On quantum solitons and their classical relatives: II. "Fermion–boson reciprocity" and classical vs quantum problem for the sine-Gordon system [a)]

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Both quantum and classical sine–Gordon fields can be built out of the fundamental free neutral massive excitations, which quantally obey the Bose–Einstein statistics. At the roots of the "boson-fermion reciprocity" invented by Coleman, lies the spin $\frac{1}{2}$ approximation of the underlying Bose system. By generalizing the coherent state methods to incorporate non-Fock quantum structures and to give account of the so-called boson transformation theory, we construct the carrier Hilbert space $\mathscr{H}_{SG}$ for quantum soliton operators. The $\hbar \to 0$ limit of state expectation values of these operators among pure coherent-like states in $\mathscr{H}_{SG}$ reproduces the classical sine–Gordon field. The related (classical and quantum) spin $\frac{1}{2}$ xyz Heisenberg model field is built out of the fundamental sine–Gordon excitations, and hence can be consistently defined on the appropriate subset of the quantum soliton Hilbert space $\mathscr{H}_{xyz}$. A correct classical limit is here shown to arise for the Heisenberg system: phase manifolds of the classical Heisenberg and sine–Gordon systems cannot be then viewed independently as a consequence of the quantum relation.

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1. A NONSINGULAR BOSON TRANSFORMATION

Suppose we have been given a classical scalar field, which obeys a differential equation:

$$\Lambda(\partial)\psi = F(\psi) \quad \Lambda(\partial) = -\nabla^2 + \partial^2/\partial x_0^2 + m^2, \tag{1.1}$$

and let $\varphi(\mathbf{x})$, $\pi(\mathbf{x})$ be the initial (time $t = 0$) data specifying an arbitrary free-field solution $\psi_0(\mathbf{x},t) = \psi_0(\mathbf{x},t,\varphi,\pi)$ of (1.1). By using the Yang–Feldman relation

$$\psi(\mathbf{x},t) = \psi_0(\mathbf{x},t) + \Lambda(\partial)^{-1}F(\mathbf{x},t,\psi_0), \tag{1.2}$$

we find that $\psi(\mathbf{x},t)$ is uniquely specified by fixing the initial free data

$$\begin{aligned}\psi(\mathbf{x},t) &= \psi(\mathbf{x},t,\varphi,\pi)\\ &= \psi_0(\mathbf{x},t,\varphi,\pi) + \Lambda(\partial)^{-1}F(\mathbf{x},t,\varphi,\pi).\end{aligned} \tag{1.3}$$

Whenever the data φ,π specify the conventional Fourier transformable plane wave solution ψ_0, we denote $\psi_0 = \psi_{\text{in}} = \psi_0(\varphi_{\text{in}},\pi_{\text{in}})$, called the in-field associated with ψ.

Let us now consider a quantum scalar (Heisenberg) field $\hat\psi$ satisfying an equation analogous to (1.1)

$$\begin{aligned}&\Lambda(\partial)\hat\psi(x) = F(x,\hat\psi) \quad x = (\mathbf{x},t)\\ &\hat\psi(x) = \hat\psi_0(x) + \Lambda(\partial)^{-1}F(x,\hat\psi_0),\end{aligned} \tag{1.4}$$

where we at once choose $\hat\psi_0(x) = \hat\psi_{\text{in}}(x)$, with $\hat\varphi_{\text{in}}(\mathbf{x}),\hat\pi_{\text{in}}(\mathbf{x})$ satisfying the (equal time) commutation relations

$$[\hat\varphi_{\text{in}}(\mathbf{x}),\hat\pi_{\text{in}}(\mathbf{y})]_- = \delta(\mathbf{x}-\mathbf{y})i\hbar. \tag{1.5}$$

Hence

$$\hat\psi(x) = \hat\psi_{\text{in}}(x) + \Lambda(\partial)^{-1}F(x,\hat\varphi_{\text{in}},\hat\pi_{\text{in}}), \tag{1.6}$$

which is understood as an equality among the matrix elements of the operators, calculated between the in-field (Fock space!) vectors. An expressions of $\hat\psi$ in terms of $\hat\varphi_{\text{in}},\hat\pi_{\text{in}}$ we call a *dynamical map*.[1-7] Armed with the plane wave quantum data $\hat\varphi_{\text{in}},\hat\pi_{\text{in}}$ obeying (1.5), we are able to construct these free (in) field states, which are generated from the vacuum by a classical current, the familiar coherent states

$$\begin{aligned}|\varphi,\pi\rangle &= \hat T_{\varphi,\pi}|0\rangle\\ &= \exp(i/\hbar)\int d^3x\,[\pi(\mathbf{x})\hat\varphi_{\text{in}}(\mathbf{x}) - \varphi(\mathbf{x})\hat\pi_{\text{in}}(\mathbf{x})]|0\rangle.\end{aligned} \tag{1.7}$$

Here an exponent can always be rewritten in terms of the canonical pair (no transition to momentum space) $a^*(\mathbf{x})$, $a(\mathbf{x})$ as $-(\alpha,a^*) + (\bar\alpha,a)$, where the complex valued function $\alpha(\mathbf{x})$ is subject to the square integrability condition

$$(\bar\alpha,\alpha) = \int d^3x\,\bar\alpha(\mathbf{x})\alpha(\mathbf{x}) = \|\alpha\|^2 < \infty, \tag{1.8}$$

with $\alpha(\mathbf{x}) = [\varphi(\mathbf{x}) + i\pi(\mathbf{x})]/\hbar\sqrt{2}$. Then

$$\begin{aligned}\hat T^{-1}_{\varphi,\pi}\hat\varphi_{\text{in}}(\mathbf{x})\hat T_{\varphi,\pi} &= \hat\varphi_{\text{in}}(\mathbf{x}) + \varphi(\mathbf{x}),\\ \hat T^{-1}_{\varphi,\pi}\hat\pi_{\text{in}}(\mathbf{x})\hat T_{\varphi,\pi} &= \hat\pi_{\text{in}}(\mathbf{x}) + \pi(\mathbf{x}),\end{aligned} \tag{1.9}$$

so that

$$\begin{aligned}\langle 0|\hat T^{-1}_{\varphi,\pi}\hat\psi(x)\hat T_{\varphi,\pi}|0\rangle &= \langle\varphi,\pi|\hat\psi(x)|\varphi,\pi\rangle\\ &= \langle 0|\psi(x,\hat\varphi_{\text{in}} + \varphi,\hat\pi_{\text{in}} + \pi)|0\rangle,\end{aligned} \tag{1.10}$$

and hence a c-number field $\psi_{\varphi,\pi}(x)$ can be associated with $\hat\psi(x)$ according to

$$\langle 0|\hat\psi_{\varphi,\pi}(x)|0\rangle = \langle 0|\hat T^{-1}_{\varphi,\pi}\hat\psi(x)\hat T_{\varphi,\pi}|0\rangle = \psi_{\varphi,\pi}(x). \tag{1.11}$$

Because of (1.5), $\hat\psi_{\varphi,\pi}(x)$ can always be rearranged to a normal ordered form plus terms following from the contractions of the in-fields during the ordering procedure. These last, by virtue of (1.5) are proportional to $\hbar$, and hence, we can consider

$$\begin{aligned}\lim_{\hbar\to 0}\psi_{\varphi,\pi}(x) &= \lim_{\hbar\to 0}[\langle 0|{:}\hat\psi_{\varphi,\pi}(x){:}|0\rangle + O(\hbar)]\\ &= \langle 0|{:}\hat\psi_{\varphi,\pi}(x){:}|0\rangle = \phi(\mathbf{x},t,\varphi,\pi),\end{aligned} \tag{1.12}$$

