On quantum solitons and their classical relatives: Spin $\frac{1}{2}$ approximation of the sine-Gordon system

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If gradient terms in the linear Bose chain Hamiltonian couple nearest neighbors only, then in the spin $\frac{1}{2}$ approximation they reduce to the spin $\frac{1}{2}$ x-y-z Hamiltonian. Hence, spin $\frac{1}{2}$ approximation reveals the Thirring model as the one, whose spectrum is completely included in the spectrum of the underlying Bose chain. Relation to the Coleman's equivalence is discussed.

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1. INTRODUCTION

There are still missing points in the most recent studies of the famous Thirring-sine-Gordon model equivalence in $1+1$ dimensions. One knows here that the spin $\frac{1}{2}$ x-y-z Heisenberg chain provides an equivalent description of the massive Thirring model put on a linear (space direction) lattice. A continuum limit of the latter can be used to recover spectral properties of the quantum sine-Gordon system (Coleman's equivalence), within the appropriate limitations on the coupling constant values of the Bose model.[1-3]

However no one has satisfactorily investigated the question of a "classical limit" of the quantum sine-Gordon field, with the special emphasis on the relation of classical and quantum soliton fields. The only exception in this context were the semiclassical quantization methods of Ref. 4, and quite inconclusive Coleman's remarks in last sections of Ref. 1.

Second, suppose, we start from a classical sine-Gordon field energy density[5]

$$H(x,t) = \tfrac{1}{2}\left\{\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 + 2m^2(1-\cos\phi)\right\}(x,t) \tag{1.1}$$

and approximate it on a linear lattice with spacing ϵ. Then

$$\begin{aligned}\int_{\mathbb{R}^1} H(x,t)\,dx &\to H \\ &= \sum_s \{[\pi_s^2 + 2m^2(1-\cos\phi_s)] - (\phi_s - \phi_{s+1})^2/\epsilon^2\} \\ &= \sum_s (H_s + V_{s,s+1})\end{aligned} \tag{1.2}$$

can be viewed to describe a linear chain of plane pendula subject to harmonic interactions among nearest neighbors.[6] The gradient terms $\{V_{s,s+1}\}_{s=0,\pm1,\cdots}$ are here responsible for the emergence of nontrivial configurations in the pendular chain. In the "single site approximation" of H by $\Sigma_s H_s$, a quantization of the chain is immediate through a simple replacement of each classical pendulum in the chain by a respective quantum one. A corresponding Schrödinger problem,[7] involves a pair of variables $\{\pi = -i\hbar(\partial/\partial\phi), \phi\}$ which though infinitesimally canonical cannot be integrated to a representation of the CCR (canonical commutation relations) algebra, in the sense of Ref. 8.

As a consequence, there is no straightforward way of getting a quantum analog of gradient terms $\{V_{s,s+1}\}$, and probably it was the main reason why the lattice analog (1,2) of the classical sine-Gordon system, has never been explicitly related to the spin $\frac{1}{2}$ x-y-z Heisenberg chain. The latter can in principle be considered as a lattice ancestor of the quantum sine-Gordon system, and as such should somehow be related to the quantization of the classical lattice problem (1.2), which is a lattice descendant of the classical sine-Gordon system.

Our aim is to establish the underlying relation. Basic result of the paper can be summarized as follows: Quantum fields on a lattice are considered as functions of the single site level raising-lowering operators. *We prove that each linear Bose chain, whose gradient term in the Hamiltonian couples nearest neighbors only, in its spin $\frac{1}{2}$ approximation, is equivalent to the lattice Thirring model.* Specification of its coupling constant relies on the explicit form of the quantum field entering the gradient term. (For a particular case of the sine-Gordon field, each quantum soliton operator appears to give rise to its own Thirring problem.[8,9] A crucial point in this investigation is an interaction of a quantum system with a nonzero temperature reservoir, without which a spin $\frac{1}{2}$ approximation makes no sense.[10]

2. QUANTUM PENDULUM AND PENDULAR CHAIN

The quantum pendulum spectral problem is conventionally expressed in terms of the Mathieu equation:

$$(2q\cos 2z - d^2/dz^2)\psi(z) = a\psi(z), \tag{2.1}$$

with $q = m^2$, $\phi = 2z$, $z\in[0,2\pi]$, $\psi = \psi(\phi)\in\mathscr{L}^2(0,4\pi)$ and a playing the role of the eigenvalue, compare, e.g., Ref. 7.

The spectrum of the quantum pendulum in the coupling constant range $q\in(0,\infty)$ is nondegenerate, and both eigenfunctions and eigenvalues exhibit a q-dependence. The Mathieu (eigen)functions:

$$\begin{aligned}&ce_{2n}(z\pm\pi) = ce_{2n}(z), \quad se_{2n+2}(z\pm\pi) = se_{2n+2}(z), \\ &ce_{2n+1}(z\pm\pi) = -ce_{2n+1}(z), \\ &se_{2n+1}(z\pm\pi) = -se_{2n+1}(z), \quad n = 0,1,2,\cdots,\end{aligned} \tag{2.2}$$

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$$\frac{1}{\pi}\int_0^{2\pi} ce_k(z)ce_l(z)\,dz = \delta_{kl} = \frac{1}{\pi}\int_0^{2\pi} se_k(z)se_l(z)\,dz,$$
$$\int_0^{2\pi} ce_k(z)se_l(z)\,dz = 0, \tag{2.3}$$

form a complete orthonormal system in $h = \mathscr{L}^2(0,4\pi)$ and hence

$$h = \bigoplus_{n=0}^{\infty} h_n = \bigoplus_{n=0}^{\infty} (h_n^{ce} \oplus h_n^{se}) = h^{ce} \oplus h^{se}. \tag{2.4}$$

In the limit $q \to 0$, $E_0(q)$ falls down to its minimum E_0, while $E_k^{ce} \to E_k^{se} \to E_k(0), E_k(0) < E_{k+1}(0)$ for $k > 0$. In the opposite limit $q \to \infty$, the spectrum $\{E_k(q)\}_{k=0,1,\cdots}$ goes to that of the doubly degenerate harmonic oscillator with the identifications:

$$E_{2n}^{ce} \to E_{2n+1}^{ce} \to E_{2n}^{\infty}, \quad E_{2n+1}^{se} \to E_{2n+2}^{se} \to E_{2n+1}^{\infty},$$

for all $n = 0,1,\cdots$.

