Quantization of spinor fields. II. Meaning of "bosonization" in 1 + 1 and 1 + 3 dimensions

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We demonstrate that the correspondence principle allowing us to relate the classical (c number) and quantum levels of spinor fields in 1 + 1 and 1 + 3 dimensions, involves free Bose systems with unbounded from below Hamiltonians. The necessary condition for the quantum spinor fields to be "bosonized" on the "physical" space is that for the related free Bose systems, only the non-negative part of the spectrum persists, due to constraints. Compared with the bosonization formulas, the number of independent Bose degrees of freedom necessary for a consistent formulation of the correspondence principle is doubled.

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1. MOTIVATION

The "classical versus quantum" problem is far from clear for the Dirac system, both on the level of quantum field theory and relativistic quantum mechanics. In quantum mechanics the Dirac equation is believed not to admit a satisfactory classical analog, allowing at the same time an interpretation as a classical field equation for the spinor system.

It seems that here, not the relativistic form of the problem, but the spin, involves the most serious difficulties. First, spin itself is derivable from the pure Galilean background.[1] Second, it is known that the spin gives nonzero modifications to particle trajectories in the macroscopic ($\hbar \to 0$) limit, in the large-distance scale.[2] Notice also that in Q.E.D. $\hbar$ enters all quantities through $e^2/\hbar c$ so that a simple $\hbar \to 0$ rule is inapplicable and should be combined rather with the nonrelativistic limit $c \to \infty$.[3]

Basic investigations of classical analogs of the Dirac equation resulted in constructing suitable relativistic theories of spinning particles followed by their quantization.[4–14] All these attempts are based on the hope that the Dirac system can be completely understood *(and described) in terms of the conventional canonical variables.*

The less conventional way, though relatively simple and elegant, is to use not the usual phase space but rather a "superspace" which, in addition to the canonical variables, involves the supplementary anticommuting (Grassmann) variables, these last giving rise to spin after quantization. The appearance of pseudomechanics [15,16] provides one with a simple way to handle the relativistic quantum mechanics of the Dirac particle. Nevertheless, as pointed out in Ref. 16, the Grassmann variant of the classical mechanics cannot be applied to the real world and acquires a physical meaning *after* quantization only.

Despite its (physically) phantom nature, the generalization of the superspace concept to the continuous (field theory) level became very popular in high-energy physics, due to its calculational simplicity. If became even more popular with the advent of graded Lie algebras and supersymmetries, though these last do not need the introduction of anticommuting c numbers into the basic formalism[17,18] Likewise, in quantum mechanics, the two basic trends are met in the quantum field theory of the Dirac particle: the canonical one preferring to look for any (more or less "classical") c number level, and the Grassmann one, for which the anticommuting function ring is used to construct a pre-quantum level for the quantum field.

Though the Grassmannian way is dominant in the physical literature, there are nevertheless quite serious investigations of the c number origin of the quantum Dirac field, which date back to Klauder's paper.[19] Its idea was developed in Ref. 20, which is referred to as Paper I of the present series. Another investigation of the non-Grassmann pre-quantum level for spin 1/2 and Fermi lattices, together with the path integration formulas for propagators, was given in Refs. 21–23. Recall that path integrals for spinning particles were considered in Ref. 12 and quite recently in Refs. 24–27, and 28. In Ref. 25, in connection with the semiclassical quantization procedure for the continuous ferromagnetic system, the notion of a true (non-Grassmann) physical path was necessary. Then a Bose quantization of the system, under suitable constraints, was shown to conform with the well-known Bethe's solutions.

The present paper follows essentially the non-Grassmann approach, extending the earlier results of Ref. 20. Our opinion is (see, e.g., Refs. 20, 27, 29–31) that *any pseudoclassical theory described in terms of the Grassmann variables hides* (or even stronger: lacks) *its true physical content, and can in principle be reformulated as a conventional* (not pseudo) *theory of some singular canonical system.* Let us now recapitulate the basic result of Ref. 20 (i.e., I of the present series) concerning the quantization of the "classical" Dirac field. Suppose we are given free Dirac spinor fields $\overset{c}{\psi}(x), \overset{c}{\bar{\psi}}(x)$, $x \in M^4$ and let $\mathscr{F}$ be the set of all functionals $\mathbb{C} \ni \Omega\,(\overset{c}{\psi}, \overset{c}{\bar{\psi}})$

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$= \Sigma_{n,m} (\omega_{nm}, \overset{c}{\psi}{}^n \overset{c}{\bar{\psi}}{}^m)$.

There exists in $\mathscr{F}$ a subset $\overset{c}{\mathscr{F}}$ (a prequantum level) of functionals Ω which is closed under the left multiplication operation $(*)$ of Ref. 20,

$$\overset{c}{\mathscr{F}} \ni \overset{c}{\Omega}_1, \overset{c}{\Omega}_2 \Rightarrow \overset{c}{\Omega}_1 (*) \overset{c}{\Omega}_2 = \overset{c}{\Omega}_{12} \in \overset{c}{\mathscr{F}}, \tag{1.1}$$

and for which the quantization prescription

$$\overset{c}{\Omega} (\overset{c}{\psi}, \overset{c}{\bar{\psi}}) = (\alpha | : \overset{c}{\Omega} (\overset{B}{\psi}, \overset{B}{\bar{\psi}}) : | \alpha) \rightarrow : \overset{c}{\Omega} (\overset{B}{\psi}, \overset{B}{\bar{\psi}}) : \tag{1.2}$$

allows us to identify all elements of the Fermi field algebra according to

$$\mathbb{1}_F : \overset{c}{\Omega} (\overset{B}{\psi}, \overset{B}{\bar{\psi}}) : \mathbb{1}_F = : \Omega (\psi, \bar{\psi}) : \longleftrightarrow \overset{c}{\Omega} (\overset{c}{\psi}, \overset{c}{\bar{\psi}}). \tag{1.3}$$

Equation (1.3) is an identity in the quantum domain generated by the free quantum Dirac field $\psi(x), \bar{\psi}(x)$. The system of units is $\hbar = c = 1$ and $\overset{c}{\mathscr{F}}$ stands for a prequantum rather than classical level.

In the above $|\alpha)$ denotes a coherent state for the subsidiary CCR algebra involved in $\overset{B}{\psi}, \overset{B}{\bar{\psi}}$, these last being obtained from $\overset{c}{\psi}, \overset{c}{\bar{\psi}}$ through replacing the classical Fourier amplitudes $\bar{\alpha}_k^{\pm}(\mathbf{p}), \alpha_k^{\pm}(\mathbf{p})$, $k = 1,2$ by the Bose generators $\overset{*}{a}{}_k^{\pm}(\mathbf{p}), a_k^{\pm}(\mathbf{p})$ (the number of internal degrees of freedom is preserved while going from bosons to fermions in this construction!).

Because the fields $\psi, \bar{\psi}$ are by definition relativistic Dirac operators, the emergence of the relativistic looking objects $\overset{B}{\psi}, \overset{B}{\bar{\psi}}$ needs some explanation in light of the spin-statistics theorem: they cannot be the Wightman fields. On the other hand, the Fermi quantization (1.3), at first sight, seems to have nothing in common with any canonical quantization procedure, despite its involving bosons (for these last a canonical procedure in principle can be expected to exist).