[a)] Main ideas of the paper are taken from my University of Alberta preprint under the title "On quantum solitons and their classical relatives."
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and quite analogously (notice that all $\hat{\psi}_{\varphi,\pi}(x)$ do commute among themselves under the sign :·: of the normal ordering)

$$\lim_{\hbar\to 0} F_{\varphi,\pi}(x) = \lim_{\hbar\to 0} \langle 0|\hat{F}_{\varphi,\pi}(x)|0\rangle = \lim_{\hbar\to 0} [\langle 0|:F(x,\hat{\psi}_{\varphi,\pi}):|0\rangle + O(\hbar)] = F(\mathbf{x},t,\phi(\varphi,\pi)) = F(x,\varphi,\pi). \quad (1.13)$$

So that, after taking the $\hbar\to 0$ limit of the Fock vacuum expectation value of (1.4), we have transformed a quantum field equation into a corresponding classical Euler one. Everything is true under the assumption that the $\hbar\to 0$ limit exists at all,

$$\lim_{\hbar\to 0} \langle 0|\Lambda(\partial)\hat{\psi}_{\varphi,\pi}(x)|0\rangle = \Lambda(\partial) \lim_{\hbar\to 0} \langle 0|\hat{\psi}_{\varphi,\pi}(x)|0\rangle$$

$$= \Lambda(\partial)\phi(x,\varphi,\pi) = \lim_{\hbar\to 0} \langle 0|\hat{F}_{\varphi,\pi}(x)|\rangle = F(x,\phi)$$

$$\Rightarrow \Lambda(\partial)\phi(x) = F(x,\phi). \quad (1.14)$$

Let us recall that in the above, it was crucial to choose the classical data φ,π which are consistent with the square integrability condition (1.8). However such data do not at all exhaust the set of those allowed by the classical Euler equation (1.14). Hence it becomes of interest to establish whether such, say *singular*, data can be used (and in what sense) to generate operators of the form $\hat{\psi}_{\varphi,\pi}(x)$, and are consistent with the formula (1.14). The mapping (1.9) we call a *nonsingular boson transformation*.

2. A SINGULAR BOSON TRANSFORMATION

A nonsingular boson transformation maps a Fock space vector into a Fock space vector again, i.e., a vacuum $|0\rangle$ into a coherent state $|\varphi,\pi\rangle$, and this last can obviously be mapped into another coherent state, say $|\varphi',\pi'\rangle$ by a unitary transformation: $\hat{T}_{\varphi',\pi'}\hat{T}^{-1}_{\varphi,\pi}|\varphi,\pi\rangle = |\varphi',\pi'\rangle$.

In the above the square integrability condition (1.8) accounts for these classical data which are consistent with the unitarity requirement $\hat{T}^{-1}_{\varphi,\pi} = \hat{T}^{*}_{\varphi,\pi}$ and thus a good behavior at space infinity of the boson transformation parameters φ,π guarantees that $\hat{T}_{\varphi,\pi}$ is a nonsingular mapping.

To introduce a *singular boson transformation*, and to understand its meaning in the quantum theory (with the 1+1 dimensional sine–Gordon perspective in mind) let us restrict the space dimensionality to one, and cover $\mathbb{R}^1$ by a countable sequence $\{\Delta_k\}_{k=0,\pm 1,\dots}$ of the noninteresting, semi-open intervals each one with length (lattice spacing) Γ small enough. Let us introduce

$$a_k^* = (1/\sqrt{\Gamma})\int \chi_k(x)a^*(x)\,dx,$$
$$a_k = (1/\sqrt{\Gamma})\int \chi_k(x)a(x)\,dx, \quad (2.1)$$

where $\chi_k(x) = {}^{1\ x\in\Delta_k}_{0\ x\notin\Delta_k}$ is a characteristic function of the set Δ_k. For any function $\lambda(x)$ on $\mathbb{R}^1$ we shall also introduce a sequence $\{\lambda_i = (1/\Gamma)\int\chi_i(x)\lambda(x)\ dx\}_{i=0,\pm 1,\dots}$ approximating $\lambda(x)$ in the sense of $\lambda(x) \cong \Sigma_i\lambda_i\chi_i(x)$. Now let a sequence $\{a_k,a_k^*\}_{k=0,\pm 1,\dots}$ generate a Fock representation of the CCR algebra

$$[a_i,a_j^*]_- = \delta_{ij}, \quad [a_i,a_j]_- = 0, \quad a_i|0\rangle = 0\forall_i, \quad (2.2)$$

whose carrier (Fock) Hilbert space we designate IDPS ($|0\rangle$), according to the direct product convention.[8–13]

For each fixed value of $j = 0, \pm 1,\cdots$ a unitary transformation

$$\hat{T}_j^\lambda = \exp(\lambda_j a_j^* - \bar{\lambda}_j a_j) \quad (2.3)$$

realizes a boson transformation

$$(\hat{T}_j^\lambda)^{-1} a_j \hat{T}_j^\lambda = a_j + \lambda_j,$$
$$(\hat{T}_j^\lambda)^{-1} a_j^* \hat{T}_j^\lambda = a_j^* + \bar{\lambda}_j \quad (2.4)$$

and under the condition $|\Sigma_i\lambda_j|^2 < \infty$, a global mapping

$$\hat{T}_\lambda = \prod_j \hat{T}_j^\lambda = \exp\Big[\sum_j(\lambda_i a_j^* - \bar{\lambda}_j a_j)\Big] \quad (2.5)$$

exists and is unitary. Moreover the previously considered $\hat{T}_{\varphi,\pi}$ (formally) emerges here as a continuum limit ($\Gamma\to 0$) of $\hat{T}_\lambda$.

Let h be a Hilbert space for an elementary quantum system, and let a^*,a be the associated raising and lowering operators. Then the direct product construction leads to the direct product space $\mathscr{H} = \Pi_j^{\otimes} h_j$ where $h_j = h\forall_j$ and the representation $\{a_j^*,a_j,\mathscr{H}\}_{j=0,\pm 1,\dots}$ is infinitely reducible. Recall that the generating vector $|0)$ for the (Fock) irreducibility sector IDPS ($|0\rangle$) reads $\Pi_j^{\otimes}(e_0)_j$, where $e_0\in h$ is the ground state of an elementary system. One easily finds that in the case $\Sigma_j|\lambda_j|^2 < \infty$, we get

$$\hat{T}_\lambda|0\rangle = \prod_j \hat{T}_j^\lambda \prod_k{}^{\otimes}(e_0)_k = \prod_k{}^{\otimes}(\hat{T}_k^\lambda e_0)_k:$$
$$= \prod_k{}^{\otimes}(|\lambda_k\rangle) = |\lambda\rangle \in \mathrm{IDPS}(|0\rangle), \quad (2.6)$$

where for each $k = 0, \pm 1,\cdots h \ni |\lambda_k\rangle$ is a coherent state: $a|\lambda_k\rangle = \lambda_k|\lambda_k\rangle$ specified by a parameter λ_k. It proves that the choice of any ill-behaving at $\pm\infty$ boson transformation parameter $\bar{\lambda}(x) = (1/\sqrt{2})[\varphi(x) + i\pi(x)]$ results in the failure of the $\Sigma_j|\lambda_j|^2 < \infty$ condition and hence the nonexistence of the global mapping operator $\hat{T}_\lambda$. This is just the singular case of interest for us.