By introducing:

$$e_{4n} = (1/\sqrt{\pi})ce_{2n}, \quad e_{4n+1} = (1/\sqrt{\pi})ce_{2n+1},$$
$$e_{4n+2} = (1/\sqrt{\pi})se_{2n+1}, \quad e_{4n+3} = (1/\sqrt{\pi})se_{2n+2}, \tag{2.5}$$

we can define densely in h the pair of the raising and lowering operators a^*, a, for quantum pendulum

$$a^* = \sum_{k=0}^{\infty} \sqrt{k+1}\, e_{k+1} \otimes e_k,$$
$$a = \sum_{k=1}^{\infty} \sqrt{k}\, e_{k-1} \otimes e_k, \tag{2.6}$$

which generate in h a Fock representation of the CCR algebra

$$[a,a^*]_- = \mathbb{1} = \sum_n e_n \otimes e_n, \quad ae_0 = 0.$$

In terms of $\{e_n\}$ the quantum pendulum Hamiltonian becomes immediately diagonalized, which allows an expression of H in terms of the true (integrable) generators a^*, a of the CCR algebra

$$H = \sum_{n=0}^{\infty} E_n e_n \otimes e_n = \sum_n E_n a_n^*{:}\exp(-a^*a){:}a^n = H(a^*,a), \tag{2.7}$$

where we have exploited the fact that: $\exp(-a^*a)$: is a projection on the ground state e_0 in h.

On the other hand, one easily finds that the operators

$$\mathbb{S}^+ = a^* \frac{\cos^2(\pi a^*a/2)}{(a^*a+1)^{1/2}},$$
$$\mathbb{S}^- = \frac{\cos^2(\pi a^*a/2)}{(a^*a+1)^{1/2}}\, a, \tag{2.8}$$

generate in h a reducible representation of CAR algebra

$$[\mathbb{S}^-,\mathbb{S}^+]_+ = \mathbb{1} = \sum_n e_n \otimes e_n,$$
$$\mathbb{S}^- e_{2n} = 0, \quad \mathbb{S}^+ e_{2n} = e_{2n+1}, \quad \forall n = 0,1,\cdots, \tag{2.9}$$
$$(\mathbb{S}^-)^2 = 0 = (\mathbb{S}^+)^2,$$

which becomes reduced on each two-dimensional sector $h_n^{ce(se)}$ of h. In particular, if to denote P_0 a projection on h_0^{ce} in h, then:

$$\mathbb{S}_0^+ = P_0 \mathbb{S}^+ P_0 = a^*{:}\exp(-a^*a){:},$$
$$\mathbb{S}_0^- = P_0 \mathbb{S}^- P_0 = {:}\exp(-a^*a){:}a, \tag{2.10}$$

are identities on h_0^{ce}, and the spin $\frac{1}{2}$ operator $\mathbf{S}$, with $S^+ = \mathbb{S}_0^+, S^- = \mathbb{S}_0^-, S^2 = -\frac{1}{2} + \mathbb{S}_0^+ \mathbb{S}_0^-$, emerges at once. Let us now consider a linear chain of elementary quantum systems. It is characterized by a countable set $\{a_s^*, a_s\}_{s=0,\pm 1,\cdots}$ of the CCR algebra generators, which form a reducible representation in the general tensor product space

$$\mathscr{H} = \prod_s^{\otimes} (h)_s, \tag{2.11}$$

where with each site of the chain, we associate a copy of the quantum pendulum Hilbert space. An irreducible, Fock component of this CCR algebra arises in the proper subspace IDPS(Ω) of $\mathscr{H}$, where

$$\Omega = \prod_s^{\otimes} (e_0)_s, \quad [a_s, a_t^*]_- = \delta_{st}\mathbb{1},$$
$$[a_s, a_t]_- = 0 = [a_s^*, a_t^*]_-, \quad a_s\Omega = 0 \forall s. \tag{2.12}$$

By comparison with (2.7) we find that a correctly quantized form of the lattice sine-Gordon Hamiltonian, should be given as a nonlinear function of generators $\{a_s^* a_s\}$:

$$H \to \hat{H} = H(a^*,a) = \sum_s \left\{ \sum_k E_k a_s^{*k}{:}\exp(-a_s^*a_s){:}a_s^k + V_{s,s+1}(a^*,a) \right\}, \tag{2.13}$$

where, in principle, the gradient term should read

$$\hat{V}_{s,s+n} = V_{s,s+1}(a^*,a) = [(1/\epsilon)(\hat{\phi}_{s+1} - \hat{\phi}_s)]^2,$$
$$\hat{\phi}_s = \phi_s(a^*,a). \tag{2.14}$$