The basic purpose of the present paper is to clarify the formal arguments of Ref. 20, by taking into account the results of Refs. 27, 29–31 and then going into the physics involved to explain the canonical quantization aspects which are inherent in (1.3), though not explicit in the formalism of Ref. 20. The concept of the Bose background for the quantum Dirac field ("bosonization", see, e.g., Ref. 29) becomes crucial at this point.The basic idea in the course of the paper is the naive version of the correspondence principle,[20] for quantum Bose systems. Take an operator expression in terms of the generators of the CCR algebra, make the so-called Bose transformation of them (translations by c-number functions), and then calculate a Fock vacuum expectation value of the result in the tree approximation [i.e., make a normal ordering before calculating $(0|\cdot|0)$]. The system of units is $\hbar = c = 1$.

Let us emphasize that the classical spinor fields $\overset{c}{\psi}, \overset{c}{\bar{\psi}}$, (1.1)–(1.3) due to the $\mathbb{1}_F(\cdot)\mathbb{1}_F$ sandwiching depend linearly on the classical amplitudes $\alpha_k^{\pm}(\mathbf{p}), \bar{\alpha}_k^{\pm}(\mathbf{p})$, $k = 1,2$. In what follows we shall admit a nonlinear dependence, which will simplify the arguments.

In Sec. 2 we demonstrate that the bosonization of the (massless) Thirring model necessitates a positivity of energy condition for the (1 + 1 dimensional) Maxwell field involved. In Sec. 3, we show that if the massive Thirring model in the charge 0 sector of the physical space is to be bosonized, then a positivity of energy condition necessarily occurs for the involved free massive neutral vector field in 1 + 1 dimensions. In Sec. 4 we prove the existence of the Maxwell-field (single potential in the Coulomb gauge) reformulation of the free quantum Dirac field, and the related algebra of observables. It appears as a consequence of the positivity of energy condition imposed on the two-potential Maxwell field Hamiltonian.

A common feature of all these cases is that a correspondence principle needs the number of Bose degrees of freedom to be doubled compared with the bosonization formulas. The underlying free field Hamiltonians have the form

$$H = H(\phi) - H(\phi'), \tag{1.4}$$

where ϕ, ϕ' are two independent free scalar (massless or massive, respectively) fields for the Thirring model, while the two independent Coulomb-gauge Maxwell fields are for the Dirac field.

In Sec. 5, we recover the two-potential Maxwell field content of the relativistic quantum mechanics of the Dirac electron. In contrast to q.f.t., a single potential formulation, seems not to be adequate here, which is inconsistent with the the Lorentz covariance properties of Dirac spinors.

2. THE MASSLESS THIRRING MODEL

A. The massless self-interacting spinor field theory in 1 + 1 dimensional space–time

$$\mathscr{L} = i\bar{\psi}\partial\!\!\!/\psi + (g/2) : j_\mu j^\mu :, \quad j^\mu = \bar{\psi}\gamma^\mu\psi,$$
$$g_{00} = -g_{11} = 1, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.1}$$

is known not to possess the spinor asymptotic fields. On the other hand, if one is to follow the general principles of quantum field theory, one expects that all Heisenberg operators should be expressed in terms of some (free) asymptotic fields.

A discussion of this problem was given in Refs. 32–35, a non-Wightman neutral massless scalar field $\phi(x)$:

$$\Box\phi(x) = 0, \quad \phi^{+}(x)|0) = 0,$$
$$[\partial_0\phi(x^0, x^1), \quad \phi(x^0, y^1)]_- = -i\delta(x^1 - y^1) \tag{2.2}$$

was shown to represent the Thirring spinors in the form

$$\psi(x) = :\exp[ia\phi(x) - ib\gamma^5\tilde{\phi}(x)]:u. \tag{2.3}$$

$a, b \in \mathbb{R}, ab = \pi$, u being a two-component constant, the normal ordering involving an order $\{\tilde{\phi}^-, \phi^-, \phi^+, \tilde{\phi}^+\}$ of the positive and negative frequency parts of $\phi(x)$ and of the related (conjugate) field $\tilde{\phi}(x)$

$$\Box\tilde{\phi}(x) = 0, \quad \partial_\mu\tilde{\phi}(x) + \epsilon_{\mu\nu}\partial^\nu\phi(x) = 0,$$
$$\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}, \quad \epsilon_{01} = 1. \tag{2.4}$$

The generator P_μ of space–time translations reads[33]

$$P_0 = H(\phi) = H(\tilde{\phi}) = \tfrac{1}{2}\int dx^1 :[(\partial_0\phi)^2 + (\partial_1\phi)^2], \tag{2.5}$$

$$P_1 = \int dx^1 :(\partial_0\phi)(\partial_1\phi):$$

and induces

$$\begin{aligned}
&[\phi^{\pm}(x), P_\mu]_- = i\partial_\mu \phi^{\pm}(x),\\
&[\tilde{\phi}^{\pm}(x), P_\mu]_- = i\partial_\mu \tilde{\phi}^{\pm}(x),\\
&[\psi(x), P_\mu]_- = i\partial_\mu \psi(x).
\end{aligned} \tag{2.6}$$

The canonical anticommutation relations for $\psi(x),\bar{\psi}(x)$ hold weakly on the vacuum and one-particle sectors of the (Bose field ϕ) state space, belonging to an indefinite metric carrier space.

B. Suppose we are to deal with a classical "photon" field A in $1+1$ dimensions:

$$\Box A_\mu = 0, \quad \partial_\mu A^\mu = 0. \tag{2.7}$$

A corresponding Hamiltonian,

$$\begin{aligned}
H &= \tfrac{1}{2}(\partial_\nu A_\mu)(\partial_\nu A^\mu)\\
&= \tfrac{1}{2}\{[(\partial_0 A_0)^2 + (\partial_1 A_0)^2] - [(\partial_0 A_1)^2 + (\partial_1 A_1)^2]\},
\end{aligned} \tag{2.8}$$

is obviously gauge invariant, hence allowing us to apply the Faddeev–Popov's path-integration arguments to this abelian gauge system.[36] Within the Hamiltonian formalism, the Lorentz condition $\partial_\mu A^\mu = \partial_0 A_0 - \partial_1 A_1$ appears as a constraint, which should still be accompanied by the supplementary "gauge fixing" condition, so that a canonical pair corresponding to one of the two allowed degrees $\{\pi_\mu, A_\mu\}_{\mu=0,1}$ can be completely eliminated from the formalism.

