A concept of spin 1/2 approximation in the quantum theory of lattice Bose systems

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We give a model independent description of criteria under which expectation values of observables associated with a finite part of the lattice Bose system can be made to converge to those of the associated Fermi or finite spin system.

1. MOTIVATION

In the series of papers Refs. 1–3 we have investigated the question of a possible "metamorphosis" of bosons into fermions from a mathematical point of view, by looking for fermion field algebras inside bicommutants of given Bose field algebras. In Refs. 4–8 we have considered a few physical cases in which the bosonlike properties of finite spin or Fermi systems arise in local observations, see also Ref. 9. It suggested that we state the problem of (almost) Fermi or spin states of the given Bose systems and further that we look for constraints which must be imposed on Bose systems to make them exhibit the Fermi-like properties.

Our idea is based on looking for the "metamorphosis" prescriptions which allow the expectation values of observables associated with a finite part of the Bose system to converge to those of the associated Fermi or finite spin system. Then locally bosons can be used as approximations of fermions or conversely. In this connection the basic observation of Refs. 7 and 8 was that the spin-$\frac{1}{2}$ approximation of the initially given Bose system should arise as a result of the "metamorphosis": Each Bose degree of freedom must be then replaced by the spin-$\frac{1}{2}$ degree.

From a physical point of view, the spin-$\frac{1}{2}$ approximation becomes of interest for these Bose systems, whose low-lying (the vicinity of the ground state energy) excitations play the dominating role: It is, for example, known that the system ϕ^4 in one, and two space–time dimensions can be made to exhibit the Fermi-like properties, and that its Fermi (i.e., spin-$\frac{1}{2}$) limit can be introduced; it is a respective one or two space–time dimensional Ising model.[9–17] Our aim is to give the model independent criteria under which the spin-$\frac{1}{2}$ approximation works for an arbitrary Bose system on the lattice.

In Sec. 2 we construct a one-parameter family of (in general non-Fock) lattice Bose systems, whose irreducibility sectors (IDPS($|f,\lambda\rangle$)) are labelled by a respective family of generating vectors $\{|f,\lambda\rangle\}_{\lambda\in[0,\infty)}$. Theorem 1 and 2 prove that each IDPS($|f,\lambda\rangle$) can be embedded into a larger space so that a one-parameter family of coupled pairs (Bose system-thermal reservoir) arises for each separately chosen and fixed (for all λ) equilibrium temperature value $\beta = 1/kT$. Then $(f,\lambda\,|a_s^*a_s|\,f,\lambda)$ equals the statistical Bose distribution, which in the limit $\lambda\to\infty$ goes over to the statistical Fermi distribution. It allows us to interpret each $\{|f,\lambda\rangle\}_{\lambda\in[0,\infty)}$ system to give an account of the one-parameter family of couplings of the lattice boson to the gas of its quanta, subject to the ${}_\beta(f,\lambda\,|a_s^*a_s\,|\,f,\lambda)_\beta = 1/\{\exp[\beta\omega_B^s(\lambda,\beta)]-1\}$ constraint for all $\lambda\in[0,\infty)$ and a singly chosen fixed value of $\beta>0$.

With each β we have associated its own β th family $\{\omega_s^B(\lambda,\beta)\}$ of frequencies of the quantum gas. In Sec. 3, we investigate the limiting properties of the expectation values [in between the elements of IDPS($|f,\lambda\rangle$)] of the normal ordered bounded operator functions $:F_J(a^*,a):$ associated with the J th finite part $\{a_s^*,a_s\}_{s\in J}$ of the Bose system, as $\lambda\to\infty$.

Theorems 6 and 7 establish here the Fermi limits for all possible $:F_J(a^*,a):$ including in this number operators of local time translations $\exp(itH_B^J)$.

An essential feature of this quantum picture is that the quantum system needs to be in contact with the temperature nonzero thermostat. Then the temperature dependent domain for quantum operators is introduced, so that if the energy separation between the energy levels is large enough, or if the gap between the lowest two and the others is large enough, we find that the two-level (spin-$\frac{1}{2}$) approximation of the quantum system allows a satisfactory reproduction of its basic properties, like, e.g., the structure of the set of transition probabilities, or expectation values of bounded operators.