In Ref. 10, we have given a description of the quantum sine-Gordon field in terms of (single-site) canonical generators $\{a_s^*, a_s\}_{s=0,\pm 1,\cdots}$ and shown that each single classical soliton field ϕ, after quantization should give rise to its associated irreducibility sector IDPS(ϕ) of $\mathscr{H}$. On the other hand, in Ref. 10, we have formulated a model independent description of criteria under which expectation values of observables associated with a finite part of the lattice Bose system, can be made to converge to these of the associated Fermi (or spin $\frac{1}{2}$) system. A leading idea behind, was that of a "spin $\frac{1}{2}$ approximation concept" for quantum Bose systems in thermal bath. We have shown that if a projection

$$P_0 = \prod_s P_0^s = \prod_s \{{:}\exp(-a_s^*a_s){:} + a_s^*{:}\exp(-a_s^*a_s){:}a_s\} \tag{2.15}$$

on the lowest two levels of each single lattice degree of freedom, happens to be a spectral projection for a Bose system

$$[P_0, \hat{H}_B]_- = 0, \tag{2.16}$$

then for a finite fraction of lattice sites, one has

$$P_0 H_B(a^*,a) P_0 = H_F(\mathbb{S}^+,\mathbb{S}^-) = \hat{H}_F \tag{2.17}$$

and a resolution of the spectral problem for the spin $\frac{1}{2}$ lattice, partly (within $P_0\mathscr{H} = \mathscr{H}_0$) resolves a spectral problem for $\hat{H}_B$. *In connection with the time development, notice that for all vectors* $|a\rangle, |b\rangle \in P_0\mathscr{H} = \mathscr{H}_0$ *there holds*

$$\langle a| \exp(i\hat{H}_B t)|b\rangle = \langle a| \exp(i\hat{H}_F t)|b\rangle, \tag{2.18}$$

see, e.g. Ref. 10. *If* $|a\rangle \in \mathscr{H}_0$, *but* $|b\rangle \in \mathscr{H}$ *the identity* (18) *holds as well.* Because in Ref. 8 we have established a connection between classical solitons and irreducibility domains for the CCR algebra generated by $\{a_s^*, a_s\}_{s=0,\pm 1,\cdots}$ quantum solitons can be specified by identifying the corresponding IDPS-generating vectors. Hence, in the infinite volume limit, the spectral problem for $\hat{H}_F$, should indeed give rise to a classification of quantum solitons.

In the present papers we are concentrated on the relation between the classical sine-Gordon equation and the equivalent lattice approximations of the quantum sine-Gordon system: the spin $\frac{1}{2}$ *x-y-z* and Thirring model.

Studies of the Fermi contents of quantum solitons will not be studied here

3. SPIN $\frac{1}{2}$ APPROXIMATION OF GRADIENTS IN THE LINEAR BOSE CHAIN, AS SPIN $\frac{1}{2}$ *x-y-z* HEISENBERG PROBLEM

(i) Take an elementary quantum system (single site Schrödinger problem in the linear chain) in its two-level (spin $\frac{1}{2}$) approximation. Within a Hilbert space h it means that we preserve two lowest levels $\{e_0, e_1\}$ of the general spectral solution $\{e_k\}_{k=0,1,\cdots}$. We characterize them by indicating an orientation of the spin $\frac{1}{2}$ arrow: $S^2 e_0 = (-\frac{1}{2})e_0$, $S^2 e_1 = (\frac{1}{2})e_1$, or $e_0 = |0\rangle \equiv \downarrow$, $e_1 = |1\rangle \equiv \uparrow$, for each fixed excitation possibility.

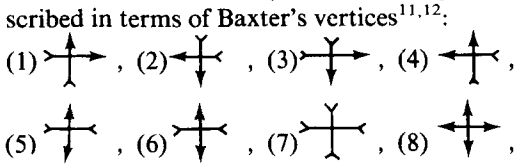
The set of all mappings between the excitation levels of the two-level problem can be described in terms of four arrow diagrams:

$$\frac{\uparrow}{\uparrow}, \frac{\uparrow}{\downarrow}, \frac{\downarrow}{\uparrow}, \frac{\downarrow}{\downarrow}.$$

Let us now consider the two chain neighbors, both in the spin $\frac{1}{2}$ approximation. Then a propagation of excitations along a linear chain can be completely given in terms of 16 arrow diagrams, describing elementary site-to-site energy exchanges $(s, \uparrow\downarrow) \longleftrightarrow (s+1, \uparrow\downarrow)$:

$$\begin{aligned}
&(1)\ \frac{\uparrow\uparrow}{\uparrow\uparrow}, \frac{\downarrow\downarrow}{\downarrow\downarrow}, \quad (2)\ \frac{\downarrow\uparrow}{\uparrow\downarrow}, \frac{\uparrow\downarrow}{\downarrow\uparrow},\\
&(2)\ \frac{\uparrow\downarrow}{\uparrow\downarrow}, \frac{\downarrow\uparrow}{\downarrow\uparrow}, \quad (4)\ \frac{\uparrow\uparrow}{\downarrow\downarrow}, \frac{\downarrow\downarrow}{\uparrow\uparrow},\\
&(5)\ \frac{\uparrow\uparrow}{\uparrow\downarrow}, \frac{\downarrow\downarrow}{\downarrow\uparrow}, \quad (6)\ \frac{\uparrow\downarrow}{\downarrow\downarrow}, \frac{\downarrow\uparrow}{\uparrow\uparrow},\\
&(7)\ \frac{\uparrow\downarrow}{\uparrow\uparrow}, \frac{\downarrow\uparrow}{\downarrow\downarrow}, \quad (8)\ \frac{\uparrow\uparrow}{\downarrow\uparrow}, \frac{\downarrow\downarrow}{\uparrow\downarrow}.
\end{aligned} \tag{3.1}$$

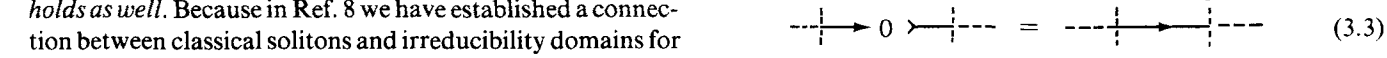
Each of these elementary transition diagrams can be described in terms of Baxter's vertices[11,12]:



(3.2)

provided, we adopt the Kadanoff-Wegner convention[13]: the right or upward arrow in the Baxter's diagram corresponds to the case when the two adjacent (with an arrow in between) spins are parallel, while a leftward or downward arrow is assigned to antiparallel spins.

We shall be composing transition diagrams in both horizontal and vertical directions according to the rule.