Assume the series $\Sigma_j|\lambda_j|^2$ to diverge, and let us consider the vector $|\lambda) \in \mathscr{H}$ given by

$$|\lambda\rangle = \prod_k{}^{\otimes}(\hat{T}_k^\lambda e_0) = \prod_k{}^{\otimes}|\lambda_k\rangle, \quad (2.7)$$

which by construction is unitarily inequivalent to $|0)$, and hence orthogonal to it. As a consequence, $|\lambda)$ generates a new irreducibility sector

$$\mathrm{IDPS}(|\lambda\rangle) = \mathscr{H}_\lambda \subset \mathscr{H} \quad (2.8)$$

for the representation $\{a_i^*,a_i\}_{i=0,\pm 1,\dots}$ of the CCR algebra in $\mathscr{H}$. The representation is obviously a non-Fock one.

Assume now a quantum operator $\hat{F} = F(a^*,a) = F(\hat{\varphi}_{\rm in}, \hat{\pi}_{\rm in})$ to map $|\lambda\rangle$ into a vector from IDPS ($|\lambda\rangle$) (otherwise $\hat{F}|\lambda)$ would be orthogonal to $|\lambda\rangle$). Then

$$\langle\lambda|F(a^*,a)|\lambda\rangle = \langle 0|F(a^* + \bar{\lambda}, a + \lambda)|0\rangle$$
$$= \langle 0|F(\hat{\varphi}_{\rm in} + \varphi, \hat{\pi}_{\rm in} + \pi)|0) = \langle 0|\hat{F}_\lambda|0\rangle, \quad (2.9)$$

and the boson transformation is realized for each $\hat{\varphi}_k^{\rm in}$ or $\hat{\pi}_k^{\rm in}$

factor appearing in the explicit expression for $\hat{F}$. Notice that the discretization can be immediately removed by taking a continuum limit $\Gamma \to 0$.

A transformation

$$R_{\varphi,\pi}: F(\hat{\varphi}_{\text{in}}, \hat{\pi}_{\text{in}}) \mapsto F(\hat{\varphi}_{\text{in}} + \varphi, \hat{\pi}_{\text{in}} + \pi), \tag{2.10}$$

with φ,π violating the condition $\|\lambda\|^2 < \infty$, we call a *singular boson transformation*. In this way, we have proved that a singular boson transformation necessarily induces a transition to a (different from the Fock one) irreducibility sector for the representation $\{\hat{\varphi}_{\text{in}}, \hat{\pi}_{\text{in}}\}$ of the CCR algebra in $\mathscr{H}$.

It is useful to know whether the two vectors $|\alpha\rangle$ and $|\gamma\rangle$, constructed according to (2.7)–(2.9) are unitarily inequivalent, i.e., whether a mapping: $\hat{F}_\lambda \to \hat{F}_\alpha$ is singular or not. For this purpose let us restrict considerations to a single quantum degree of freedom in the discretized (2.4) case. Namely, we have

$$|0\rangle = e_0, \quad |\alpha\rangle = \hat{T}_\alpha |0\rangle = \hat{T}_\alpha \hat{T}_\gamma^{-1} |\gamma\rangle = \hat{T}_{\alpha\gamma} |\gamma\rangle, \tag{2.11}$$

with

$$\begin{aligned} \hat{T}_\alpha &= \exp(\bar{\alpha} a - \alpha a^*) \\ &= \exp(|\alpha|^2/2)\exp(\bar{\alpha} a)\exp(-\alpha a^*) \\ &= \exp(-|\alpha|^2/2)\exp(-\alpha a^*)\exp(\bar{\alpha} a), \end{aligned} \tag{2.12}$$

and $\hat{T}_\gamma^{-1} = \hat{T}_{-\gamma}$ so that, by using the Baker–Hausdorff formula, we get

$$\begin{aligned} \hat{T}_\alpha \hat{T}_\gamma^{-1} &= \exp(\bar{\alpha} a - \alpha a^*)\exp(-\bar{\gamma} a + \gamma a^*) \\ &= \exp[(\bar{\alpha} - \bar{\gamma})a - (\alpha - \gamma)a^*]\exp(\bar{\alpha}\gamma - \alpha\bar{\gamma})\tfrac{1}{2} \\ &= \exp[(\bar{\alpha} - \bar{\gamma})a]\exp[(\gamma - \alpha)a^*] \\ &\quad \times \exp[(1/2)(|\alpha|^2 + |\gamma|^2 - 2\alpha\bar{\gamma})] \end{aligned} \tag{2.13}$$

i.e.,

$$|\langle 0|\hat{T}_\gamma^{-1}\hat{T}_\alpha|0\rangle|^2 = \exp(-|\alpha - \gamma|^2). \tag{2.14}$$

It proves that a necessary condition for the global mapping $\Pi_k \hat{T}^k_{\alpha\gamma}$ to be nonsingular, is

$$\sum_k |\alpha_k - \gamma_k|^2 < \infty, \tag{2.15}$$

whose continuum limit is simply

$$\int d^3x\, |\alpha(\mathbf{x}) - \gamma(\mathbf{x})|^2 = \tfrac{1}{2}\int d^3x\, \{[\varphi_\alpha(\mathbf{x}) - \varphi_\gamma(\mathbf{x})]^2 + [\pi_\alpha(\mathbf{x}) - \pi_\gamma(\mathbf{x})]^2\} < \infty. \tag{2.16}$$

In this limit a direct product Π_i^∞ becomes a continuous direct product. In that case only, a transition from $|\gamma\rangle$ to $|\alpha\rangle$ can be realized by a unitary transformation, within the same irreducibility sector IDPS ($|\alpha\rangle$), say. Otherwise quantum operators $F_\alpha(\hat{\varphi}_{\text{in}}\hat{\pi}_{\text{in}}), F_\gamma(\hat{\varphi}_{\text{in}}\hat{\pi}_{\text{in}})$ are associated with different (non-Fock) irreducibility $\mathscr{H}_\alpha, \mathscr{H}_\gamma$ respectively in $\mathscr{H}$, because we have

$$\sum_k |\alpha_k - \gamma_k|^2 \to \infty \Rightarrow \langle\gamma|\alpha\rangle \to 0, \tag{2.17}$$

i.e., an orthogonality property holds true for $|\alpha\rangle$, $|\gamma\rangle$ if a function $(\alpha - \gamma)(\mathbf{x})$ is not square integrable.