2. FIELD-RESERVOIR INTERACTION ON THE LINEAR BOSE LATTICE: ALMOST FERMI, BOSE DISTRIBUTIONS IN QUANTUM THEORY

Let us consider a countable sequence of the identical elementary quantum systems, each one described in terms of a separable Hilbert space $h_s = h$, $s = 0, \pm 1, \pm 2,\ldots$ enumerating single systems (sites of a linear lattice). Let $f_s\in h_s$, we introduce a notion of a product vector $f=\Pi_{s=-\infty}^{+\infty}\otimes f_s \in \Pi_s\otimes h_s$, $\|f\| = \Pi_s\|f_s\| < \infty$ and of the incomplete direct product space IDPS(f) $=\Pi_s^f\otimes h_s$ generated by the product vector f.[18] The reducible representation of the CCR algebra in $\Pi_s\otimes h_s$ is assumed to be generated by a sequence $\{a_s^*, a_s;\ [a_s, a_{s'}^*]_-\subset\delta_{ss'}\mathbf{1}_\beta,\ [a_s,a_{s'}]_- = 0 = [a_s^*, a_{s'}^*]_-\}$ such that a unique (up- to unitarity) state $|0)$, $a_s|0) = 0\ \forall s$ exists in $\Pi_s\otimes h_s$. Its irreducible component recieved by the restriction to a particular IDPS(f), we denote $\{a_s^*,a_s,f\}_{s=0,\pm1,\ldots}$. Let $\{a_s^*,a_s,f\}_{s=0,\pm1,\ldots}$ be associated with some self-interacting lattice Bose system, whose dynamics is governed by the Hamiltonian of the form $H=\Sigma_s\{H_s^0+\Sigma_{t\neq s}W_{st}^{\rm int}\}$, provided the finite volume restriction and suitable boundary conditions are imposed. Both H_s^0 and $W_{st}^{\rm int}$ can be completely expressed in terms of a_s^*, a_s; for an example of the ϕ_2^4:

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$$H = H_B = V \sum_{s=-N}^{+N} \left\{ \frac{p_s^2}{2} + \frac{m^2+2}{2} x_s^2 + \gamma x_s^4 - x_s x_{s+1} \right\},$$
$$[p_s, x_r]_- = -i\delta_{sr}, \quad m,\gamma > 0. \tag{2.1}$$

We make now a simplifying assumption and consider H in its single-site approximation by $H^0 = \Sigma_s H_s^0$. For a while let us restrict considerations to a single site of the lattice. Let A be an observable associated with a single elementary quantum system whose total Hamiltonian is denoted by $\mathscr{H}$. We look (possibly in h) for the state vector $|0(\beta))$ with the property:

$$\begin{aligned} \langle A \rangle_\beta &= (0(\beta)|A|0(\beta)) \\ &= Z^{-1}(\beta) \sum_n (n|A|n)\exp(-\beta E_n), \\ \mathscr{H}|n) &= E_n|n), \quad (n|m) = \delta_{nm}, \\ \sum_n |n)(n| &= \mathbb{1}_B, Z(\beta) = \operatorname{Tr}\exp(-\beta\mathscr{H}), \end{aligned} \tag{2.2}$$

so that $|0(\beta))$ becomes a temperature dependent state if $\beta = 1/kT$. However we have in this connection the following no–go observation.

Lemma 1: There is no state $|0(\beta))$ in h.

Proof: Indeed, if we expand $|0(\beta))$ in the energy basis (a nondegenerate discrete spectrum for $\mathscr{H}$ is assumed), $|0(\beta)) = \Sigma_n |n) f_n(\beta)$, then the following identity immediately follows,[18–21]

$$\bar{f}_n(\beta) f_m(\beta) = Z^{-1}(\beta)\exp(-\beta E_n)\delta_{nm}, \tag{2.3}$$

which cannot be reconciled with the c-number properties of the functions f_n, f_m.