If we choose the supplementary condition in the form

$$\epsilon_{\mu\nu}F^{\mu\nu} = 0 \Rightarrow \partial_0 A_1 = \partial_1 A_0, \tag{2.9}$$

then, together with the Lorentz one, it leads to

$$\partial_\mu A_0(x) + \epsilon_{\mu\nu}\partial^\nu A_1(x) = 0, \tag{2.10}$$

which is a classical version of the definition (2.4) of the conjugate field, provided we identify $A_0 = \phi$, $A_1 = \tilde{\phi}$. Notice that (2.10) implies

$$H = H(\phi) - H(\tilde{\phi}) = 0 \Rightarrow H(\phi) = H(\tilde{\phi}). \tag{2.11}$$

With no recourse to the explicit Hamiltonian formalism, if we adopt the field (2.7) as a classical relative of the asymptotic one for the Thirring model, we can make the canonical quantization step by using a generating functional for the Thirring model Green's functions (the antisymmetry question is here left aside, see however Refs. 20 and 29)

$$W(\eta,\bar{\eta}) = \int d\mu(A) \times \exp i\left\{\int d^2x\left[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] + \bar{\eta}\psi + \eta\bar{\psi}\right\}\delta(\partial_\mu A^\mu), \tag{2.12}$$

where $\eta,\bar{\eta}$ are the (commuting ring!) spinor sources, and the classical (c-number) expression for ψ reads

$$\psi(A,x) = \psi(x) = \exp[iaA_0(x) - ib\gamma^5 A_1(x)]u, \quad ab = \pi. \tag{2.13}$$

Notice at this point that a formal integration with respect to A_1, provided we denote

$$A_1(x) = \int_{-\infty}^{x} d\xi\, \partial_\xi A_1(\xi), \tag{2.14}$$

replaces $A_1(x)$ in (2.12) by

$$A_1(x) \rightarrow \int_{-\infty}^{x} d\xi\, \partial_0 A_0(\xi) \tag{2.15}$$

thus giving (put $A_0 = \phi$)

$$\psi(x) = \psi(\phi,x) = \exp\left[ia\phi(x) - ib\gamma^5 \int_{-\infty}^{x} d\xi\, \dot{\phi}(\xi)\right], \tag{2.16}$$

which agrees with Mandelstam's formula, see Refs. 24, 29, and 35, if one replaces (2.16) by the normal ordered-operator expression.

3. MASSIVE THIRRING MODEL

A. Suppose we deal with a quantum massive Thirring model in the charge 0 sector[37]:

$$H = \int dx[-i(\psi_1^*\partial_x\psi_1 - \psi_2^*\partial_x\psi_2) + m_0(\psi_1^*\psi_2 + \psi_2^*\psi_1) + 2g_0\psi_1^*\psi_2^*\psi_2\psi_1], \tag{3.1}$$

where

$$\begin{aligned}
&[\psi_i(x), \psi_j^*(y)]_+ = \delta_{ij}\delta(x-y),\\
&[\psi_i(x), \psi_j(y)]_+ = 0.
\end{aligned} \tag{3.2}$$

As the analysis of the diagonalization problem for H[37] shows, the irreducibility domains for the CAR algebra (3.2) can be looked for within the general (continuous direct product) Hilbert space $\mathscr{H} = \Pi_x \otimes (h)_x$ containing the Fock state $|0)$ together with a corresponding Fock irreducibility sector $\mathscr{H}(|0)) \subset \mathscr{H}$

$$\psi_i(x)|0) = 0, \quad \forall x \in \mathbb{R}, \quad i = 1,2. \tag{3.3}$$

The most general form of the eigenstate of H with a finite number of quasiparticles reads

$$\begin{aligned}
|\alpha_1,\dots,\alpha_n) &= \int dx_1 \cdots \int dx_n \chi(x_1,\dots,x_n) \prod_{i=1}^{n} \psi^*(x_i,\alpha_i)|0),\\
\chi(x_1,\dots,x_n) &= \exp\Big(im_0 \sum_i x_i \sinh\alpha_i\Big)\\
&\quad\times \prod_{1\leqslant i<j\leqslant n}^{i=1} [1 + i\lambda(\alpha_i,\alpha_j)\epsilon(x_i - x_j)],
\end{aligned} \tag{3.4}$$

$$\lambda(\alpha_i,\alpha_j) = -\tfrac{1}{2}g_0 \tanh\tfrac{1}{2}(\alpha_i - \alpha_j),$$

$$\psi(x,\alpha) = \psi_1(x)\exp(\alpha/2) + \psi_2(x)\exp(-\alpha/2),$$

where α can take either the value $\alpha = \beta$ or $\alpha = i\pi - \beta$ with $\tanh\beta = k/E$ (β being the rapidity of a particle with momentum k and energy E). Here

$$H(\alpha_1,\dots,\alpha_n) = \Big(\sum_i m_0 \cosh\alpha_i\Big)|\alpha_1,\dots,\alpha_n) \tag{3.5}$$

and $m_0\cosh(i\pi - \beta) = -m_0\cosh\beta$. As a consequence, the spectrum of H on this set of eigenstates is unbounded from below, and thus interpreted as "unphysical."

The physical part of the spectrum can however be recovered within $\mathscr{H}$, provided that we abandon the Fock sector for $\{\psi_i^*,\psi_i\}_{i=1,2}$ and use a procedure of "filling the Dirac sea" under the periodic boundary conditions for eigenstates. The energy of a state must then be measured relative to the ground state within the appropriate physical sector $\mathscr{H}_{\text{phys}} \subset \mathscr{H}$. Irrespective of the change of the sector from $\mathscr{H}(|0))$ to $\mathscr{H}_{\text{phys}}$, all observables of the system can be viewed as bounded (in $\mathscr{H}(|0))$ or $\mathscr{H}_{\text{phys}}$, respectively) functions of the fundamental fields $\psi_i^*(x)$, $\psi_i(x)$, $i = 1,2$, where creation and annihilation operators occur due to

$$b_i(k) = \int \frac{dx}{(2\pi)^{1/2}} \exp(-ikx)\, \psi_i(x). \tag{3.6}$$

Notice that in the absence of the self-coupling ($g_0 = 0$), after a canonical transformation

$$\begin{aligned} B_1(k) &= \cos\theta(k) b_1(k) + \sin\theta(k) b_2(k), \\ B_2(k) &= -\sin\theta(k) b_1(k) + \cos\theta(k) b_2(k), \\ \tan 2\theta(k) &= m_0/k, \end{aligned} \tag{3.7}$$

(3.1) converts into

$$H_0 = \int dk\, (k^2 + m_0^2)^{1/2} (B_1^*(k) B_1(k) - B_2^*(k) B_2(k)), \tag{3.8}$$

where under $B_i(k)|0) = 0\ \forall i,k$ the spectrum of H_0 is obviously nonpositive in $\mathscr{H}(|0))$.

B. Let us introduce a neutral massive vector field $U_\mu(x)$ in 1 + 1 dimensions (which is not a Proca one, unless one imposes the subsidiary condition):

$$(\Box + m_0^2) U_\mu(x) = 0. \tag{3.9}$$

Its Hamiltonian is

$$\begin{aligned} H &= \tfrac{1}{2}\{\partial_\nu U_\mu \partial_\nu U^\mu + m_0^2 U_\mu U^\mu\} \\ &= \tfrac{1}{2}[(\partial_\nu U_0)^2 + m_0^2 U_0] - \tfrac{1}{2}[(\partial_\nu U_1)^2 + m_0^2 U_1] \\ &= \int dk\, (k^2 + m_0^2)^{1/2} [a_0^*(k) a_0(k) - a_1^*(k) a_1(k)], \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} &[a_i(k), a_j^*(p)]_- = \delta_{ij}\delta(k-p), \\ &[a_i(k), a_j(p)]_- = 0, \\ &a_i(k)|0) = 0, \quad \forall i,k, \quad i = 0,1, \end{aligned} \tag{3.11}$$

thus constituting a Fock representation of the CCR algebra. By virtue of Ref. 38, we can identify the Fock vacuum $|0)$ with that of the free Thirring model, as in Refs. 20 and 38, the Fermi generators $B_i^*(k)$, $B_i(k)$ can be completely given in terms of the Bose generators $a_i^*(k)$, $a_i(k)$ (3.11). The coincidence of the number (two) of internal degrees of freedom is crucial in this construction of the CAR within the CCR algebra. Then $\mathbb{1}_F H \mathbb{1}_F = H_0$, where $\mathbb{1}_F$ is the Fermi operator unit, H is (3.10), while H_0 is (3.8), see Refs. 20 and 29.