3. SINE–GORDON SOLITONS AND THE BOSON TRANSFORMATION PARAMETERS

A. Let us consider a classical sine–Gordon system in 1 + 1 dimensions[14,15]

$$\begin{aligned} &\mathscr{L}(x,t) = (1/2)[\partial_x^2\psi - \partial_t^2\psi + 2m^2(1 - \cos\psi)](x,t), \\ &\partial^2\psi = m^2\sin\psi, \quad \partial^2 = \partial_x^2 - \partial_t^2, \end{aligned} \tag{3.1}$$

where $\hbar = c = 1$. The classical particle spectrum of (3.1) is well known to consist of the three kinds of elementary excitations: a fundamental neutral particle with mass m, a charged particle with mass $8m$, and a neutral particle with mass varying within an interval $[0,16m]$. Except for fundamental particles, which can be identified through their counterpart to the total energy–momentum of the field only, the remaining two arise in the large time asymptotics of the so-called soliton solutions of the sine–Gordon equation, which we introduce via the formula

$$\begin{aligned} &\psi_N(x,t) = \arccos\{1 - (2/m^2)(\partial_x^2 - \partial_t^2)\ln f_N(x,t)\}, \\ &f_N(x,t) = \det(M_{ij}), \quad i,j = 1,2,\ldots,N \\ &M_{ij} = [2/(a_i + a_j^*)]\cosh[(\theta_i + \theta_j^*)/2]. \end{aligned} \tag{3.2}$$

The parameters a_i are allowed to be complex valued, provided that for each complex a_i in the sequence $\{a_1,\ldots,a_N\}$ there appears an associated a_j with the property $a_j = a_i^*$, a_i being real otherwise. The additional restrictions on a_i are as follows:

$$\begin{aligned} &a_i \neq a_j, \quad i \neq j, \\ &|a_i|^2 = (1 - v_i)/(1 + v_i), \quad |v_i| < 1. \end{aligned} \tag{3.3}$$

Moreover, if a_i is real, a corresponding θ_i is given by

$$\theta_i = \pm m\gamma_i(x - v_i t) + \delta_i = \theta_i(x,t), \tag{3.4}$$

where $\operatorname{sgn}(a_i)$ is chosen to coincide with the sign appearing in the expression for θ_i, and $\gamma_i^2 = (1 - v_i^2)^{-1}$.

If a_i is complex and then accompanied by $a_j = a_i^*$ in the parametric sequence $\{a_i\}_{i=1,\ldots,N}$, then we have

$$\begin{aligned} \theta_i &= \theta_j^* = \theta(a_j) = \theta_R + i\theta_I \\ &= (m\gamma/|a|)[a_R(x + vt) + ia_I(vx + t)] + \delta, \end{aligned} \tag{3.5}$$

where $\delta = \delta_i = \delta_R + i\delta_I$ is a complex phase, while $a = a_i = a_R + ia_I$, $\delta_i = \delta_j^*$.

One can easily verify that $\exp\theta_i(x,t) := \lambda_i(x,t)$ with θ_i given by either (3.4) or (3.5), is a solution of the free-field equation

$$\partial^2\lambda(x,t) = m^2\lambda(x,t), \quad \partial^2 = \partial_x^2 - \partial_t^2. \tag{3.6}$$

Real solutions (not plane waves in general!) of (3.6) we identify with the mass m fundamental neutral fields associated with the sine–Gordon system. Notice that having λ complex, we have also $\bar{\lambda}$, and then $\frac{1}{2}(\lambda + \bar{\lambda})$ is the underlying mass m field.

Recall that to each real a_i there corresponds a one-soliton solution, while to each complex pair $a_i = a_j^*$ there corresponds a bound (bion, breather) solution, both in the large time asymptotics of $\psi_N(x,t) = \psi_N(a_1,\ldots,a_N,x,t)$. By looking at (3.2), and doing some algebra (with care however, as one must interchange infinite summations) one easily finds that after expanding $\psi_N(x,t)$ with respect to $\exp(\pm\theta)$ factors, for each θ_i a corresponding $\exp(-\theta_i)$ factor disappears, and hence the following formal power series expansion is valid for the classical N-soliton field

$$\psi_N(x,\tau) = \sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}\psi^{n_1\cdots n_N}\exp(n_1\theta_1)\cdots\exp(n_N\theta_N)$$
$$= \sum_{n_1}\cdots\sum_{n_N}\psi^{n_1\cdots n_N}\lambda_1^{n_1}(x,t)\cdots\lambda_N^{n_N}(x,t), \quad (3.7)$$

with $\theta_i = \theta_i(x,t)$ for $i = 1,2,...,N$ and $\psi^{n_1\cdots n_N}$ being the N-fold tensor coefficient, in which there are absorbed all the m, a_i dependent factors arising in the course of calculations, and $\psi_i(x,t) = \frac{1}{2}[\lambda_i(x,t) + \bar{\lambda}_i(x,t)]$ satisfies (3.6).

B. We are now at the point, in which considerations of the previous sections can be taken into account. Let $\hat{\psi}_{\rm in}(x,t)$ be the quantum in-field (i.e., plane wave) solution of the equation

$$(\partial^2 - m^2)\hat{\psi}_{\rm in}(x,t) = 0. \quad (3.8)$$

Let us furthermore consider the set of operator functional power series of the form

$$\hat{\psi}(x,t) = \psi(x,t,\hat{\varphi}_{\rm in},\hat{\pi}_{\rm in}),$$
$$F(x,t,\hat{\varphi}_{\rm in},\hat{\pi}_{\rm in}) = \hat{F}(x,t), \quad (3.9)$$

where $\hat{\varphi}_{\rm in}(x)$, $\hat{\pi}_{\rm in}(x)$ are the initial quantum data for $\hat{\psi}_{\rm in}(x,t)$

$$[\hat{\varphi}_{\rm in}(x),\hat{\pi}_{\rm in}(y)]_- = i\hbar\delta(x-y). \quad (3.10)$$

We replace them by operators

$$a^*(x) = (1/\sqrt{2})[\hat{\varphi}_{\rm in}(x) - i\hat{\pi}_{\rm in}(x)],$$
$$a(x) = (1/\sqrt{2})[\hat{\varphi}_{\rm in}(x) + i\hat{\pi}_{\rm in}(x)], \quad (3.11)$$

which after the smearing operation (2.1) define a reducible in the direct product space $\mathscr{H}$ representation of the CCR algebra

$$[a_s,a_t^*]_- = \hbar\delta_{st},$$
$$[a_s,a_t]_- = 0 = [a_s^*,a_t^*]_-.$$

We say that an operator $\hat{\psi}(x,t) = \psi(a^*,a,x,t)$ satisfies the quantum sine–Gordon equation with the operator-valued source $\hat{F}(x,t) = F(a^*,a,x,t)$ if and only if an operator identity

$$\partial^2\hat{\psi}(x,t) = \hat{F}(x,t) \quad (3.12)$$

holds true in the sense

$$\langle 0|\partial^2\hat{\psi}(x,t)|0\rangle = \partial^2\langle 0|\hat{\psi}(x,t)|0\rangle = (0|\hat{F}(x,t)|0), \quad (3.13)$$

$$\lim_{\hbar\to 0}\partial^2\langle 0|\hat{\psi}(x,t)|0\rangle = \lim_{\hbar\to 0}\langle 0|\hat{F}(x,t)|0\rangle$$
$$= \partial^2\phi(x,t) = m^2\sin\phi(x,t), \quad (3.14)$$

where $|0\rangle$ stands for the Fock vacuum for $\{\hat{\varphi}_{\rm in},\hat{\pi}_{\rm in}\} \equiv \{a^*,a\}$. It means that each particular solution $\phi(x,t)$ of (3.1) should arise as a vacuum expection value in the tree approximation of some corresponding quantum operator $\hat{\psi}_\phi(x,t)$

$$\phi(x,t) = \lim_{\hbar\to 0}\langle 0|\hat{\psi}_\phi(x,t)|0\rangle. \quad (3.15)$$

C. Because on the quantum level we deal with a single fundamental mass m "meson" in-field $\hat{\psi}_{\rm in}(x,t)$, one easily finds by comparison with Secs. 1 and 2 that a sufficient condition for $\hat{\psi}_\phi(x,t)$ to generate power series expansions of the form (3.7), (3.9) in the sense of (3.13), (3.14) is the identity

$$\hat{\psi}_\phi(x,t) = \psi(x,t,\hat{\varphi}_{\rm in} + \varphi,\hat{\pi}_{\rm in} + \pi) := \psi(x,t,a^* + \bar{\lambda},a + \lambda), \quad (3.16)$$

i.e.,

$$\langle 0|\hat{\psi}_\phi(x,t)|0\rangle := \langle\lambda|\hat{\psi}(x,t)|\lambda\rangle \equiv \langle\varphi,\pi|\hat{\psi}(x,t)|\varphi,\pi\rangle, \quad (3.17)$$

which is a straightforward (albeit formal) generalization of the discrete procedures of Sec. 2.

