/4}(\sin\pi v - q^2\sin 3\pi v + q^6\sin 5\pi v - \cdots)$$
$$= i\sum_{n=-\infty}^{+\infty}(-1)^n q^{(n-1/2)^2}e^{(2n-1)\pi vi}, \quad (4.12)$$

$$\theta_4(v,q) = 1 - 2(q\cos 2\pi v - q^4\cos 4\pi v + q^5\cos 6\pi v - \cdots) = \sum_{n=-\infty}^{+\infty} (-1)^n q^{n^2} e^{2n\pi i v}. \quad (4.13)$$

One should also realize that the elliptic modulus k is a function of q and for small l' reads

$$k^2 \cong 16q\,\frac{1+4q^2}{1+8q+24q^2}. \quad (4.14)$$

(C) Let us analyze the $l'\to 0$ behavior of the xyz model R matrix (3.19) upon replacements (4.2), (4.6), (4.9) and provided we demand that $\eta = 0$ is mapped into π. Then

$$\Theta(2\eta)\to 1,\quad \Theta(\lambda-\lambda')\to 1,\quad \Theta(\lambda-\lambda'+2\eta)\to 1, \quad (4.15)$$

but

$$\begin{aligned} H(\lambda-\lambda') &\to 2q^{1/4}\sin[\tfrac{1}{2}i(\beta'-\beta)+\pi] \\ &= 2q^{1/4} i \sinh[\tfrac{1}{2}(\beta-\beta')], \\ H(2\eta) &\to 2q^{1/4} i \sin(\pi-\mu) = 2q^{1/4}\sin\mu, \quad (4.16) \\ H(\lambda-\lambda'+2\eta) &\to 2q^{1/4} i \sinh[\tfrac{1}{2}(\beta-\beta')-i\mu]. \end{aligned}$$

Consequently,

$$\begin{aligned} a(\lambda,\lambda') &= \Theta(2\eta)\Theta(\lambda-\lambda')H \\ &\quad\times(\lambda-\lambda'+2\eta)\to 2q^{1/4} i\sinh[\tfrac{1}{2}(\beta-\beta')-i\mu], \\ b(\lambda,\lambda') &= H(2\eta)\Theta(\lambda-\lambda') \\ &\quad\times\Theta(\lambda-\lambda'+2\eta)\to 2q^{1/4}\sin\mu, \quad (4.17) \\ c(\lambda,\lambda') &= \Theta(2\eta)H(\lambda-\lambda') \\ &\quad\times\Theta(\lambda-\lambda'+2\eta)\to 2q^{1/4} i\sinh[\tfrac{1}{2}(\beta-\beta')], \\ d(\lambda,\lambda') &= H(2\eta)H(\lambda-\lambda')H(\lambda-\lambda'+2\eta) \\ &\to -8q^{3/4} i\sin\mu\sinh[\tfrac{1}{2}(\beta-\beta')]\sinh\{\tfrac{1}{2}(\beta-\beta')-i\mu\}. \end{aligned}$$

Now let us make a reflection $\mu\to-\mu$ in the above formulas [compare, e.g., also (4.6)] and divide all of them by $a(\lambda,\lambda',\mu\to-\mu)$. We get

$$\begin{aligned} a(\lambda,\lambda') &\to 1, \\ b(\lambda,\lambda') &\to \frac{i\sin\mu}{\sinh[\frac{1}{2}(\beta-\beta')+i\mu]}, \quad (4.18) \\ c(\lambda,\lambda') &\to \frac{\sin[\frac{1}{2}(\beta-\beta')]}{\sinh[\frac{1}{2}(\beta-\beta')+i\mu]}, \\ d(\lambda,\lambda') &\to 4q^{1/2}\sin\mu\sinh[\tfrac{1}{2}(\beta-\beta')]. \end{aligned}$$

Now, it suffices to recall (4.6) with $\beta = 2v$ to noticing that, with an accuracy up to $q^{1/2}$ corrections, the following approximate relationship holds true for the R matrices of the xyz and sine–Gordon models:

$$R_{xyz}(\lambda,\lambda')\frac{1}{a(\lambda,\lambda')} \cong R_{sG}(\lambda,\lambda'), \quad (4.19)$$

where the right-hand side is l' independent. Though the approximation improves with $l'\to 0$, we cannot here make the limit $l' = 0$, like

$$\lim_{l'\to 0} R_{xyz}\frac{1}{a(\lambda,\lambda')} = R_{sG}(\lambda,\lambda'). \quad (4.20)$$

Namely, because of (4.3)–(4.7), the mass factor $m_0 = \frac{1}{16}(l'^2/\delta)\sin\mu$ appearing there as well as the identification $l'^2/16\delta = m^2\delta/4$ of (4.9) depends on l'. A nontrivial limit arises only if we simultaneously let $\delta\to 0$, so that $l'^2/\delta\to$const, i.e., if with $l'\to 0$, we recover the continuum limit of the lattice theory. But at the same time $m^2\delta$ should be kept constant, which means that we are forced to let m go to ∞ with $\delta\to 0$. The need for such an effect was explicitly recovered in Ref. 5; then only the spin-$\frac{1}{2}$ approximation of the sine-Gordon model in terms of the xyz model becomes reliable.