This means that all functions $\Omega(\psi^*,\psi)$ can be rewritten as functions $\Omega(\psi^*,\psi) = F(B^*,B) = G(a^*,a)$. Hence all Fermi field observables can be given as observables of the system of two independent neutral scalar fields with the same mass m.

Let us emphasize that the spectrum of H, (3.10) analogously to that of H_0 (3.8), is unbounded from below within the Fock space. However, for the Proca field a subsidiary condition

$$\partial_\mu U^\mu = 0 \tag{3.12}$$

would remove this unboundedness difficulty, as in Refs. 32 and 33; then

$$\begin{aligned} U_\mu &= -m_0^2 \epsilon_{\mu\nu} \partial^\nu U, \\ U &:= \epsilon^{\mu\nu} \partial_\mu U_\nu, \\ (\Box &+ m_0^2) U = 0, \end{aligned} \tag{3.13}$$

which proves that a system $\{(\Box + m_0^2) U_\mu = 0 = \partial_\mu U_\mu\}$ is equivalent to a neutral massive scalar field. The restriction $\partial_\mu U^\mu = 0$ makes the spectrum of H positive definite, at the same time reducing a two-component system to a single-component one.

C. Let us here comment that the very same positivity requirement for H, (3.1) makes the massive Thirring model equivalent to the quantum sine-Gordon system (within suitable limitations on the coupling constants values).

As found in Refs. 31 and 39, both classical and quantum sine-Gordon fields (including solitons) do exhibit a neutral massive free-field structure, hence the field U can be quite naturally embedded in the sine-Gordon framework. We conjecture that a positivity condition for U_μ induces a positivity condition for (3.1).

D. On the other hand, by using a boson transformation concept[39] (generalized coherent states come into account here), Thirring model observables

$$\Omega(\psi^*,\psi) = G(a^*,a), \tag{3.14}$$

if Bose transformed,

$$G(a^*,a) \to G(a^* + \bar{\alpha}, a + \alpha), \tag{3.15}$$

give rise in the tree approximation to the following (Fock) vacuum expectation values:

$$\begin{aligned} &(0|G(a^* + \bar{\alpha}, a + \alpha)|0) \to (0|{:}G(a^* + \bar{\alpha}, a + \alpha){:}|0) = G(\bar{\alpha},\alpha), \\ &(0|{:}\psi(a^* + \bar{\alpha}, a + \alpha){:}|0) = \overset{c}{\psi}(\bar{\alpha},\alpha). \end{aligned} \tag{3.16}$$

By exploiting this procedure, the quantum Thirring model Hamiltonian goes over to the classical Thirring model Hamiltonian of exactly the same form, with $\overset{c}{\psi} = \overset{c}{\psi}(\bar{\alpha},\alpha)$. In the above, :.: denotes a normal ordering with respect to Bose variables.

The classical Thirring model is known[40] to be a completely integrable system, whose classical spectrum reads

$$H = \int_0^\infty [\rho_1(s) - \rho_2(s)] [p^2(s) + m_0^2]^{1/2} ds^2 + \sum_{n=1}^{N} (A_n^2 + M_n^2)^{1/2}, \quad (3.17)$$

where momenta of mass m_0 particles with densities ρ_1, ρ_2 are given by $p(s) = (m_0/2)(s^{-2} + s^2)$. The remaining (discrete) part of the spectrum is due to solitons.

Hence the two neutral scalar fields are necessarily present on the classical Thirring level: the procedure (3.14)–(3.16) does require the two scalar fields to define $G(\bar{\alpha},\alpha)$ while for the sine-Gordon system, the same procedure[39] would recover a single neutral field structure (which is consistent with the classical spectral solution of Ref. 41).

E. Our conclusion is that the two, entirely different classical, completely integrable models, i.e., the massive Thirring and sine-Gordon systems, both exhibit the neutral free field structure, whose respective quantum images can coincide (Coleman's equivalence) provided a positivity condition for U_μ induces a positivity condition on (3.1).

This removes the redundant degree of freedom from the theory, thus replacing a two-component Bose-field formulation, which is characteristic for the Thirring model, by a single scalar-field formulation, corresponding to the sine-Gordon system. *Warning*: In contrast to the massless Thirring model, the underlying massive free scalars are not asymptotic fields at all. Nevertheless, all quantum and classical observables of the Thirring and sine-Gordon systems can be completely expressed in terms of them. For a review of analogous (quasiparticle) structures, see, e.g., Ref. 29.

4. ELECTROMAGNETIC (FREE FIELD) STRUCTURE OF THE QUANTIZED DIRAC FIELD

A. Previous 1 + 1 dimensional considerations can be summarized in the shorthand notions of the "free massless neutral field structure of the Thirring model" and the "free massive neutral field structure of the massive (and the related sine-Gordon) model."

Needless to say, in both massive and massless Schwinger models (quantum electrodynamics in 1 + 1 dimensions) no free fermions might occur in the asymptotic particle spectrum.

For the massless model, if it is provided with a subsidiary condition (to guarantee positivity of the spectrum on the physical subspace of the general indefinite metric Hilbert space) the only field of importance remains the free Proca field $U_\mu(x)$ (and hence a massive scalar U).[32,42] In the case of the massive model the spectrum consists of the neutral massive bosons identified with those of the massive sine-Gordon system (no definite free field structure of it is known to me, but it would surely be a scalar Bose one).

B. Our wisdom about quantum electrodynamics in 1 + 3 dimensions and hence the quantized Dirac and electromagnetic fields, follows from Refs. 43 and 44.

We shall work with the so-called scattering representations of the electromagnetic (free, asymptotic, in–out) field algebra; they have an energy momentum operator satisfying the relativistic spectrum conditions and can be defined in the charged sectors of the physical Q.E.D. Hilbert space. With an appropriate definition of the charge operator Q in $\mathscr{H}_{\rm phys}$ following from $\Box A_\mu(x) = j_\mu(x)$ with $j_\mu(x)$ interpreted as an electric current, one can prove that a Hilbert space $\mathscr{H}_{\rm in} \subset \mathscr{H}_{\rm phys}$ is in the domain of Q and $Q\mathscr{H}_{\rm in} = 0$. Then one[44] proves that no asymptotic (free) charged field $\psi_{\rm in}$ can exist in $\mathscr{H}_{\rm phys}$, which is local with respect to $F_{\mu\nu}^{\rm in}$. A scattering representation algebra we denote $\pi(\mathscr{A})$.