To circumvent this difficulty, we shall use a trick of Refs. 19–21 and introduce a subsidiary tilde system, playing the role of the reservoir:

$$\begin{aligned} &\tilde{\mathscr{H}}|\tilde{n}) = E_n|\tilde{n}), \quad (\tilde{n}|\tilde{m}) = \delta_{nm}, \quad \sum_n |\tilde{n})(\tilde{n}| = \tilde{\mathbb{1}}_B, \\ &|n,\tilde{n}) \in h \otimes \tilde{h}, \quad (\tilde{n},n'|A|m,\tilde{m}') = (n'|A|m)\delta_{nm'}, \\ &f_n(\beta) := |\tilde{n})\exp(-\beta E_n/2)\cdot Z^{-1/2}(\beta), \\ &|0(\beta)) = Z^{-1/2}(\beta)\sum_n |n,\tilde{n})\exp(-\beta E_n/2). \end{aligned} \tag{2.4}$$

For an elementary spin $\frac{1}{2}$ it implies

$$\begin{aligned} &\mathscr{H} = \omega b^* b, \quad [b,b^*]_+ = \mathbb{1}_F, \\ &|0(\beta))_F = \{1/[1+\exp(-\beta\omega)]^{1/2}\}\{|0) \\ &\qquad + \exp(-\beta\omega/2)\cdot b^*\tilde{b}^*|0)\} \\ &\qquad = \exp(-iG)|0), \end{aligned} \tag{2.5}$$

$$|0) = e_0 \otimes \tilde{e}_0 = |0,\tilde{0}), \quad G = -i\Theta(\beta)\{\tilde{b}b - b^*\tilde{b}^*\},$$
$$\cos\Theta(\beta) = [1+\exp(-\beta\omega)]^{-1/2}$$

(h becomes a two-dimensional space), so that:

$${}_F(0(\beta)|b^*b|0(\beta))_F = \exp(-\beta\omega)/[1+\exp(-\beta\omega)] \tag{2.6}$$

follows. For a boson system (elementary Schrödinger one)

$$\begin{aligned} &\mathscr{H} = \omega a^* a, \quad [a,a^*]_- \subset \mathbb{1}_B, \\ &|0(\beta))_B = [1-\exp(-\beta\omega)]^{1/2} \\ &\qquad \times\exp[a^*\tilde{a}^*\exp(-\beta\omega/2)]|0) \\ &\qquad = \exp(-iG)|0), \end{aligned} \tag{2.7}$$

$$G = -i\Theta(\beta)(\tilde{a}a - a^*\tilde{a}^*),$$
$$\cosh\Theta(\beta) = [1-\exp(-\beta\omega)]^{-1/2},$$

we get

$${}_B(0(\beta)|a^*a|0(\beta))_B = \exp(-\beta\omega)/[1-\exp(-\beta\omega)]. \tag{2.8}$$

Assume now that the experimental limitations impose a lower (say positive) bound $\omega_0 \leqslant \omega_s$ on the observable frequency spectrum and let $\beta \gg 0$ (low temperature limit). Then

$${}_B(0(\beta)|a^*a|0(\beta))_B \simeq \exp(-\beta\omega) \simeq {}_F(0(\beta)|b^*b|0(\beta))_F \tag{2.9}$$

and thus the temperatures increase restores, up to a significant level, a difference between the Bose and Fermi distributions in the above. Our aim in this place is to construct the mechanism, which is capable of compensating the difference between the Fermi and Bose distributions so that both cases become indistinguishable within experimental accuracy limits, at nonzero finite temperatures. Define now a mapping U_λ in h according to:

$$\begin{aligned} &U_\lambda f = f_\lambda \in h, \quad |k) = e_k, \\ &(e_k, e_l) = \delta_{kl}, \quad \sum_k \tilde{e}_k \otimes e_k = \mathbb{1}_B, \\ &U_\lambda e_k = e_k^\lambda = \left(\frac{1}{1+\lambda}\right)^{\Sigma_{j=1}^{k}(j-1)/2} \cdot e_k, \\ &f = \sum_k f_k e_k, \quad k = 0,1,\cdots, \quad \lambda \in [0,\infty), \end{aligned} \tag{2.10}$$

so that $\|e_k^\lambda\| = [1/(1+\lambda)]^{\Sigma_{j=1}^{k}(j-1)}$ for all $k > 1$, $\|e_0^\lambda\| = \|e_0\|$, $\|e_1^\lambda\| = \|e_1\|$. Assume further to have fixed a countable sequence $\{f_s\}_{s=0,\pm 1,\cdots}$ of state vectors in h, and define