(3.3)

and basically interesting for us will be the transitions between spin-up-spin-down configurations of a linear chain of the identical two-level systems, which can be described in terms of so-called "transition matrices":

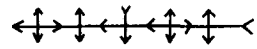
$$T(|\text{conf } 1\rangle) = |\text{conf } 2\rangle, \tag{3.4}$$

like e.g., the mapping

$$\begin{array}{c}
\cdots\ s\ s+1\ s+2\ s+3\ s+4\ s+5\ \cdots \\
\cdots\ |\uparrow|\uparrow|\uparrow|\downarrow|\downarrow|\downarrow|\ \cdots
\end{array}
\equiv
\begin{array}{l}
\cdots|1\rangle|1\rangle|1\rangle|0\rangle|0\rangle|0\rangle\cdots \\
\cdots e_1\otimes e_1\otimes e_1\otimes e_0\otimes e_0\otimes e_0\cdots
\end{array} \tag{3.5}$$

$$\begin{array}{c}
\cdots\ s\ s+1\ s+2\ s+3\ s+4\ s+5\ \cdots \\
\cdots\ |\downarrow|\uparrow|\downarrow|\uparrow|\downarrow|\uparrow|\cdots
\end{array}
\equiv
\begin{array}{l}
\cdots e_0\otimes e_1\otimes e_0\otimes e_1\otimes e_0\otimes e_1 \\
\cdots|0\rangle|1\rangle|0\rangle|1\rangle|0\rangle|1\rangle\cdots,
\end{array}$$

which in the language of Baxter's diagrams reads



(3.6)

(ii) One immediately finds that whenever the two configurations of the infinitely long spin $\frac{1}{2}$ chain differ in the infinite number of single site entries, then the respective vectors $|\text{conf } 1\rangle$, $|\text{conf } 2\rangle$ are orthogonal within the direct product space $\mathscr{H}_0 = \Pi_s{}^{\otimes}(h_0)_s$, $(h_0)_s = P_0^s(h)_s$, P_0^s $= :\exp(-a_s^* a_s): + a_s^* :\exp(-a_s^* a_s): a_s$ projecting on the linear span of e_0, e_1 in $(h)_s$.

These two vectors give thus rise to the two unitarily inequivalent irreducibility sectors for the spin $\frac{1}{2}$ algebra

$$\begin{aligned}
&\{\mathfrak{S}_s^+, \mathfrak{S}_s^-\}_{s=0,\pm 1,\cdots},\\
&[\mathfrak{S}_s^\pm, \mathfrak{S}_t^\pm]_- = 0, \quad s \neq t,\\
&[\mathfrak{S}_s^+, \mathfrak{S}_s^-]_+ = P_0^s,
\end{aligned} \tag{3.7}$$

IDPS ($|\text{conf}1,2\rangle$), respectively. If not orthogonal, the two configurations are represented by product vectors within the same irreducibility sector.

Suppose we have chosen some configuration, i.e., $|\text{conf}\rangle$, and hence IDPS ($|\text{conf}\rangle$). Mappings between configurations within IDPS ($|\text{conf}\rangle$) influence at most a finite (though arbitrary) number of spins $\frac{1}{2}$ in the chain, and can be represented by operators acting in IDPS ($|\text{conf}\rangle$).

To investigate the structure of such configuration-to-configuration mappings within any fixed IDPS ($|\text{conf}\rangle$), we can exploit the original Baxter's study of transfer matrices for the eight-vertex model,[11,12] vertex configurations being just the Baxter's diagrams mentioned before.

(iii) To proceed along the Baxter's lines, we must admit a thermal coupling of the linear chain with the environment. (See, e.g., for the notion of a "field-reservoir interaction" of ours.[14]) Then we must assign to the vertices, the appropriate Boltzmann weights

$$\omega_1 = \omega_2 = a = \exp(-\beta\epsilon_1), \quad \omega_3 = \omega_4 = \exp(-\beta\epsilon_2), \tag{3.8}$$

$\omega_5 = \omega_6 = c = \exp(-\beta\epsilon_3)$, $\omega_7 = \omega_8 = d = \exp(-\beta\epsilon_4)$,

where energies $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4$, we shall for a while leave unspecified, β being the inverse temperature of our open system.

Let us now form a related set of parameters:

$$w_1 = \tfrac{1}{2}(c+d), \quad w_2 = \tfrac{1}{2}(c-d),$$
$$w_3 = \tfrac{1}{2}(a-b), \quad w_4 = \tfrac{1}{2}(a+b), \tag{3.9}$$

and notice that transitions between different configurations of the N-particle segment of the linear chain, which are realized in a sequence of N steps, can be represented by the Baxter's "transfer matrix,"[11,13,3] on the Baxter's $N \times N$, eight-vertex lattice, provided the toroidal boundary conditions are taken into account. One knows that an operator representative T of the transfer matrix in IDPS ($|\text{conf}\rangle$) commutes with the spin $\frac{1}{2}$ x-y-z model Hamiltonian,[15,11,12] so that the most general Hamiltonian, responsible for quantum fluctuations around a fixed configuration $|\text{conf}\rangle$ of the chain is one of the form

$$H_{xyz} = -\sum_s \sum_a J_a S^a_s S^a_{s+1}, \tag{3.10}$$

where $a = x,y,z$.