(D) The limiting properties of the transfer matrix eigenvalue are not straightforward if we use the form (3.32) of Λ. However,

$$\begin{aligned} e^{2\pi i\Theta} h^N(x-\eta)\Lambda_{xyz}(\Theta,\lambda,\lambda_1,\dots,\lambda_n) &= \Lambda'(\Theta,\lambda,\lambda_1,\dots,\lambda_n) \\ &= e^{4\pi i\Theta}\left[\frac{h(\lambda+\eta)}{h(\lambda-\eta)}\right]^N \prod_{k=1}^{n}\frac{h(\lambda-\lambda_k-2\eta)}{h(\lambda-\lambda_k)} \\ &\quad+\prod_{k=1}^{n}\frac{h(\lambda-\lambda_k+2\eta)}{h(\lambda-\lambda_k)} \end{aligned} \quad (4.21)$$

exhibits quite reasonable limiting behavior: By virtue of (4.3)–(4.7) we arrive at

$$\Lambda' \underset{l'\to 0}{\to} \exp(ikL)\prod_{k=1}^{n}\frac{\sinh(\lambda-\lambda_k+i\mu)}{\sinh(\lambda-\lambda_k)} + \prod_{k=1}^{n}\frac{\sinh(\lambda-\lambda_k+i\mu)}{\sinh(\lambda-\lambda_k)} \quad (4.22)$$

so that

$$e^{2\pi i\Theta}h^N(\lambda-\eta)\Lambda_{xyz}(\Theta,\lambda,\lambda_1,\dots,\lambda_n) \underset{l'\to 0}{\to} \exp[-d(\lambda)N]\cdot\Lambda_{sG}(\lambda,\lambda_1,\dots,\lambda_n). \quad (4.23)$$

In the above we can again approach $l' = 0$ while letting δ go to 0, (4.23) recovers a resolution of the eigenvalue problem $A(\lambda)\Omega = e^{a(\lambda)}\Omega$, $D(\lambda)\Omega = e^{d(\lambda)}\Omega$ in the $\{l'\to 0,\delta\to 0\}$ limit. Because tha data $\{R, a(\lambda), d(\lambda)\}$ are defining objects for the model, we find that indeed the simultaneous $\{l'\to 0, \delta\to 0\}$ limit of the xyz Heisenberg model defined in $P\mathscr{H}\subset\mathscr{H}$ gives rise to the (continuous) sine–Gordon model, in full agreement with Ref. 1. However, let us recall that the xyz model eigenvectors were explicitly constructed in terms of $\sigma_n^+ = Pa_n^*P$, $\sigma_n^- = Pa_nP$. The limiting procedure does not destroy the Pauli exclusion principle which is coded in the property $(\sigma_n^+)^2 = 0 = (\sigma_n^-)^2\ \forall n$, and because of the finite volume (L is kept fixed) we have not destroyed the Fock-ness property

$$\sigma_n^-\Omega_0 = a_k\Omega_0 = 0\quad\forall k,\quad \Omega_0 = \prod_{k=1}^{N}{}^{\otimes} e_k^-$$

[compare, e.g., (3.22)]. Consequently, in the continuum limit we still remain in the proper subspace $P\mathscr{H}$ of the Fock space, where P is a continuum generalization of the former discrete projection, in complete analogy with the case of the nonlinear Schrödinger model.

On the other hand, the representation (3.10) of the fundamental sine–Gordon algebra satisfies its continuum limit as well, being however defined in the whole of $\mathscr{H}$ through bounded in $\mathscr{H}$ Weyl operators. It means that in the continuum we can represent the fundamental sine–Gordon structure $\{R, a(\lambda), d(\lambda)\}$ in at least two equivalent ways: first in all of $\mathscr{H}$ and second in the proper subspace $P\mathscr{H}$ of $\mathscr{H}$. In the language of representation theory, it means that the continuum limit of the fundamental representation (3.10) is reducible and has at least one nontrivial component, namely this on $P\mathscr{H}$. Hence though the spectral problem resolved in Ref. 1 gives an exact spectrum of the sine–Gordon model, it does not suffice to recover all of its eigenfunctions. We conjecture that the WKB/massive Thirring spectrum of the sine–Gordon system is exact and infinitely degenerate.