Under an additional assumption that the whole classical N-soliton sector, N fixed, can be generated [via (3.12)–(3.17)] by using a *single quantum field* denoted $\hat{\psi}_N(x,t)$, the conditions (3.16), (3.17) are replaced by

$$\hat{\psi}_{N,\phi}(x,t) = \psi_N(x,t,a^* + \bar{\lambda}_N,a + \lambda_N), \quad (3.18)$$

with

$$\bar{\lambda}_N := \sum_{i=1}^{N}\exp\theta_i = \lambda_N. \quad (3.19)$$

An operator $\hat{\psi}_N(x,t)$ we call a *quantum N-soliton operator*. Its domain in $\mathscr{H}$, i.e., the N-soliton sector in the state space, will be established below.

Recall that to have a comparison with arguments of Sec. 2, one should replace continuous translations $\bar{\lambda}(x)$, $\lambda(x)$ by the approximating sequences $\{\bar{\lambda}_i,\lambda_i\}_{i=0,\pm 1,\pm 2,\ldots}$. But first, one should notice that $\varphi(x)$,$\pi(x)$ arise as the initial classical data for the free field solution of $(\partial^2 - m^2)\psi(x,t) = 0$ of the form

$$\psi_N(x,t) = \sum_{i=1}^{N}\tfrac{1}{2}[\bar{\lambda}_i(x,t) + \lambda_i(x,t)]. \quad (3.20)$$

The consistency of the choice (3.20) of the boson transformation parameter was analytically checked in a slightly different framework in Ref. 5, see, e.g., also Ref. 3, for 1- and 2-solitons. An explicit form of the expansion coefficients $\psi^{n_1\cdots n_N}$ can easily be found for these particular fields.

D. In this way we have demonstrated that both quantum and classical soliton fields, can in principle be built out of the fundamental free neutral mass m excitations. Hence the three types of basic sine–Gordon excitations are only outwardly independent: All of them can be reduced to exhibit a more fundamental mass m neutral free field structure. This concept lies at the foundation of our studies of the Bose → Fermi metamorphosis in the whole series of papers[16–20] among which Ref. 20 is devoted to the study of spin $\frac{1}{2}$ approximation of the sine–Gordon system, and explanation of its relation to the spin $\frac{1}{2}$ xyz Heisenberg and Thirring models.

4. COHERENTLIKE DOMAINS FOR QUANTUM SOLITON OPERATORS

A. The time evolution of a quantum operator $\hat{\psi}_\phi(x,0) \to \hat{\psi}_\phi(x,t)$ must be consistent with the "classical limit" formula (3.15). Hence for a concrete N-soliton solution of (3.1), we can rewrite (3.15), as

$$\lim_{\hbar\to 0}\langle 0|\hat{\psi}_{N\phi}(x,t)|0\rangle = \lim_{\hbar\to 0}\langle\varphi,\pi,t|\hat{\psi}_N(x)|\varphi,\pi,t\rangle$$
$$= \lim_{\hbar\to 0}\langle\varphi,\pi|\hat{\psi}_N(x,t)|\varphi,\pi\rangle = \phi_N(x,t), \quad (4.1)$$

where $|\varphi,\pi,t\rangle$ differs from $|\varphi,\pi)$ through replacing the initial data φ,π by the "time dependent" ones according to

$$\lambda_N = \lambda_N(x,t) = \sum_{i=1}^{N} \exp\theta_i(x,t). \tag{4.2}$$

$\theta_i(x,t)$ is fixed by specifying the $2N$ parameters $\{\underset{\sim}{a},\underset{\sim}{\delta}\} = \{a_1,...,a_N,\delta_1,...,\delta_N\}$ according to (3.2)–(3.5). Notice that the whole time dependence giving rise to a correct classical result, can be absorbed in the state vectors.

If we look at the difference $\lambda_N(x,t) - \lambda_N(x,0) = f(x,t)$ we see at once that $f(x,t)$ is not a square integrable function on $\mathbb{R}^1$ for $t \neq 0$, hence (see, e.g., Sec. 2) for neither instant of time $t \neq 0$, is a quantum state $|\varphi,\pi,t\rangle$ unitarily equivalent to $|\varphi,\pi)$.[21–23] They both belong to the different (orthogonal) irreducibility sectors IDPS $(|\varphi,\pi\rangle)$ and IDPS $(|\varphi,\pi,t\rangle)$, respectively.

On the other hand, one finds immediately that a time development of $\theta_i(x)$ into $\theta_i(x,t)$ simply results in the time dependent phase shift of $\theta_i(x)$, which can thus be completely absorbed in the time dependent phase parameter

$$\delta_i \to \delta_i + \delta_i(t) \Rightarrow \theta_i \to \theta_i(x,t) = \theta_i(x) + \delta_i(t). \tag{4.3}$$

Consequently

$$\exp[\theta_i(x,t)] = \exp[\theta_i(x,t) + \delta_i(t)]. \tag{4.4}$$

Because for a fixed choice of soliton parameters $\{a_1,...,a_N\}$ the phases $\{\delta_1,...,\delta_N\}$ are still completely arbitrary, we have found that the quantum N-soliton time development

$$\hat{\psi}_N(x) \to \hat{\psi}_N(x,t), \tag{4.5}$$

can be described by giving a time dependent trajectory in the set of initial data

$$\psi_i(x,a,\delta) = x_i(x) \to \psi_i(x,t) = \psi_i(x,a,\delta'), \tag{4.6}$$
$$\delta' = \delta + \delta(t).$$

Hence for a given N-soliton solution $\phi_N(x,t)$ of (3.1) we have

$$|\varphi,\pi,t\rangle = |\varphi,\pi,\underset{\sim}{a},\underset{\sim}{\delta},t) = |\varphi,\pi,\underset{\sim}{a},\underset{\sim}{\delta}') \tag{4.7}$$

at a fixed instant of time t, with $\underset{\sim}{a} = \{a_1,...,a_N\}$ so that

$$\langle\varphi,\pi|\hat{\psi}_N(x,t)|\varphi,\pi) = (\varphi',\pi'|\hat{\psi}_N(x)|\varphi',\pi') \tag{4.8}$$

at t fixed, and IDPS $(|\varphi,\pi\rangle)$, IDPS $(|\varphi',\pi'\rangle)$ being orthogonal. In the way we have replaced a time development problem for a quantum soliton operator $\hat{\psi}(x,t)$ by a transition through an infinity of unitarily inequivalent and non-Fock representations of the in-field canonical algebra. Each irreducible representation exists in its own IDPS $(|\varphi,\pi,\underset{\sim}{a},\underset{\sim}{\delta}\rangle)$, where the $\underset{\sim}{\delta}$ parametrization amounts to a time development, see, e.g., also Ref. 22.