An explicit form of the underlying H_{xyz} operator can be deduced for its N-site version (N arbitrary)[11]:

$$H_{xyz} = -J_z(\text{sn}\zeta)\left\{\tfrac{1}{2}\sum_{s=1}^{N}\sum_a p'_a \mathfrak{S}^a_s \mathfrak{S}^a_{s+1}\right\}, \tag{3.11}$$

with

$$\mathfrak{S}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathfrak{S}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \mathfrak{S}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$J_z = J_x/J_y = \text{cn}(2\zeta)/\text{dn}(2\zeta), \tag{3.12}$$
$$\text{cn}(2\zeta) = J_x/J_z,$$

and

$$\begin{aligned} P_1 &= \tfrac{1}{2}(w_1 - w_2 - w_3 + w_4) \Rightarrow p'_1 = \text{cn}(2\zeta)/\text{sn}(2\zeta), \\ P_2 &= \tfrac{1}{2}(-w_1 + w_2 - w_3 + w_4) \Rightarrow p'_2 = \text{dn}(2\zeta)/\text{sn}(2\zeta), \\ P_3 &= \tfrac{1}{2}(-w_1 - w_2 + w_3 + w_4) \Rightarrow p'_3 = 1/\text{sn}(2\zeta), \\ P_4 &= \tfrac{1}{2}(w_1 + w_2 + w_3 + w_4) \Rightarrow p'_4 \\ &= (\text{cn}(2\zeta) - \text{dn}(2\zeta) - 1)/\text{sn}(2\zeta), \end{aligned} \tag{3.13}$$

$\text{sn}(V,l)$, $\text{cn}(V,l)$, $\text{dn}(Vl)$, being the elliptic functions of the modulus l

$$\begin{aligned} l &= [(s_3 - s_2)(s_4 - s_1)/(s_4 - s_2)(s_3 - s_1)]^{1/2} \\ &\quad \times [(J_z^2 - J_y^2)/J_z^2 - J_x^2)]^{1/2}, \\ &\text{sn}^2(v,l) + \text{cn}^2(v,l) = 1, \\ &l^2\,\text{sn}^2(v,l) + \text{dn}^2(v,l) = 1, \end{aligned} \tag{3.14}$$

where the particular value ζ of $V \in \mathbb{R}^+$ is chosen to be fixed by

$$\text{sn}(\zeta,l) = [(s_3 - s_1)/(s_4 - s_1)]^{1/2}. \tag{3.15}$$

The four parameters $\{s_i\}_{i=1,2,3,4}$ are related to the initial Baxter's parameters $\{w_i\}_{i=1,2,3,4}$ through formulas:

$$w_j^2 = p(\xi - s_j), \tag{3.16}$$

where

$$\begin{aligned} \xi = \xi(v) = [&(s_3 - s_1)s_4 - s_3(s_4 - s_1)\,\text{sn}^2(v,l)]/ \\ &\cdot[s_3 - s_1 - (s_4 - s_1)\,\text{sn}^2(v,l)] \end{aligned} \tag{3.17}$$

and p is one more free parameter of the theory.

(iv) At $\{w_j\}$ fixed, both p, and $\{s_j\}$ exhibit the V-dependence and the nonuniqueness of their choice can be removed by assuming that we evaluate them as the initial $V = 0$ data corresponding to $\{w_j\}_{j=1,2,3,4}$ and at a fixed value of the constant ζ. Then a connection (3.16) between $\{w_j\}$ and $\{s_j\}$ is unique.

Let us here emphasize an important Baxter's observation[11] that the two transfer operators $T_{\{s\}}$, $T_{\{s'\}}$ commute if $\{s_j\} = \{s'_j\}$. It implies that by varying a single parameter $\zeta \in \mathbb{R}^+$ of the theory we have classified all noncommuting transfer operators: $T_{\{s\}} := T_\zeta$.

Commuting transfer operators are associated with the same spin $\frac{1}{2}$ x-y-z Hamiltonian, hence a related one-parameter family of Heisenberg Hamiltonians $H_{xyz}(\zeta)$ emerges, each one determining its own spin $\frac{1}{2}$ algebra irreducibility sector IDPS(ζ). Notice that by fixing ζ, we have fixed l and:

$$\begin{aligned} J_x &= \text{cn}^2(2\zeta)/\text{dn}(2\zeta), \\ J_y &= \text{cn}(2\zeta)/\text{dn}^2(2\zeta), \\ J_z &= \text{cn}(2\zeta)/\text{dn}(2\zeta), \end{aligned} \tag{3.18}$$

i.e., the coupling constants of the H_{xyz} problem.

All this means that: If the Boltzmann weights a,b,c,d, of the eight-vertex problem, are once established and fixed, there exists a one-parameter $\zeta \in \mathbb{R}^+$ family of inequivalent spin $\frac{1}{2}$ x-y-z Heisenberg problems responsible for propagation of excitations along a linear chain of spins $\frac{1}{2}$ (or a linear Bose chain in the spin $\frac{1}{2}$ approximation)

$$s_j = s_j(\zeta) = f_j(\zeta,w), \quad j = 1,2,3,4, \tag{3.19}$$

and $\{w_i = w_i(a,b,c,d)\}_{i=1,2,3,4}$ everthing at a fixed inverse temperature β of the reservoir.

The nonuniqueness can be removed if $\{a,b,c,d,\zeta\}$ arise as a five-valued function of a single common parameter λ: $a = a(\lambda)$, $b = b(\lambda)$, $c = c(\lambda)$, $d = d(\lambda)$, $\zeta = \zeta(\lambda)$, as then $s_j = s_j(\lambda)$ is uniquely determined by giving the value of λ, at β fixed.

4. DETERMINATION OF BOLTZMANN WEIGHTS

(i) Within the Kadanoff–Wegner parametrization, the set a,b,c,d of Boltzmann factors can be deduced in terms of Ising parameters $\mathfrak{S}_{jk} = \pm 1$, according to the general prescription:

$$\begin{aligned} \exp(&K^*\mathfrak{S}_{jk}\mathfrak{S}_{j+1,k+1} + K^*\mathfrak{S}_{j+1,k}\mathfrak{S}_{j,k+1} \\ &+ \lambda\mathfrak{S}_{j+1,k+1}\mathfrak{S}_{j+1,k}\mathfrak{S}_{j,k+1}\mathfrak{S}_{jk}), \end{aligned} \tag{4.1}$$

where j enumerates neighboring transition diagrams (rows of the Baxter's square lattice), while k enumerates the neighboring spins in the linear chain (columns of the Baxter's lattice). For each fixed (j,k)th transition diagram we have