An energy operator P_μ of $\pi(\mathscr{A})$ can be decomposed into $P_\mu = P_{\mu\,{\rm as}} + P_{\mu\,{\rm ch}}$ where $P_{\mu\,{\rm as}}$ is associated with $\pi(\mathscr{A})''$ and hence describes the energy momentum of the asymptotic electromagnetic field configuration. $P_{\mu\,{\rm ch}}$ is associated with $\pi(\mathscr{A})'$ and thus describes charges and fields without electromagnetic interactions.[44] Here an important relation holds true:

$$Sp\, P_{\mu\,{\rm ch}} \subseteq Sp P_{\mu\,{\rm as}} = \bar{V}^{+}, \quad (4.1)$$

so that the spectrum of charges, in principle, can be completely recovered within the spectrum of the asymptotic electromagnetic field configuration. Compare here, e.g., also Refs. 29–31, where quite analogous conclusions were drawn in our studies of the Bose–Fermi "metamorphosis."

Moreover, the charged (infra) states do necessarily generate non-Fock irreducibility sectors of the asymptotic field algebra; this conforms well with the traditional infinite direct product constructions, where the Hilbert space $\mathscr{H} = \Pi_x \otimes (h)_x$ carries a reducible representation of the field algebra $\mathscr{A}$, and one must specify the generating vector to select a definite irreducibility sector for $\mathscr{A}$ and thus to specify $\pi(\mathscr{A})$. One should also know that only the radiation field associated with the charge 0 sector of $\mathscr{H}_{\rm phys}$ is Lorentz covariant.

For the physics, the following is essential.[44] The probability distributions of the momenta of the asymptotic charged infraparticles in a scattering state can, in principle, be determined by measurements of the asymptotic electromagnetic field alone. This suggests that even though the charges in Q.E.D. are not confined (in contrast to 1 + 1 dimensional theories) the whole Dirac theory should admit a reconstruction in terms of the asymptotic free electromagnetic fields.

C. In Ref. 20, we constructed elements of the Dirac field algebra in terms of free Fermi fields. The correspondence principle leading to a non-Grassmann (commuting ring of spinors) Dirac level, involved there a CCR algebra, with the same number of internal degrees of freedom, as here of the CAR, namely,

$$\{a_i^*, a_i, |0\rangle\}_{i=1,2,3,4} \to \{b_i^*, b_i, |0\rangle\}_{i=1,2,3,4},$$
$$[a_i(\mathbf{p}), a_j^*(\mathbf{k})]_- = \delta_{ij}\delta(\mathbf{k}-\mathbf{p}),$$
$$[b_i(\mathbf{p}), b_j^*(\mathbf{k})]_+ = \delta_{ij}\delta(\mathbf{k}-\mathbf{p}),$$
$$[a_i(\mathbf{p}), a_j(\mathbf{k})]_- = 0, \quad [b_i(\mathbf{p}), b_j(\mathbf{k})]_+ = 0,$$
$$a_i^*(\mathbf{k})|0\rangle = b_i^*(\mathbf{k})|0\rangle,$$
$$a_i(\mathbf{k})|0\rangle = b_i(\mathbf{k})|0\rangle = 0, \forall, i, \mathbf{k}, \tag{4.2}$$

see Ref. 38. It was the main cause of problems with a physical interpretation of the underlying bosonization in $1+3$ dimensions.

D. However, the lesson of $1+1$ dimensional models suggests that this seemingly unphysical Bose background of the quantum Dirac system should be converted into a physical one if suitable domain constraints would enter. We claim that this role is played again by the positivity-of-energy condition.

A classical Maxwell field can be described in terms of the two independent four potentials,[45,46] M_μ, N_μ so that

$$\partial_\nu F_{\mu\nu} = j_\mu, \quad \partial_\nu \tilde{F}_{\mu\nu} = 0,$$
$$F_{\mu\nu} = M_{\mu\nu} - \tilde{N}_{\mu\nu} = \partial_\mu M_\nu - \partial_\nu M_\mu - \epsilon_{\mu\nu\alpha\beta}\partial^\alpha N^\beta, \tag{4.3}$$
$$\tilde{F}_{\mu\nu} = N_{\mu\nu} + \tilde{M}_{\mu\nu} = \partial_\mu N_\nu - \partial_\nu N_\mu + \epsilon_{\mu\nu\alpha\beta}\partial^\alpha M^\beta,$$

and the gauge freedom is significantly enlarged

$$M_\mu \to M_\mu + \partial_\mu \Lambda(x), \quad N_\mu \to N_\mu + \partial_\mu \chi(x),$$
$$M_\mu \to M_\mu^0 + M_\mu, \quad N_\mu \to N_\mu^0 + N_\mu, \tag{4.4}$$
$$\partial_\mu M_\nu^0 - \partial_\nu M_\mu^0 - \epsilon_{\mu\nu\alpha\beta}\partial^\alpha N^{0\beta} = 0.$$

Consequently, by a proper choice of gauge ($N_\mu^0 = -N_\mu$ plus the Lorentz or Coulomb one), we can remove the redundant degrees of freedom, thus reducing the problem to a single potential one. This is the case for sourceless and electric examples. In the presence of magnetic sources the second potential cannot be eliminated.

Obviously, due to the gauge freedom, the Lorentz condition can be imposed on both potentials: $\partial_\mu M^\mu = 0 = \partial_\mu N^\mu$ and the second kind of gauge freedom still allows us to make an appropriate (like, e.g., the Coulomb one) gauge choice for the system.

The Lagrangian for the two-potential system (4.3) can be introduced according to convention

$$\begin{aligned}\mathscr{L} &= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= -\tfrac{1}{4}M_{\mu\nu}M^{\mu\nu} + \tfrac{1}{4}N_{\mu\nu}N^{\mu\nu} + \partial^\mu(\epsilon_{\mu\nu\rho\sigma}\partial^\rho M^\sigma N^\nu) \\ &= \mathscr{L}_M - \mathscr{L}_N + \partial^\mu(\epsilon_{\mu\nu\rho\sigma}\partial^\rho M^\sigma N^\nu).\end{aligned} \tag{4.5}$$