Because for a fixed classical N-soliton $\phi_N(x,\underset{\sim}{a},\underset{\sim}{\delta})$ the soliton data $\underset{\sim}{a}$ remain unchanged during the time evolution, it is convenient to collect all possible choices of $\underset{\sim}{\delta}$'s for quantum solitons, by taking a direct integral.

$$\int_{\oplus}\cdots\int_{\oplus} d\mu(\delta_1,...,\delta_N)\,\mathrm{IDPS}(|\varphi,\pi,\underset{\sim}{a},\underset{\sim}{\delta}\rangle):$$
$$= \mathrm{DPS}(|\varphi,\pi,\underset{\sim}{a}\rangle), \tag{4.9}$$

where (i) the measure $d\mu(\underset{\sim}{\delta})$ equals $d\delta_1 \cdots d\delta_N$ if all $a_i, i = 1,2,...,N$ are real, (ii) if there is any conjugate pair $a_i = a_j^*$ in the set $\{a_1,...,a_N\}$ the integral $\int_{\oplus}\int_{\oplus} d\delta_i\, d\delta_j$ should be replaced by $\int\int_{\oplus} \delta(\delta_i - \delta_j^*)\, d\delta_i\, d\delta_j$, where the integration is carried out over a complex plane.

In the new Hilbert space (*nonseparable*) $\mathrm{DPS}(|\underset{\sim}{a}\rangle)$, a time evolution of the quantum operator $\hat{\psi}_\phi(x,t)$, $\phi = \phi(\underset{\sim}{a})$ can be unitarily implemented.

B. In the above by using the notation $\mathrm{DPS}(|\underset{\sim}{a}\rangle)$, we have explictly indicated that all entering boson transformation parameters are considered at a fixed choice of the sequence $\underset{\sim}{a} = \{a_1,...,a_N\}$. However we are allowed to vary the parameters $\{a_1,...,a_N\}$ at a fixed value of N, within the variability interval $(0,\infty) \ni |a_i|$ and under the demand that $a_i \neq a_j$ for $i \neq j$ which is a fundamental restriction used in producing any solution (3.2) of the classical sine–Gordon equation. While varying $\underset{\sim}{a}$'s at N fixed, we are still within a classical N-soliton sector, while quantally we go through mutually orthogonal Hilbert spaces. Notice that states $|\underset{\sim}{a},\underset{\sim}{\delta})$, $|\underset{\sim}{a}',\underset{\sim}{\delta})$ are always orthogonal, despite how close two sets $\underset{\sim}{a} = \{a_1,...,a_N\}$, $\underset{\sim}{a}' = \{a_1',...,a_N'\}$ of soliton parameters are. In this connection see e.g. (3.4) and (3.5) and note that neither function of the form

$$f_{\lambda\lambda'}(x) = \exp(m\lambda x) - \exp(m\lambda' x) \tag{4.10}$$

is square integrable on the real line $\mathbb{R}^1$, when $\lambda \neq \lambda'$. As a consequence, each classical N-soliton sector gives rise to a quantum soliton sector consisting of the infinite family of mutually inequivalent and non-Fock irreducibility domains $\mathrm{IDPS}(|\underset{\sim}{a},\underset{\sim}{\delta}\rangle)$ for the in-field algebra, each one being specified by giving the values of parameters $\underset{\sim}{a}$ and $\underset{\sim}{\delta}$, such that for all underlying $|\underset{\sim}{a},\underset{\sim}{\delta})$, $\lim_{k\to 0}(\underset{\sim}{a},\underset{\sim}{\delta}|\hat{\psi}_N(x,t)|\underset{\sim}{a},\underset{\sim}{\delta}) = \phi_N(x,t)$.

Let us now call a direct integral

$$\int_{\oplus} da\, \mathrm{DPS}(|a\rangle) = \mathscr{H}_1 \tag{4.11}$$

a *quantum one-soliton sector* for the sine–Gordon system. Here $|a| \in (0,\infty)$ and both 1-solitons and 1-antisolitons are included. The Hilbert space is by construction nonseparable, and the time evolution of quantum one-soliton is unitarily implemented in it. Let us consider a function

$$\epsilon^2(a_1,a_2) = \begin{cases} 1 & a_1 \neq a_2, \\ 0 & \text{otherwise}, \end{cases} \tag{4.12}$$

and then introduce a double direct integral

$$h_2^s := \int_{\oplus}\int da_1\, da_2\, \epsilon^2(a_1,a_2)\mathrm{DPS}(|a_1,a_2\rangle), \tag{4.13}$$

and the accompanying complex direct integral

$$h_2^b = \int_{\oplus}\int da_1\, da_2\, \delta(a_1 - a_2^*)\epsilon^2(a_1,a_2)\mathrm{DPS}(|a_1,a_2\rangle),$$
$$|a_1|,|a_2| \in (0,\infty). \tag{4.14}$$

The integral (4.13) gives acount of the asymptotically decomposable classical 2-solutions, while (4.14) gives account of the bound solitions (bions, breathers). The $\epsilon^2(a_1,a_2)$ factor in (4.13) is necessary to exclude the real coinciding parameters. The direct integral

$$h_2^s \oplus h_2^b = \mathscr{H}_2 \tag{4.15}$$

we call a *quantum* 2-*soliton* sector of the sine–Gordon system.

For any $N > 2$, a classical soliton solution can arise for a number $r = \max\{r \leqslant N/2, r = 1,2,3,\cdots\}$ of bion constituents.

Therefore, a general *quantum N-soliton* sector of the sine–Gordon system reads as follows

$$\Big\{\Big[\int_\oplus da_1 \cdots \int_\oplus da_N\Big] \oplus \Big\{\sum_{(i,j)} \int_\oplus da_1 \cdots \Big[\int_\oplus \int da_i\, da_j\, \delta(a_i - a_j^*)\Big] \cdots \int_\oplus da_N\Big\}$$
$$\oplus \Big\{\sum_{(i,j)} \sum_{(k,l)} \int_\oplus da_1 \cdots \Big[\int_\oplus \int da_i\, da_j \delta(a_i - a_j^*)\Big] \cdots \Big[\int_\oplus \int da_k\, da_l \delta(a_k - a_l^*)\Big] \cdots \int_\oplus da_N\Big\}$$
$$\oplus \cdots \oplus \Big\{\sum_{(i,j)} \sum_{(k,l)} \cdots \sum_{(m,n)} \int_\oplus da_1 \cdots \Big[\int_\oplus \int da_i\, da_j \delta(a_i - a_j^*)\Big]$$
$$\cdots \Big[\int_\oplus \int da_k\, da_l\, \delta(a_k - a_l^*)\Big] \cdots \Big[\int_\oplus \int da_m\, da_n \delta(a_m - a_n^*)\Big] \cdots \int_\oplus da_N\Big\} [\epsilon^2(a_1,\cdots,a_N) \mathrm{DPS}(|a_1,\cdots,a_N\rangle)] = \mathscr{H}_N, \tag{4.16}$$

where the maximal number of double direct integrals in the N-fold one equals an integer $\max\{r \leqslant N/2\}$ and each double integral is carried over the complex area $|a_i|,|a_j| \in (0,\infty)$, while all single ones are carried over a real open interval $|a_i| \in (0,\infty)$.