$$\begin{aligned} -\beta\epsilon_1 &= K^+ + K^- + \lambda, \\ -\beta\epsilon_2 &= \lambda - (K^+ + K^-), \\ -\beta\epsilon_3 &= K^+ - K^- + \lambda, \\ -\beta\epsilon_4 &= -(K^+ - K^-) - \lambda, \end{aligned} \tag{4.2}$$

so that a complete partition function of the Baxter's lattice

$$\sum_{(\mathfrak{S}=\pm 1)} \prod_{j,k} \exp(K^+ \mathfrak{S}_{jk}\mathfrak{S}_{j+1,k+1} + K^- \mathfrak{S}_{j+1,k}\mathfrak{S}_{j,k+1} + \lambda \mathfrak{S}_{jk}\mathfrak{S}_{j+1,k+1}\mathfrak{S}_{j+1,k}\mathfrak{S}_{j,k+1}) \tag{4.3}$$

describes the two interpenetrating (crossed bonds) Ising lattices with Ising variables $\mathfrak{S}_{jk}, \mathfrak{S}_{j+1,k}$ attached at each (j,k)th site, and an interaction between the lattices arising due to each set of four nearest-neighbor spins, see e.g., Refs. 13, 16, and 17. Notice that the (j,k)th diagram:

$$\begin{array}{c|c} (j+1,k) & (j+1,k+1) \\ \hline (j,k) & (j,k+1) \end{array} \tag{4.4}$$

gives account of the crossed mappings between spin-up and spin-down states of the two neighboring spins $\frac{1}{2}$ in the linear chain.

Depending on elementary exchange energies during site-to-site interactions of the nearest chain neighbors, the spin $\frac{1}{2}$ approximation of the gradient terms occuring on the Bose Hamiltonian, gives rise to:

(a) single Ising system if either $K^+ \neq 0$,

$K^- = \lambda = 0$, or $K^- \neq 0, K^+ = \lambda = 0$;

(b) two independently living Ising systems if

$K^+ \neq 0, K^- \neq 0$, but $\lambda = 0$;

(c) the general spin $\frac{1}{2}$ x-y-z Heisenberg system in the either case.

It needs suitable limitations on the exchange energy values, like, e.g.:

(a) $\epsilon_1 = -\epsilon_2 \Rightarrow K^+ = (\beta/2)(\epsilon_2 - \epsilon_3), K^- = (\beta/2)(\epsilon_2 + \epsilon_3)$, i.e., we need either $\epsilon_2 = \epsilon_3$ or $\epsilon_2 = -\epsilon_3$;

(b) $\epsilon_1 = -\epsilon_2$, but $\epsilon_2 \neq \pm \epsilon_3$.

(ii) Notice that a two-particle Hilbert space $(h_0)_s \otimes (h_0)_{s+1}$ is four-dimensional. With P_0 projecting on h_0 in h, we find that $V^0_{s,s+1} = P_0^s P_0^{s+1} V_{s,s+1} P_0^s P_0^{s+1}$ acts in $(h_0)_s \otimes (h_0)_{s+1}$ invariantly and can immediately be diagonalized, thus leading to the four real eigenvalues, which we identify with the exchange energies $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$, respectively.

The particular assignments of energy values of the transition diagrams, according to (3.1), (3.2), (3.8) rely on the form of respective eigenvectors. Let us denote:

$$|1\rangle = e_0 \otimes e_0, \quad |2\rangle = \frac{1}{\sqrt{2}}(e_0 \otimes e_1 + e_1 \otimes e_0),$$
$$|3\rangle = \frac{1}{\sqrt{2}}(e_0 \otimes e_1 - e_1 \otimes e_0), \quad |4\rangle = e_1 \otimes e_1. \tag{4.5}$$

Let $\{\alpha_j^k\}$ be a complex 4×4 matrix:

$$\sum_k \bar{\alpha}_i^k \alpha_j^k = \delta_{ij}, \quad \sum_i \bar{\alpha}_i^k \alpha_i^l = \delta_{kl} \tag{4.6}$$

and let the four orthonormal vectors

$$|\alpha, j\rangle = \sum_i \alpha_j^i |i\rangle \tag{4.7}$$

be the eigenvectors of $V^0_{s,s+1}$:

$$V^0_{s,s+1}|\alpha, j\rangle = \epsilon_j |\alpha, j\rangle. \tag{4.8}$$

Then an expectation value

$$\epsilon_j = \langle \alpha, j | V^0_{s,s+1} | \alpha, j\rangle = \sum_{k,l} \bar{\alpha}_j^k \alpha_j^l \langle k | V^0_{s,s+1} | l\rangle \tag{4.9}$$

explicitly reveals which transition diagrams give a nonzero counterpart to ϵ_j. An identification is here immediate: $\langle k | V^0_{s,s+1} | l\rangle$ corresponds to the arrow diagram describing a transition from a configuration associated with $|k\rangle$, to this associated with $|l\rangle$.

(iii) Notice that $\langle k | V^0_{s,s+1} | l\rangle$ makes use of the single site basis for the elementary Schrödinger problem. If the basis exhibits any parametric dependence (which is the case for examples of the anharmonic oscillator or quantum pendulum), then all $\langle k | V^0_{s,s+1} | l\rangle$'s do exhibit also, and hence $\epsilon_j = \epsilon_j(\lambda) \Rightarrow w_j = w_j(\lambda)$ arises immediately (provided there is a single parameter, like the coupling constant involved). Compare, e.g., concluding remarks of the previous section.

Let us emphasize that to find Boltzmann weights, one needs to have an explicit operator expression for $V_{s,s+1}$ in terms of the single site raising-lowering operators: $\hat{V} = V(a^*, a)$.