The constraints

$$\partial_0 N_0 = \partial_0 M_0 = 0 = \partial_\mu M^\mu = \partial_\nu N^\nu \tag{4.6}$$

reduce both potentials to the radiation gauge. In the presence of one more constraint

$$\partial_\mu(\epsilon_{\mu\nu\rho\sigma}\partial^\rho M^\sigma N^\nu) = 0, \tag{4.7}$$

the Lagrangian $\mathscr{L}$ can be replaced by $\mathscr{L} = \mathscr{L}_M - \mathscr{L}_N$, which is quite analogous to the Lagrangians (2.8) and (3.7) of the previous sections. Then the constraint

$$M_\mu N^\mu = 0 \tag{4.8}$$

allows a complete elimination of one of the two potentials, leaving us in (4.5) with a single potential in the radiation gauge; notice that both (4.7) and (4.8) are consistent with the demand $N_\mu = 0\ \forall\mu$, and because the gauge freedom is removed from the system, we conclude that $\mathscr{L} = \mathscr{L}_M - \mathscr{L}_N$ if supplied with six constraints (4.6)–(4.8) becomes a single potential system in the radiation gauge. In the path integral framework this observation can be compactly written as follows[36]:

$$\begin{aligned}&\int \exp\{iS[A]\}\prod_x \delta(\partial_\mu A^\mu)\delta(\partial_0 A^0)\prod_\mu dA_\mu(x) \\ &= \int\int \exp\{iS[M,N]\}\prod_x \delta(\partial_\mu M^\mu) \\ &\quad\times \delta(\partial_0 M^0)\delta(\partial_\nu N^\nu)\delta(\partial_0 N^0) \\ &\quad\times \delta(M_\mu N^\mu)\delta(\partial^\mu\epsilon_{\mu\nu\rho\sigma}\partial^\rho M^\sigma N^\nu) \\ &\quad\times \prod_\mu dM_\mu(x)\prod_\nu dN_\nu(x),\end{aligned} \tag{4.9}$$

with

$$S[M,N] = \int d^4x(\mathscr{L}(M) - \mathscr{L}(N)),$$
$$S[A] = \int d^4x\, \mathscr{L}(A).$$

E. Let us emphasize that the constraints (4.7) and (4.8) play the same role as the constraints (2.10) and (3.12) in cases of the massless and massive Thirring models, respectively; they transform an involved quantum Bose system with a nonpositive spectrum into a manifestly positive one, but at the price of diminishing the number of (Bose) degrees of freedom. Without the "mixing" conditions (4.7) and (4.8) we deal with a doubled Maxwell field in the radiation gauge, whose quantized Hamiltonian can be equivalently written as

$$\begin{aligned}H &= H(M) - H(N) \\ &= \int d^3k\, |\mathbf{k}| \sum_{\lambda=1}^{2} [a_M^*(\mathbf{k},\lambda)a_M(\mathbf{k},\lambda) \\ &\quad - a_N^*(\mathbf{k},\lambda)a_N(\mathbf{k},\lambda)],\end{aligned} \tag{4.10}$$

where

$$H(N) = \tfrac{1}{2}\int d^3x{:}\mathbf{N}^2 + (\nabla\times\mathbf{N})^2{:}$$
$$= \int d^3k\, |\mathbf{k}| \sum_{\lambda=1}^{2} a_N^*(\mathbf{k},\lambda)a_N(\mathbf{k},\lambda), \tag{4.11}$$
$$[a_N(\mathbf{k},\lambda), a_N^*(\mathbf{k}',\lambda')]_- = \delta_{\lambda\lambda'}\cdot\delta(\mathbf{k}-\mathbf{k}'),$$
$$[a_N(\mathbf{k},\lambda), a_N(\mathbf{k}',\lambda')]_- = 0,$$
$$[a_N^{\#}(\mathbf{k},\lambda), a_M^{\#}(\mathbf{k}',\lambda')]_- = 0,$$
$$a_N(\mathbf{k},\lambda)|0\rangle = a_M(\mathbf{k},\lambda)|0\rangle = 0,$$
$$a_N(\mathbf{k},\lambda) = i\int d^3x\, \exp(i\mathbf{k}\mathbf{x})\overleftrightarrow{\partial_0}\boldsymbol{\epsilon}(\mathbf{k},\lambda) \times \mathbf{A}(x)/[2|\mathbf{k}|(2\pi)^3]^{1/2},$$

with $\{\boldsymbol{\epsilon}(\mathbf{k},\lambda), \mathbf{k}/|\mathbf{k}|\}_{\lambda=1,2}$ forming a basis system in E^3. Quantally, the constraints (4.7) and (4.8) become the dominant ones within an infinite (continuous) direct product space carrying a reducible representation of the CCR algebra (4.11). The metric in $\mathscr{H}$ is indefinite.[44]

We identify the quartet of Bose generators

$$\begin{aligned} &a_M(\mathbf{k},1) = a_1(\mathbf{k}), \quad a_M(\mathbf{k},2) = a_2(\mathbf{k}),\\ &a_N(\mathbf{k},1) = a_3(\mathbf{k}), \quad a_N(\mathbf{k},2) = a_4(\mathbf{k}),\\ &[a_k(\mathbf{p}),a_i^*(\mathbf{q})]_- = \delta_{ki}\delta(\mathbf{p}-\mathbf{q}),\\ &[a_k(\mathbf{p}),a_i(\mathbf{q})]_- = 0,\\ &a_k(\mathbf{p})|0) = 0 \quad \forall \mathbf{p}, \quad k = 1,2,3,4, \end{aligned} \tag{4.12}$$

with the one, (4.2), used in the Bose construction of Fermi generators for the quantum Dirac field[38,20] in $1+3$ dimensions.

The previous analysis shows that the Dirac operators $\psi = \psi(a^*,a)$, $\bar\psi = \bar\psi(a^*,a)$ which are thus completely given in the two-potential Maxwell framework, after imposing the quantum images of constraints (4.7) and (4.8) become equivalent to the corresponding single-potential operators $\psi(A)$, $\bar\psi(A)$ with A_μ in the radiation gauge. It conforms well with Luther's observation,[47] that a two-component Bose system (field) should suffice to generate a free Dirac field in $1+3$ dimensions (both in the massive and massless cases).

F. To construct a classical (c-number) analog of the quantum Dirac field, see, e.g., Ref. 20, one has to work in the two-potential Maxwell framework, so that the conventional,[48] Hamiltonian density,

$$H = \psi^*(+\alpha\nabla + \beta m)\psi, \quad \psi = \psi(a^*,a), \tag{4.13}$$

after making the boson transformation[39]: $a^* \to a^* + \bar\gamma$, $a \to a + \gamma$ and then taking the Fock vacuum expectation value in the tree approximation (with respect to the Bose degrees), leads to

$$(0|{:}H(a^* + \bar\gamma, a+\gamma){:}_B|0) = H(\bar\gamma,\gamma) = \overset{c}{\psi}{}^*(\alpha\nabla + \beta m)\overset{c}{\psi}, \tag{4.14}$$

where $\overset{c}{\psi}$ is a classical (c-number) Dirac field, ${:}\,{:}_B$ designating a normal ordering of Bose creation and annihilation operators $\{a_i^*(\mathbf{k}),a_i(\mathbf{k})\}_{i=1,2,3,4}$. Here

$$\overset{c}{\psi}(\bar\gamma,\gamma) = (0|{:}\psi(a^* + \bar\gamma, a+\gamma){:}_B|0), \tag{4.15}$$

which significantly differs from the definition of $\overset{c}{\psi}$ given in Ref. 20, by being nonlinear in $\bar\gamma,\gamma$.

Remark 1: Apart of the manifest Bose background, both classically and quantally, Dirac fields preserve their canonical identity as separate (from the involved Maxwell fields) objects. Quite an analogous property was observed in Ref. 39 for the sine-Gordon system: quantum and classical solitons exhibit a free (massive neutral) field structure, but within the canonical formalism they can be viewed independently of the underlying free fields.