C. Let us emphasize once more that the factor $\epsilon^2(a_1,\ldots,a_N)$ gives account of the specifics for classical solitons "exclusion principle" by virture of which neither parameter in the sequence $\{a_1,\ldots,a_N\}$ can coincide with any other.[24]

By taking a direct sum

$$\bigoplus_{N=1}^{\infty} \mathscr{H}_N = \mathscr{H}_{\mathrm{SG}}, \tag{4.17}$$

we get a particular subspace $\mathscr{H}_{\mathrm{SG}} \subset \mathscr{H}$ of the Bose in-field direct product space $\mathscr{H}$, which includes all possible *quantum soliton sectors* for the sine–Gordon system. $\mathscr{H}_{\mathrm{SG}}$ we call a *quantum soliton Hilbert space* for the sine–Gordon system in $1+1$ dimensions. For other approaches to the sectorial structures associated with nonlinear field equations see Refs.[25–27],

5. RELATION TO THE SPIN $\frac{1}{2}$ *xyz* HEISENBERG MODEL

A. The spin $\frac{1}{2}$ *xyz* Heisenberg chain arises naturally in the so-called[20] spin $\frac{1}{2}$ approximation of the sine–Gordon system [in the case of the lattice quantization of (3.1), under an assumption of the in-field structure of all the field operators]. Strictly speaking the Heisenberg chain Hamiltonian replaces the nearest neighbor coupling (gradient) term in the sine–Gordon chain Hamiltonian. The basic assumption was that all lattice field operators can be expressed in terms of the single-site generators $\{a_s^*,a_s\}_{s=0,\pm 1,\ldots}$ of the CCR algebra in the direct product space $\mathscr{H}$ constructed for a linear chain of quantum pendula. We identify these generators with the in-field ones introduced in the previous sections.

Recall that $\mathscr{H} = \Pi_s^\otimes (h)_s$ and the spin $\frac{1}{2}$ approximation appears by projecting on the lowest two energy levels of each single site (sth) Schröedinger problem in the linear chain

$$\mathfrak{S}_s^+ = P_s a_s^* P_s, \quad \mathfrak{S}_s^- = P_s a_s P_s,$$
$$P_s = :\exp(-a_s^* a_s): + a_s^* :\exp(-a_s^* a_s): a_s, \tag{5.1}$$
$$h \ni |\psi\rangle = \sum_{k=0}^{\infty} f_k |k\rangle \rightarrow P|\psi\rangle = \sum_{k=0,1} f_k |k\rangle .$$

The projection operation can be equivalently described by a simultaneously fulfilled sequence of single-site constraints:

$$\hat{n}_s(\hat{n}_s - 1)|\psi\rangle_s = 0, \quad \forall_s,$$
$$\hat{n}_s = a_s^* a_s, \quad \hat{n}_s|k\rangle_s = k|k\rangle_s. \tag{5.2}$$

Then the lattice sine–Gordon Hamiltonian

$$H = \sum_s \{ [\pi_s^2 + 2m^2(1 - \cos\phi_s)] - (\phi_s - \phi_{s+1})^2/\epsilon^2$$
$$= \sum_s (H_s + V_{s,s+n}), \tag{5.3}$$

if supplemented by the constraints (5.2) plus the periodic boundary conditions,[20] converts into

$$P\sum_s V_{s,s+1} P = P\Big(H - \sum_s H_s\Big)P = H_{xyz} = -\sum_{a,s} J_a \hat{S}_s^a \hat{S}_{s+1}^a, \tag{5.4}$$

with $P = \Pi_s P_s$ and H_{xyz} denoting the spin $\frac{1}{2}$ *xyz* Heisenberg model Hamiltonian. The coupling constants $\{J_a\}_{a=1,2,3}$ rely here on the explicit a_s^*,a_s dependence of the quantum lattice field $\hat{\phi}_s = \phi_s(a^*,a)$ entering the gradient term, see Ref. 20.

All the spin $\frac{1}{2}$ generators $\{\mathbf{S}_s\}_{s=0,\pm 1,\ldots}$ are obviously built out of the fundamental generators $\{a_s^*,a_s\}$,[18,19] and constitute an irreducible representation of the SU(2) algebra.

B. We wish to define the spin operators and the spin Hamiltonian on the domain $\mathscr{H}_{xyz}$ belonging to the carrier space $\mathscr{H}_{\mathrm{SG}}$ of quantum soliton operators. But for this purpose an irreducible spin $\frac{1}{2}$ realization of the H_{xyz} is inappropriate (no unique choice of the irreducibility domain in $\mathscr{H}_{\mathrm{SG}}$).

Let us introduce the following functions of $\{a_s^*,a_s\}$:

$$\underline{\hat{S}}_k^+ = (2s)^{1/2}(1 - a_k^* a_k/2s)^{1/2} a_k$$
$$\underline{\hat{S}}_k^- = (2s)^{1/2} a_k^* (1 - a_k^* a_k/2s)^{1/2} \tag{5.5}$$
$$\underline{\hat{S}}_k^3 = s - a_k^* a_k \quad s = 1|2,1,3|2,\ldots,$$

which satisfy in $\mathscr{H}$ the following operator identities

$$[\underline{\hat{S}}_i^+, \underline{\hat{S}}_j^-]_- = 2\underline{\hat{S}}_i^3 \delta_{ij} \hbar, \quad [\underline{\hat{S}}_i^3, \underline{\hat{S}}_j^\pm]_- = \pm 2\underline{\hat{S}}_i^3 \delta_{ij} \hbar. \tag{5.6}$$

It is the Holstein–Primakoff realization of the SU(2).[18,19] In the single site Hilbert space h_i, let us specify a proper subspace $P_s h_i$ by demanding that $P_s h_i$ (P_s is a projection) is a linear span of all eigenvectors $|n)$ of $a_i^* a_i$ for which $a_i^* a_i |n) = n|n\rangle$, $n \leqslant 2s$. One can easily check that the operator $\hat{\underline{S}}_i$ acts on $P_s h_i$ invariantly. Thus if restricted to $P_s h_i$, $\hat{\mathbf{S}}_i = \underline{P}_s \hat{\underline{S}}_i P_s$ generates an irreducible spin $s = n/2$ representation of the SU(2) group Lie algebra.