Remark: For the particular case of the ϕ^4_2 theory in $1+1$ dimensions:

$$H = \sum_{j=1}^N \left[-\tfrac{1}{2}\frac{\partial^2}{\partial \phi_j^2} + \frac{\tau}{2}\phi_j^2 + \tfrac{1}{4}\phi_j^4 + \frac{c}{2}(\phi_{j+1} - \phi_j)^2 \right], \tag{4.10}$$

with $\phi \in \mathbb{R}^1$, the spin $\frac{1}{2}$ approximation arises in the single site anharmonic oscillator basis, and due to the simplest possible form of the gradient term

$$\phi_{j+1} - \phi_j = \frac{1}{\sqrt{2}}[(a_{j+1}^* + a_{j+1}) - (a_j^* + a_j)],$$

i.e., $\phi_j = (1/\sqrt{2})(a_j^* + a_j)$ one gets an immediate expression of $P_0 H P_0$ in terms of Fermi variables c^*, c (which in turn are associated with spin $\frac{1}{2}$ variables via the Jordan–Wigner formulas), according to Ref. 18:

$$P_0 H P_0 = \sum_{j=1}^N \{\epsilon c_j^* c_j + \Delta[1 - (c_j^* - c_j)(c_{j+1}^* + c_{j+1})]\}, \tag{4.11}$$

where $\Delta = c|\langle 1|\phi|0\rangle| = c\langle 0|\phi^2|0\rangle$, $\epsilon = E_1 - E_0$ and $|0\rangle$, $|1\rangle$ are vectors of the single-site basis.

By comparing with Ref. 19, one immediately recognizes that for large systems the gradient term $\Delta[1 - (c_j^* - c_j)(c_{j+1}^* + c_{j+1})]$ up to an irrelevant constant Δ, coincides with the one-dimensional Ising term $-J\Sigma_j s_j^x s_{j+1}^x$ provided $\Delta = J/4$.

A dependence of J on coupling constants $\{\tau, c\}$ of the ϕ_2^4 theory is here manifest, as entering J via the $\langle 0|\phi^2|0\rangle = |\langle 1|\phi|0\rangle|$ factor in the expression for Δ.

5. QUANTUM SINE-GORDON SYSTEM AS THE MASSIVE THIRRING MODEL: COMMENTS ON COLEMAN'S EQUIVALENCE

(i) Recall that by virtue of Sec. 4(ii), for the particular case of the quantum sine-Gordon chain, the (m,β) paramatrization enters the Heisenberg model coupling constants $\mathbf{J}$ by evaluating at β fixed, the explicit values of $V^0_{s,s+1}$ eigenvalues $\{\epsilon_i\}_{i=1,2,3,4}$ The single site problem is quantum pendulum here with m being the coupling constant, and hence indeed $J = J(m,\beta)$.

By rescaling the variables H,ϕ of (1.2) according to

$$\phi = \beta_C \phi', \quad H = \beta_C^2 H', \tag{5.1}$$

we arrive at the Coleman's form,[1] of the sine-Gordon energy density

$$H' = \left\{\left(\frac{\partial\phi'}{\partial t}\right)^2 - c\left(\frac{\partial\phi'}{\partial x}\right)^2\right\}\frac{m^2}{\beta_C^2}(1 - \cos\beta_C\phi'), \tag{5.2}$$

where $c = -1$ in the Coleman's case, and β_C is the Coleman's coupling constant β of Ref. 1.

The single-site quantum pendulum problem arising in connection with the rescaled pendulary chain, is related to the previous one by

$$E'_k = E_k/\beta_C^2, \quad \forall k = 0,1,\cdots, \tag{5.3}$$

and just the E'_0, E'_1 eigenvalues are used in the spin $\frac{1}{2}$ approximation procedure of previous sections.

(ii) It has been proved by Luther,[2] that in case of a very weak anisotropy of the x-y-z model

$$J_x \cong J_y \Rightarrow \mathrm{sn}(2\zeta)\cdot\mathrm{dn}(2\zeta) \cong 1, \tag{5.4}$$

one should make an identification

$$J_z = \beta_C^2/8\pi, \tag{5.5}$$

with the restriction that $\beta_C^2 \in [0,8\pi]$ as within this interval only a continuum limit can be taken for the lattice problem, and shown to lead to the quantum sine-Gordon system of the form (5.2). A massive Thirring model appears here as a mediating step both on the lattice and continuous levels.

By assuming $J_z = J_z(m,\beta)$ we have a unique connection between the (m,β) sine-Gordon chain and the $(\mathbf{J})$ Heisenberg problem, compare, e.g., Sec. 3(iv). But then, we must have

$$\beta_C^2 = \beta_C^2(m,\beta) \tag{5.6}$$

and hence in the Coleman's framework varying β_C means: (a) varying β at m fixed, (b) varying m at β fixed, (c) varying both m and β simultaneously.

In fact, the case (a) only was considered in Ref. 1, as then one can "forget" about $\beta = 1/kT$.

(iii) Take $\beta_C^2(m,\beta) \in \mathbb{R}^+$ and assume that at m fixed, $\beta_C^2(m,\beta)$ grows monotonically when $T \to 0$ (i.e., $\beta \to \infty$), and decreases in the opposite extreme, of $T \to \infty$ (i.e., $\beta \to 0$). In terms of the Coulomb gas of charges $\pm q$,[20] at thermal equilibrium, $\beta_C = (4\pi\beta)^{1/2}q, q = q(m)$ and there is a natural critical temperature β_0 corresponding to the Coleman's bound $\beta_C^2 = 8\pi$. Namely for $\beta_C^2 \in [0,8\pi)$ the Coulomb gas lives in its plasma phase, while for $\beta_C^2 \in [8\pi,\infty)$ a dipole gas occurs, and then the system must be kept on the lattice as ultraviolet cutoff cannot be consistently removed from the theory.

(iv) The very same Coulomb gas picture arises while varying m at β fixed. In that case, there is a critical value of $m = m_0$, i.e., $q_0 = q(m_0)$ at which a transition from plasma to dipoles occurs. A correct variability interval for the plasma phase is here $1/m \in [0,1/m_0)$, while $1/m \in [1/m_0,\infty)$ for the dipole phase. In this connection let us recall an old Lenard's result,[21] see, e.g., also Refs. 22 and 23 that in the zero-space dimension quantum pendulum stands for an equivalent of the Coulomb gas problem where the dipole and plasma cases (no phase transition here!) appear at the opposite extremes of m: $m = 0$ and $m = \infty$, respectively.