Remark 2: The Hamiltonian (4.13) is not positive definite and the normal ordering ${:}\,{:}_F$ with respect to the four Fermi generators $\{b_i(a^*,a,\mathbf{k}),b_i(a^*,a,\mathbf{k})\}_{i=1,\dots,4}$ is known to convert (4.13) into a conventional positive definite operator ${:}H{:}_F = H_{\rm Dirac}$. This Hamiltonian should be identified with $P_{\mu\,{\rm ch}}$ in Ref. 44

In the path-integral framework, the generating functional for spinor Green's functions of the Dirac field reads then (compare, e.g., also Ref. 20)

$$\begin{aligned} W(\bar\eta,\eta) = \int\int \exp\Big\{ i\int d^4x\Big[\overset{c}{\bar\psi}(i\partial\!\!\!/ + m)\overset{c}{\psi} + \bar\eta\psi + \eta\bar\psi\Big]\Big\}\\ \times \prod_x \delta(\partial_\mu M^\mu)\delta(\partial_0 M^0)\delta(\partial_\nu N^\nu)\delta(\partial_0 N^0)\\ \times \delta(M_\mu N^\mu)\delta(\partial^\mu(\epsilon_{\mu\nu\rho\sigma}\partial^\rho M^\sigma N^\nu))\\ \times \prod_\mu dM_\mu(x)\prod_\nu dN_\nu(x), \end{aligned} \tag{4.16}$$

where

$$\overset{c}{\mathscr{L}} = \overset{c}{\bar\psi}(i\partial\!\!\!/ + m)\overset{c}{\psi} = \overset{c}{\pi}\overset{c}{\dot\psi} - \overset{c}{H}, \quad \overset{c}{\pi} = i\overset{c}{\psi}{}^* \tag{4.17}$$

and $\overset{c}{\psi},\overset{c}{H}$ are given by (4.14) and (4.15).

By taking into account the constraints it is possible to perform the four integrations with respect to $\Pi_\mu\, dN_\mu(x)$ so that we are left with a single potential radiation gauge for $\overset{c}{\mathscr{L}},\overset{c}{\psi}$.

G. Notice that because of the Coulomb gauge adopted, there is no manifest Lorentz invariance in either (4.16) or (4.9). However, Lorentz covariance properties are correct, as can be most easily seen in the single potential-radiation gauge framework for Dirac operators.

Let $\Lambda \in \mathscr{L}^\uparrow_+$, then the Maxwell field transforms as follows:

$$F^\Lambda_{\mu\nu}(x) = \tau_\Lambda(F_{\mu\nu}(\Lambda x)) = \Lambda^{\ \alpha}_{\mu}\Lambda^{\ \beta}_{\nu}F_{\alpha\beta}(x), \tag{4.18}$$

while the Dirac field transforms according to

$$\begin{aligned} \psi_\Lambda(x) = \sigma_\Lambda(\psi(\Lambda x)) = S^{-1}(\Lambda)\psi(\Lambda x)\\ \times = U(\Lambda)\psi(x)U(\Lambda)^{-1}, \end{aligned} \tag{4.19}$$

with $S(\Lambda)$ a finite-dimensional representative of Λ acting on spinor indices. Here,

$$S^{-1}\gamma^\mu S = \Lambda^{\mu}_{\ \nu}\gamma^\nu \tag{4.20}$$

and γ^μ is the Dirac matrix.

If, according to Sec. 4F, we adopt a single potential Coulomb gauge structure $\psi = \psi(A)$ of the quantized field ψ, then a Lorentz transformation $U(\Lambda)$ must be accompanied by the Lorentz transformation $V(\Lambda)$ of the Maxwell potential, induced by (4.18):

$$A'^{\mu}_{\Lambda}(x) = V(\Lambda)A^\mu(x)V(\Lambda)^{-1} = \Lambda^{\mu}_{\ \nu}A^\nu(\Lambda x). \tag{4.21}$$

However, the Coulomb gauge does not persist in the transformation unless an accompanying gauge transformation $W(\alpha,\Lambda)$ is performed on A'^{μ}_{Λ} to restore the gauge:

$$A^\mu_\Lambda(x) = W(\alpha,\Lambda)A'^{\mu}_{\Lambda}(x)W(\alpha,\Lambda)^{-1} = A'^{\mu}_{\Lambda} - \partial^\mu\alpha_\Lambda(\Lambda x). \tag{4.22}$$

α_Λ is an operator-valued gauge "parameter."

This means that the Lorentz mapping $U(\Lambda)$ of free Dirac operators induces (and inversely is induced by) the following mapping in the asymptotic electromagnetic field algebra:

$$\Lambda:\psi\to\psi_\Lambda\Longleftrightarrow\psi(A)\mapsto\psi(A_\Lambda),$$
$$A^\mu_\Lambda(x)=W(\alpha,\Lambda)V(\Lambda)A^\mu(x)V(\Lambda)^{-1}W(\alpha,\Lambda)^{-1}. \tag{4.23}$$

Let us mention that (charged) generating vectors for irreducible representations of the scattering algebra $\pi(\mathscr{A})$ transform covariantly under the Lorentz group in consistency with (4.23).[44]

5. ASYMPTOTIC MAXWELL FIELD CONTENT OF THE FREE DIRAC SYSTEM, CONTINUED: RELATIVISTIC QUANTUM MECHANICS OF THE DIRAC ELECTRON

A. Assume that we work within a two-potential framework (4.10) so that, due to the Coulomb gauge, a quartet of canonical Bose operators $\{a_i^*(\mathbf{k}),a_i(\mathbf{k})\}_{i=1,2,3,4}$ (4.12) emerges, with the N_μ, M_μ dependence fixed by (4.11).

Let $f(\mathbf{p})$ be a normalized momentum space function

$$\int\frac{d^3k}{|\mathbf{k}|}\bar f(\mathbf{k})f(\mathbf{k})=1. \tag{5.1}$$

We introduce

$$\int\frac{d^3k}{|\mathbf{k}|^{1/2}}a_i(\mathbf{k})\bar f(\mathbf{k}):=a_i^f,\quad i=1,2,3,4$$
$$[a_i^f,a_k^{f*}]_-=\delta_{ik},\quad [a_i^f,a_k^f]_-=0, \tag{5.2}$$

where the manifest f dependence $a_i=a_i^f$ occurs. We shall fix the choice of $f=f(\mathbf{p})$ and the f index will be omitted for simplicity.

With the CCR algebra generators (5.2), provided we take a Fock representation: $a_i|0)=0\ \forall i=1,2,3,4$, we can construct the new operators $\{\mathbf{s},\boldsymbol{\zeta}\}$:

$$s_k-\zeta_k=i\epsilon_{ijk}a_j^*a_k,\quad s_k+\zeta_k=i(a_k^*a_4-a_4^*a_k). \tag{5.3}$$

They satisfy the following commutation relations[27]:

$$[s_i,N]_-=0=[\zeta_i,N]_-=[N,\mathbf{s}^2]_-,\qquad N=\sum_{i=1}^4a_i^*a_i,\quad \mathbf{s}^2=\boldsymbol{\zeta}^2, \tag{5.4}$$

and in addition, the SU(2)×SU(2) ones:

$$[s_i,\zeta_i]_-=0,$$
$$[s_i,s_j]_-=i\epsilon_{ijk}s_k,\quad [\zeta_i,\zeta_k]_-=i\epsilon_{ijk}\zeta_k. \tag{5.5}$$

All operator identities are valid in the Hilbert space $h=h_f$ constructed from the Maxwell field Fock vacuum $|0)$ by applying $\{a_i^*,a_i\}_{i=1,2,3,4}$.