$$\begin{aligned} &|f) = \prod_s \otimes (f \otimes \tilde{e})_s, \\ &\prod_s (h \otimes \tilde{h})_s \ni |f,\lambda) = \prod_s \otimes (f^\lambda \otimes \tilde{e}_0)_s, \\ &f_s^\lambda = U_\lambda f_s / \|U_\lambda f_s\|, \\ &\|U_\lambda f_s\|^2 = \sum_k |f_s^k|^2 [1/(1+\lambda)]^{\Sigma_{j=1}^{k}(j-1)}. \end{aligned} \tag{2.11}$$

Then the following holds true:

Theorem 1: Assume the product vector $|f,\lambda)$ in $\Pi_s \otimes (h \otimes \tilde{h})_s$ to be constructed so that: $f_s^0 \in \mathbb{R}, f_s^0 > 0, \forall s$, $|f,\lambda) = \Pi_s \otimes (f^\lambda \otimes \tilde{e}_0)_s$,

$$\sum_s \ln\left[\frac{\|f_s\|}{f_s^0}\left(1 + \sum_{k=1}^{\infty} \frac{k|f_s^k|^2}{\|f_s\|^2}\right)^{1/2}\right] < \infty. \tag{2.12}$$

Then for each fixed value of $\lambda \in [0,\infty)$ there exists in IDPS($|f,\lambda)$) an associated vector $|\Theta,\lambda)$ satisfying

$$(\Theta,\lambda|a_s^* a_s|\Theta,\lambda) = \sinh^2\Theta_s(\lambda) = (f,\lambda|a_s^* a_s|f,\lambda). \tag{2.13}$$

Proof: In the basis $\{e_k\}_{k=0,1,\cdots}$ of h we have

$$h = h_s \ni f_s^\lambda = \sum_{k=0}^{\infty} f_s^k e_s^\lambda (1/\|U_\lambda f_s\|), \tag{2.14}$$

so that for all s the formula

$(f,\lambda|a_s^* a_s|f,\lambda)$

$$= \sum_{u=1}^{\infty} \frac{k\,|f_s^k|^2}{\|U_\lambda f_s\|^2}\left(\frac{1}{1+\lambda}\right)^{\sum_{j=1}^{k}(j-1)} \tag{2.15}$$

can be used to define a sequence $\{\Theta_s = \Theta_s(\lambda)\}_{s=0,\pm 1,\ldots}$ by demanding

$$(f,\lambda\,|a_s^*a_s\,|f,\lambda) := \sinh^2\Theta_s(\lambda). \tag{2.16}$$

Any choice of the set $\{\Theta_s(\lambda)\}_{s=0,\pm 1,\ldots}$ subject to the above identity determines a state $|\Theta,\lambda)$ in $\Pi_s \otimes (h\otimes \tilde{h})_s$ and, because $|f,\lambda)$ is a state in the same space [i.e., $\Pi_s \otimes (h\otimes\tilde{h})_s$] the sufficient condition to allow $|\Theta,\lambda)$ and $|f,\lambda)$ to belong to the same IDPS = IDPS($|f,\lambda)$) is $(f,\lambda\,|\Theta,\lambda)\neq 0$.[18]

This last condition holds true, if for all fixed $\lambda\in[0,\infty)$ there is

$$\prod_s \frac{1}{\cosh\Theta_s(\lambda)}\frac{f_s^0}{\|U_\lambda f_s\|} = \exp\left[-\sum_s \ln\left(\frac{\|U_\lambda f_s\|}{f_s^0}\cosh\Theta_s(\lambda)\right)\right]\neq 0, \tag{2.17}$$

which needs

$$\sum_s \ln\left\{\frac{\|U_\lambda f_s\|}{f_s^0} \times\left[1+\sum_k |f_s^k|^2\frac{k}{\|U_\lambda f_s\|^2}\left(\frac{1}{1+\lambda}\right)^{\sum_{j=1}^{k}(j-1)}\right]^{1/2}\right\}<\infty. \tag{2.18}$$

Notice that each argument under the sign of ln exceeds the value 1 for all λ, and so by taking $\lambda>0$ we have improved a convergence of the series. Hence if the inequality of Theorem 1 holds true, we have guaranteed the fulfillment of the last inequality for all λ.