(v) It suggest the way to understand the Coleman's equivalence in terms of the single-site (i.e., quantum pendulum) data. Let us notice that for large m the quantum pendulum spectral problem admits an equivalent description in terms of the anharmonic oscillator, spectral problem which we choose in the form of[24]:

$$H_\nu = \tfrac{1}{2}[p^2 + \nu(q^2 - (2\nu)^{-1})^2], \quad \nu \to 0. \tag{5.7}$$

In the (weak) limit $\nu \to 0$, the corresponding spectral problem is doubly degenerate and gives

$$\lim_{\nu\to 0} E_\nu^{2j} = \lim_{\nu\to 0} E_\nu^{2j+1} = \sqrt{2}(j + \tfrac{1}{2}) = \epsilon_j. \tag{5.8}$$

For quantum pendulum, the $m \to \infty$ limit gives

$$E_{2n}^{ce} \to E_{2n+1}^{ce} \to (2n + \tfrac{1}{2})\epsilon = (\epsilon/\sqrt{2})\epsilon_{2n},$$
$$E_{2n+1}^{se} \to E_{2n+2}^{se} \to \{(2n+1) + \tfrac{1}{2}\}\epsilon = (\epsilon/\sqrt{2})\epsilon_{2n+1}, \tag{5.9}$$

where ϵ is a fixed positive factor, insensitive to the $m \to \infty$ limit.[25,9]

Whenever to rescale H according to Ref. 24, with an appropriate redefinition of the original constants of Ref. 24:

$$H_\nu = H'_\nu[\beta'(g-1)^{1/2}]^{-1} \cong H'_\nu\beta_C^2,$$
$$g = m/\beta'^3, \quad \nu = g(g-1)^{-3/2}, \quad \beta' = \beta'(\beta), \tag{5.10}$$

then, the $\nu \to 0$ limit is just the same as $m \to \infty$ limit at fixed β. Obviously the $m \to \infty$ limit does make sense for β_C^2 itself, as

$$\beta_C^2 = \beta_C^2(\beta,m) = \frac{\beta'^{1/2}}{(m - \beta^3)^{1/2}} \tag{5.11}$$

and it does exist for the rescaled Hamiltonian $H'_\nu = H_\nu/\beta_C^2$. Namely, if $\nu \to 0$, then

$$M_\nu = (E_\nu^1)' - (E_\nu^0)' \leqslant A \exp(-Bg^{1/2}), \tag{5.12}$$

where A, B are fixed, g-independent constants. Then $\beta_C^2 = 8\pi$ would imply

$$m = m_0(\beta) = \beta'^3 + \beta'/(8\pi)^2, \tag{5.13}$$

i.e., to each value of β, provided β is small enough [to insure that $m_0(\beta)$ is sufficiently large], we find its own corresponding $m_0(\beta)$.

The small β-large $m_0(\beta)$ limitation means that we need a fixed upper bound M_ν^0:

$$A \exp(-Bg^{1/2}) \leqslant M_\nu^0 \ll 1, \tag{5.14}$$

which induces an appropriate balance of β and $m_0(\beta)$ values.

Let then, at $\beta_C^2 = 8\pi$, $A \exp(-Bg^{1/2}) = M_\nu^0 \ll 1$. It fixes the range of variability for β's at m fixed (or m's at β fixed), within which the bound M_ν^0 is not passed by the energy interval value M_ν.

(vi) By recalling the properties of quantum pendulum when varying m at β fixed, we find that with the lowering of m a separation between lowest two levels of quantum pendulum $\Delta = E^1 - E^0$ increases, while a separation $\Delta' = E^2 - E^1$ between next lowest decreases. Hence a bound M_ν^0 can be consistently interpreted as this value of Δ, beginning from which a two-level approximation of the pendulary chain fails, and a three-level one (at least) should be taken into account.

Remark 1: In terms of the Coulomb gas in zero-space dimension, the dipole gas appears when a spin 0 approximation is reliable (lowest eigenvalue contribution to the partition function is of interest). The plasma appears when a spin $\frac{1}{2}$ approximation becomes reliable (double degeneracy of the lowest eigenvalue occurs).

While working in 1 +1 dimensions, the spin $\frac{1}{2}$ approximation is the lowest one, exhibiting a nontrivial gradient structure, and the nontrivial dipole one, is just the spin 1 (or 2,3,4,···) approximation of the chain as then next eigenvalue contributions to the partition function are taken into account.

Remark 2: In connection with the form of gradient terms in the spin 1 approximation of the linear chain, let us notice that SU(3) is the largest symmetry group for the three-level system [while SU(2) was for a two-level one], and hence the most general form of the nearest neighbor gradient term should be that of the current–current interaction type, with the number eight of coupling constants involved. The structure of the gradient term may vary depending on the explicit choice of the model. For example, a three-level approximation if applied to a system of coupled two-dimensional oscillators in the plane rotor approximation,[26] involves gradient terms of the form $\cos(\phi_{j+1} - \phi_j)$. Their image in the spin 1 approximation is simply $J(S_{j+1}^x S_j^x + S_{j+1}^y S_j^y)$, i.e., the spin 1 x-y model coupling

Remark 3: Coming back to the spin $\frac{1}{2}$ approximation framework, let us notice that the minimal form of gradient terms of lattice Bose systems, which in the two-level approximation, give rise to the Heisenberg chain, has been derived in Ref. 27.

Some other aspects of the two-level approximation trick in connection with the Bose contents of spinor fields can be found in Refs. 28 and 29. The whole program of my investigations on the spin $\frac{1}{2}$ approximation concept was initiated in Ref. 30.

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