(E) The existence of a representation of the sine–Gordon algebra in $P\mathscr{H}$ implies the existence of a representation in the continuum limit of any among the proper subspaces of $\mathscr{H}$ received by composing projections on two arbitrary (neighboring) excitation levels at each single site. In this connection, let us notice that single-site operators,[22]

$$\sigma^+ = a^* \frac{\cos^2(\pi a^* a/2)}{(a^*a+1)^{1/2}}, \quad \sigma^- = \frac{\cos^2(\pi a^* a/2)}{(a^*a+1)^{1/2}} a \qquad (4.24)$$

generate in $h = \mathscr{L}^2(R^1)$ a reducible representation of the CAR algebra:

$$[\sigma^-,\sigma^+]_+ = 1 = \sum_n e_n \otimes e_n, \quad (e_n,e_k) = \delta_{nk},$$
$$\sigma^- e_{2n} = 0, \quad \sigma^+ e_{2n} = e_{2n+1} \quad \forall\ n = 0,1,\cdots,$$
$$(\sigma^-)^2 = 0 = (\sigma^+)^2 \qquad (4.25)$$

provided $\{e_n\}$, is a complete basis system in $\mathscr{L}^2(R^1)$. For the sine–Gordon model $\mathscr{L}^2(R^1)$ is the Hilbert space of the quantum pendulum.[6] The representation (4.24) becomes reduced on each two-dimensional subspace of $h = \oplus_{n=0,1,\dots} h(n)$, $h(n) = \mathscr{L}(e_{2n}, e_{2n+1})$. Our construction allows to identify a representation of (4.24) related to the single-site projection $p(0) = :\exp(-a^*a): + a^*:\exp(-a^*a):a$, i.e., $P = P(0) = \Pi_{i=1}^N p_j(0)$. We can as well use a projection $P = P(k)$ with $p(k)h(k) = h(k)$ and the spin-$\frac{1}{2}$ algebra $\{P(k)\sigma_j^\pm P(k) = \sigma_j^\pm(k)\}_{k=0,1,\dots}^{j=1,2,\dots,N}$. In case of the nonlinear Schrödinger model we were able to give a detailed description of how fermions arise in the Bose system. In the present case, things are less straightforward, but once any operator quantity is given in terms of $\{\sigma_j^\pm\}$ then the Jordan Wigner transformation allows us to rewrite it in terms of pure fermionic variables. Let us exploit a reducible representation of the CAR given in Ref. 22:

$$b_i = \exp\left(i\pi \sum_{k=1}^{i-1} N_k\right)\cdot(N_k+1)^{-1/2}\cos^2(\pi N_i/2)\cdot a_i,$$
$$b_i^* = a_i^*(N_i+1)^{-1/2}\cos^2(\pi N_i/2)\exp\left[-i\pi\sum_{k=1}^{i-1} N_k\right], \qquad (4.26)$$
$$N_k = a_k^* a_k,$$

where

$$[b_i,b_j^*]_+ = \delta_{ij}, \quad [b_i,b_j]_+ = 0. \qquad (4.27)$$

The representation follows from the previously introduced (4.24), $\sigma^\pm \to \sigma_j^\pm$, via

$$b_j = \left(\prod_{k=1}^{j-1} c_k\right)\sigma_j^-, \quad \sigma_j^* = \left(\prod_{k=1}^{j-1} c_k^*\right)\sigma_j^+,$$
$$c_k = \exp(i\pi N_k) \Rightarrow [c_k,\sigma_j^\pm]_+ = 0, \quad c_k^2 = 1. \qquad (4.28)$$

Here $[N_k, P(j)]_- = 0 \forall k,j$; hence we can immediately reduce the representation of the CAR (4.26) according to $b_j \to b_j(k) = P(k) b_j P(k)$. In particular, due to $P(0) = \Pi_{j=1}^N P_j(0)$, we arrive at the well-known Jordan Wigner formulas:

$$P(0) b_i P(0) = b_i(0) = \exp\left(i\pi \sum_{k=1}^{i=1} \sigma_k^+ \sigma_k^-\right)\cdot\sigma_i^-,$$
$$P(0) b_i^* P(0) = b_i^*(0), \qquad (4.29)$$

which are easily invertible. For the derivation of (4.29) one should notice that $(N_i+1)^{-1/2}\cos^2(\pi N_i/2) = 1$ on the allowed domain. If we supply (4.26) with

$$\tilde{b}_i^* = \left(\prod_{k=1}^{i-1} c_k^*\right)\sigma_i^-, \quad \tilde{b}_i = \left(\prod_{k=1}^{i-1} c_k\right)\sigma_i^+, \qquad (4.30)$$

we arrive at the two-component Fermi system (the massive Thirring model demands it):

$$[b_j(k), b_i^*(l)]_+ = \delta_{ji}\delta_{kl}, \quad [b_j(k), b_i(l)]_+ = 0,$$
$$b_j(1) = b_j, \quad b_j(2) = \tilde{b}_j. \qquad (4.31)$$

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