Theorem 2: Under notations of Theorem 1, let us introduce $\Omega_s^B(\lambda) = \ln[\coth^2\Theta_s(\lambda)]$ and $\Omega_s^F = 2\ln(|f_s^0/f_s^1|)$ provided $f_s^1\neq 0$. Then for all s the quantity

$$n_s^B(\lambda) = (\Theta,\lambda\,|a_s^*a_s\,|\Theta,\lambda) \tag{2.19}$$

converges to

$$n_s^F = 1/(1+\exp\Omega_s^F) \tag{2.20}$$

as $\lambda\to\infty$.

Proof: Because of

$$\Omega_s(\lambda) = \ln\frac{1+\sinh^2\Theta_s(\lambda)}{\sinh^2\Theta_s(\lambda)}$$

we have $\sinh^2\Theta_s(\lambda) = 1/[\exp\Omega_s(\lambda)-1]$. But by virtue of (2.15) the limit

$$\lim_{\lambda\to\infty}\Omega_s(\lambda) = \ln\frac{2|f_s^1|^2+|f_s^0|^2}{|f_s^1|^2} \tag{2.21}$$

immediately follows, so that indeed

$$\lim_{\lambda\to\infty} 1/[\exp\Omega_s(\lambda)-1] = 1/(1+|f_s^0|^2/|f_s^1|^2) = 1/(1+\exp\Omega_s^F), \tag{2.22}$$

which proves the theorem.

Now, notice that $\Omega_s^B(\lambda)$, Ω_s^F arise as dimensionless quantities, thus providing us with a continuous spectrum of dimensional frequencies for each fixed value of $\beta\omega_s(\lambda,\beta) = \Omega_s(\lambda)\in\mathbb{R}^+$. Here, we may demand $\beta = 1/kT$ to be a common fixed factor, characterizing a thermal equilibrium of any (possibly finite) fraction J of lattice sites. Then by fixing β we have associated with each sth site, $s\in J$, a frequency curve $\omega_s(\lambda,\beta)$ for which an immediate identity,

$$\omega_s(\lambda,\beta) = \ln\left(\frac{1+n_s^B(\lambda)}{n_s^B(\lambda)}\right)^{1/\beta}, \tag{2.23}$$

follows from the definition of Ω_s^B. In the dimensional description, the "enforcing" of the spin $\frac{1}{2}$ approximation of the given Bose system, can be understood as the transition through a λ family of frequencies $\{\omega_s(\lambda,\beta)\}_{s\in J}$ at a fixed thermal equilibrium. Since for $\lambda\gg 1$, $n_s^B(\lambda)$ can be made not to differ significantly from n_s^F, the notion of an almost-Fermi, Bose distribution seems to be suitable for $n_s^B(\lambda)$ in that case.

3. SPIN-$\frac{1}{2}$ APPROXIMATION ON THE BOSE LATTICE

We consider a fixed finite subset $J\ni s$ of elementary quantum systems (sites) in our infinite assembly, subject to a particular field-reservoir interaction of Theorems 1 and 2.

We are interested now in investigating effects of the λ constraint of the previous section on the finite part of the system only. Let us extract from $|f,\lambda)$ a finite tensor product part $|f,\lambda)_J = \Pi_{s\in J}\otimes f_s^\lambda$ and define a vector $|\lambda)$, associated with $|f,\lambda)_J$ by putting

$$(\lambda\,|a_s^*a_s\,|\lambda) = \sinh^2\Theta_s(\lambda)\cdot\prod_{s\in J}\|U_\lambda f_s\|^2. \tag{3.1}$$