As shown in Ref. 27, by using $\mathbf{s},\boldsymbol{\zeta}$ and the quantum mechanical momentum operator $\mathbf{p}=\{p_k=-i\partial/\partial x_k\}_{k=1,2,3}$, the Hamiltonian

$$H=2m\zeta_3+4\zeta_1(\mathbf{s}\cdot\mathbf{p}), \tag{5.6}$$

if provided with a domain restriction

$$(N-1)|\phi\rangle=0 \tag{5.7}$$

in the Hilbert space $h\otimes\mathscr{H}$ ($\mathscr{H}$ carrying a Schrödinger representation of the CCR algebra $\{x_i,p_i\}_{i=1,2,3}$) becomes equivalent to Dahl's Hamiltonian[14]:

$$H_F=2m\zeta_{F3}+4\zeta_{F1}(\mathbf{s}_F\cdot\mathbf{p}) \tag{5.8}$$

Here, operators

$$H_F=P_{1/2}HP_{1/2},\quad \mathbf{s}_F=P_{1/2}\mathbf{s}P_{1/2},\quad \boldsymbol{\zeta}_F=P_{1/2}\boldsymbol{\zeta}P_{1/2},$$
$$P_{1/2}=\prod_{k=1}^4\{:\exp(-a_k^*a_k):+a_k^*:\exp(-a_k^*a_k):a_k\}-\prod_{k=1}^4:\exp(-a_k^*a_k): \tag{5.9}$$

act invariantly in $h_{1/2}\otimes\mathscr{H}$, $h_{1/2}=P_{1/2}h$.

The matrix realization of H_F in $h_{1/2}\otimes\mathscr{H}$ coincides with the well-known Dirac expression

$$H_F\psi=(m\beta+\boldsymbol{\alpha}\cdot\mathbf{p})\psi=i\,\partial\psi/\partial t, \tag{5.10}$$

provided ψ is a bispinor composed of the expansion coefficients of the generalized vector $|\phi\rangle=|\phi,\mathbf{x},t\rangle=\Sigma_{k=1}^4\psi_k(\mathbf{x},t)|k\rangle$ in the basis of $h_{1/2}:|k\rangle=a_k^*|0)$.

B. If we smear $|\phi,\mathbf{x},t\rangle$ with a space-time dependent test function $g=g(\mathbf{x},t)$

$$|g\cdot\phi\rangle=\int d^4x\,g(x)|\phi,x\rangle=\sum_{i=1}^4\psi_i(g)|i\rangle, \tag{5.11}$$

then the only effect of the Lorentz transformation on $|g\cdot\phi\rangle$ following from the Lorentz invariance of (5.10), is due to the unitary mapping $U(\Lambda)$ inducing a base change in $h_{1/2}$

$$U(\Lambda)|g\cdot\phi\rangle=|g\cdot\phi\rangle_\Lambda=\sum_{k=1}^4\psi_k(g)|k,\Lambda\rangle, \tag{5.12}$$

with

$$|k,\Lambda\rangle=\sum_{l=1}^4T(\Lambda)_{kl}|l\rangle \tag{5.13}$$

such that [compare, e.g., (4.20)]

$$|g\cdot\phi\rangle_\Lambda=\sum_{k=1}^4\psi_k^\Lambda(g)|k\rangle=\sum_{k=1}^4\sum_{l=1}^4(S(\Lambda)_{kl}^{-1}\psi_l)|k\rangle. \tag{5.14}$$

$S(\Lambda)$ is a finite dimensional representative of Λ in E^4. Equation (5.13) allows a nontrivial mixing of the Maxwell degrees of freedom, which preserves the Coulomb gauge of both Maxwell potentials after a Lorentz mapping. Notice, however, that the correct spinor transformation law under Λ can here be generated by supplementing the conventional Lorentz transformations of M_μ, N_μ with the appropriate "mixing" gauge transformations plus the ones necessary to restore the Coulomb gauge [like (4.22)]. Notice that the constraint (5.7) still does not remove the gauge freedom.

Remark 1: In a fixed Lorentz frame, the above quantization procedure for the Dirac system can be realized in the formal path integration framework[27]: $\hbar=c=1$

$$\bar\gamma_k=(1/\sqrt{2})(\rho_k-i\pi_k),$$
$$\gamma_k=(1/\sqrt{2})(\rho_k+i\pi_k),\quad k=1,2,3,4$$
$$Z=\int\prod_{k=1}^4\mathscr{D}\rho_k\mathscr{D}\pi_k\prod_{i=1}^3\mathscr{D}p_i\mathscr{D}x_i\prod_t\delta(\rho^2+\pi^2-2)\times\delta((\rho,\pi))(\pi^2-\rho^2)\exp\left[i\int dt\left\{\sum_{k=1}^4\pi_k\dot\rho_k+\sum_{i=1}^3p_i\dot x_i-H_{\rm cl}(\rho,\pi,\mathbf{p})\right\}\right],$$
$$(\rho,\pi)=\sum_{k=1}^4\rho_k\pi_k, \tag{5.15}$$

where the "classical" Hamiltonian follows from the quantized one: $H_F = P_{1/2}HP_{1/2}$, $[P_{1/2},H]_- = 0$ by making the boson transformation of all Bose creation–annihilation operators and then calculating the Fock vacuum expectation value of the obtained quantities in the tree approximation:

$$H_{cl}(\rho,\pi,\mathbf{p}) = (0|:H(a^* + \bar{\gamma}, a + \gamma, A^* + \bar{\beta}, A + \beta):_B|0).$$

Originally, we used in Ref. 27 an incorrect form of the action which in addition to H_{cl} included more terms linear in momentum.)

Remark 2: Recall that if we consider the quantum field theory version of the Dirac system, the single particle Hamiltonian (4.10) is put in between the quantized Dirac fields, in its matrix form. Hence the whole Maxwell content of a single particle theory is lost.

Remark 3: In contrast to quantum field theory, it seems that the internal space of the Dirac particle needs a two-potential Maxwell framework rather than a single-potential one, at least to preseve the Lorentz covariance properties of the whole procedure. Notice that the impossibility of removing one potential among the two is usually connected with the presence of magnetic charges in the system.

Remark 4: The J. Math. Phys. referee has acquainted me with some papers which are relevant to this paper. I would like to point out the DeFacio, Hammer, and Tucker approach[49,50] to the quantization of relativistic fields: The equation of motion for the field and the resulting conserved current are the only data needed to derive all the necessary commutators or anticommutators. No *a priori* need for the canonical formalism appears in this construction of arbitrary spin electrodynamics. Application to the Dirac system which is minimally coupled to the vector meson field is of major importance for us.

The existence of a classical c-number set of solutions to the field equations is absolutely required for the existence of a quantized theory. In this connection, and in connection with Secs. 4 and 5 of this paper I must recall the paper by Gross[51] on the classical Dirac-photon system, together with the quantum investigations of Refs. 52–55.

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