Let us now introduce a coherent state domain in h_i, so that a single site coherent state $|\alpha)_i$ is given. Then

$$\lim_{\hbar \to 0} {}_i\langle \alpha | \hat{\underline{S}}_i | \alpha \rangle_i = \mathscr{s}_i(\alpha), \tag{5.7}$$

where

$$\begin{aligned} \mathscr{s}_i^+(\alpha) &= (2s)^{1/2}(1 - |\alpha_i|^2/2s)^{1/2}\alpha_i, \\ \mathscr{s}_i^-(\alpha) &= (2s)^{1/2}(1 - |\alpha_i|^2/2s)^{1/2}\bar{\alpha}_i, \\ \mathscr{s}_i^3(\alpha) &= s - |\alpha_i|^2, \end{aligned} \tag{5.8}$$

and moreover because (5.7) and (5.8) holds true for all $i = 0, \pm 1, \ldots$, the direct product state expectation in terms of $|\alpha\rangle$ leads to:

$$\lim_{\hbar \to 0} \langle \alpha | H_{xyz} | \alpha \rangle = -\sum_{a,s} J_a \mathscr{s}_s^a(\alpha) \mathscr{s}_{s+1}^a(\alpha). \tag{5.9}$$

For the product state $|\alpha\rangle = \Pi_i |\alpha\rangle_i = |\alpha_1 s\rangle$ whose translation parameters satisfy

$$|\alpha_i|^2 \leqslant 2s \quad \forall i = 0, \pm 1, \cdots \tag{5.10}$$

the $\{\mathbf{s}_i(\alpha)\}$ admit the following parametrization

$$\begin{aligned} |\alpha_i|^2 &:= s(1 - \cos\Theta_i), \\ \mathrm{Re}\alpha_i &= \sin\Theta_i \cos\phi_i / [(1 + \cos\Theta_i)/s)]^{1/2}, \\ \mathrm{Im}\alpha_i &= \sin\Theta_i \sin\phi_i / [(1 + \cos\Theta_i)/s]^{1/2}, \end{aligned} \tag{5.11}$$

so that

$$\begin{aligned} \mathscr{s}_i^1 &= s\sin\Theta_i \cos\phi_i, \\ \mathscr{s}_i^2 &= s\sin\Theta_i \sin\phi_i, \\ \mathscr{s}_i^3 &= s\cos\Theta_i, \end{aligned} \tag{5.12}$$

and[28-32]

$$\lim_{s \to \infty} \lim_{\hbar \to 0} \langle \alpha, s | \frac{\hat{\mathbf{S}}_i}{s} | \alpha, s \rangle = \mathbf{s}_i, \tag{5.13}$$

$$\lim_{s \to \infty} \lim_{\hbar \to 0} \langle \alpha, s | \frac{H_{xyz}}{s^2} | \alpha, s) = H_{xyz}^{\mathrm{cl}} = -\sum_{a,s} J_a s_s^a s_{s+1}^a,$$

with

$$s^1 = \sin\Theta\cos\phi, \quad s^2 = \sin\Theta\sin\phi, \quad s^3 = \cos\Theta,$$

provided

$$|\alpha) := |\alpha, s), \quad s = \tfrac{1}{2}, 1, \tfrac{3}{2}, \cdots.$$

C. The above procedures allow us to select a subset $\mathscr{H}_{xyz}$ in $\mathscr{H}_{\mathrm{SG}}$ which via $\hbar \to 0$ limit followed by the $s \to \infty$ limit of state expectations allows us to reproduce a classical level for the spin $\frac{1}{2}$ *xyz* Heisenberg model. The underlying soliton parameters (i.e., the boson transformation ones) are given by (5.11) with $s = \frac{1}{2}$.

In this way not only does the quantum sine–Gordon system recover the spin $\frac{1}{2}$ *xyz* model spectrum and the state space, the classical levels of both systems also exhibit the relation, as a consequence of the quantum one.

Remark 1: Because, after taking the continuum limit, the boson transformation parameters $\alpha(x,t)$ are themselves parametrized by the classical spins $\mathbf{s}(x,t)$, the sine–Gordon solitons can be viewed as functions of these spins also. It seems that classical motions on the phase manifold of the classical *xyz* Heisenberg system have their image in the corresponding submanifold of the sine–Gordon phase space, obviously if the respective α's obey (5.10)–(5.12).

Remark 2: In the above we have mainly discussed a discrete (lattice) version of the *xyz* model. Obviously, under an appropriate limiting operation, a continuous model can always be formally received. However, for the general case (i.e., with no restrictions on the coupling constants of the sine–Gordon system) the continuum limit may not exist on the quantum level (compare Refs. 19 and 20).

Remark 3: Let us comment that quite analgous application of the tree approximation for the Bose constructed Fermi system (however without a subsequent $s \to \infty$ limit as in the above) allows us to obtain an appropriate (commuting ring) c-number level for the Dirac field,[17] with no recourse to the Grassmann algebra methods. An analog of the infinite spin limit was then constructed for the Dirac system[33] at least on the level of relativistic quantum mechanics.

Remark 4: By taking advantage of the "psudoparticle"[34] structure of the Thirring model (the $N = \int dx \psi^+ \psi = 0$ sector)

$$\begin{aligned} H = \int dx \, \{ &-i(\psi_1^+ \partial_x \psi_1 - \psi_2^+ \partial_x \psi_2) \\ &+ m_0(\psi_1^+ \psi_2 + \psi_2^+ \psi_1) + 2g\psi_1^+ \psi_2^+ \psi_2 \psi_1 \}, \end{aligned} \tag{5.14}$$

$$\{\psi^+(x), \psi(y)\} = \delta(x - y)I, \quad \psi|0\rangle = \langle 0|\psi^+ = 0,$$

the eigenstates of H are found to be in the form

$$|\psi\rangle = \sum_{\{a\}} \int dx_1 \ldots dx_i \, \chi^{a_1 \cdots a_i}(x_1, \ldots, x_i) \psi_{a_1}^+(x_1) \cdots \psi_{a_i}^+(x_i)|0\rangle, \tag{5.15}$$

with $\chi^a(\underline{x})$ being the totally antisymmetric wavefunction. Because ψ, ψ^+ are the (free) pseudoparticle fields, we can at once repeat step by step all considerations of the papers[16,17] on quantization of spinor fields. Namely, the Thirring model can be considered as a spin $\frac{1}{2}$ approximation of the subsidiary two-component Bose system (field). The tree approximation, if applied to this mediating Bose level, immediately recovers the classical Thirring model (defined on the c-number commuting ring of spinor functions) by considering the associated-with-fermions Bose transformed operators and then taking the vacuum expectation values in the tree approximation. For this (non Grassmann) classical Thirring model complete integrability was proved and the soliton solutions found in Ref. 35. It neatly disproves the physical (!) utility of Grassmann algebra methods for the case of the Thirring model.

Remark 5: The above-mentioned two-component Bose field, (a four-component one in case of the Dirac system in 1 + 3 dimensions), acquires a physical meaning if considered in the framework of the "field-reservoir" interaction[19,20] where the spin $\frac{1}{2}$ approximation procedure for the sine–Gordon system does automatically involve an additional to the fundamental one $\{a_s^*, a_s\}_{s=0,\pm 1,\ldots}$, field of the reservoir $\{\tilde{a}_s^*, \tilde{a}_s\}_{s=0,\pm 1,\ldots}$, which all together give rise to $\psi_1, \psi_1^+; \psi_2, \psi_2^+$ in the continuum limit. Notice that $\{a_s^*, a_s\}$

describe elementary excitations of the sine–Gordon field, while $\{\tilde{a}_s^*,\tilde{a}_s\}$ describe those induced in the reservoir. Quantized ψ,ψ^+ are given in terms of the Bose constructed Fermi generators $b_\alpha = b_\alpha(a^*,a,\tilde{a}^*,\tilde{a})$, $b_\alpha^* = b_\alpha^*(a^*,a,\tilde{a}^*,\tilde{a})_{\alpha=1,2}$ so that the Lorentz invariance of the Thirring model becomes the Lorentz invariance of the field-reservoir system in the spin $\frac{1}{2}$ approximation.

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