Notice that such a step is impossible in the infinite J limit. Here

$$|\lambda) = \prod_{s\in J}\otimes\, U_\lambda f_s = \left\{\prod_{s\in J} U_\lambda^s\right\}\prod_{s\in J}\otimes f_s := U_\lambda|0), \tag{3.2}$$

$$|0) = \lim_{\lambda\to 0}|\lambda),$$

and the redundant $\tilde{e}_0\otimes\cdots\otimes\tilde{e}_0$ terms in the tensor product were for simplicity omitted. Analogously, with $|\Theta,\lambda)_J$ in hand, we get

$$(\vartheta\,|a_s^*a_s\,|\vartheta) = \prod_{s\in J}\|U_\lambda f_s\|^2\ {}_J(\Theta,\lambda\,|a_s^*a_s\,|\Theta,\lambda)_J\,. \tag{3.3}$$

Let us add in this place the following:

Lemma 2: Denote P_E, a spectral projection on the subspace h_E of h consisting of states whose energy is bounded by E. For all finite values of E the map U_λ is invertible in h_E. Moreover there exists the map

$$U_{\lambda\lambda'}: f_{\lambda'}\to f_\lambda,\quad U_{\lambda\lambda'} = U_{\lambda\lambda'}^{-1}\neq U_{\lambda'\lambda}^*\,. \tag{3.4}$$

Proof: By restricting to h_E we guarantee that for $f\in h_E$ the vectors $U_\lambda^{-1}f$ will also belong to h_E. Here

$$U_\lambda^{-1}e_k = e_\lambda^k = (1+\lambda)^{\sum_{j=1}^{k}(j-1)/2}\cdot e_k\,. \tag{3.5}$$

Obviously in h_E, $f_\lambda = U_\lambda f = U_\lambda U_{\lambda'}^{-1}U_{\lambda'}f = U_{\lambda\lambda'}f_{\lambda'}$ and $U_{\lambda\lambda'}^{-1}f_\lambda = U_{\lambda'}U_\lambda^{-1}U_\lambda f = f_{\lambda'}$. Consequently $(\lambda\,|a_s^*a_s|\lambda) = (0|U_\lambda^*a_s^*a_sU_\lambda|0)$, where $|0) = \lim_{\lambda\to 0}|\lambda)$.

The existence of U_λ^{-1} allows us to generate λ "motions" by the use of operations $U_{\lambda\lambda'}$. Therefore, we should in where

$$|d) = \lim_{\lambda\to\infty} |d,\lambda), \quad :F_J(\sigma^+,\sigma^-): = :F_J(a^*\to\sigma^+, a\to\sigma^-): .$$

Proof: Immediate by arguments of Theorem 5, if we notice that $\lim_{\lambda\to\infty} |d,\lambda) = \mathbb{1}_F^J|d) = |d)$, $:F_J(\sigma^+,\sigma^-):$ we call a Fermi limit of $:F_J(a^*,a):$.

Suppose, we have given an IDPS-generating vector $|f,\lambda)$. Let $H_B^J(\lambda)$ be a normal ordered with respect to $\{a_s^*, a_s\}_{s\in J}$ polynomial (or a bounded function) generator of the time translations for the lattice Bose system $\{a_s^*, a_s, |f,\lambda)\}$, $\lim_{\lambda\to\infty} H_B^J(\lambda) = \underline{H}_B^J$, $\lim_{\lambda\to 0} H_B^J(\lambda) = H_B^J$. By denoting $H_B^J(\lambda, a^*, a) = H_B^J(\lambda)$ we indicate a (possible) λ-dependence of the expansion coefficients of the generator H_B^J associated with IDPS$(|f,\lambda))$.

Theorem 7: Suppose $\mathbb{1}_F^J$ to be a projection on some spectral subspace of $\lim_{\lambda\to\infty} H_B^J(\lambda) = \underline{H}_B^J$. Then

$$\lim_{\lambda\to\infty} (d,\lambda) \exp[iH_B^J(\lambda)t]|d,\lambda) = (d|\exp(iH_F^J t)|d) \tag{3.15}$$

for all $|d,\lambda)\in d_\lambda^J$ and $H_F^J = \underline{H}_B^J\,(a^*\to\sigma^+, a\to\sigma^-)$.

Proof: We have

$$\begin{aligned}\lim_{\lambda\to\infty} (d,\lambda|\exp[iH_B^J(\lambda)t]|d,\lambda) &= (d|\mathbb{1}_F^J \exp(i\underline{H}_B^J t)\mathbb{1}_F^J|d) \\ &= \sum_n \frac{(it)^n}{n!}(d|\mathbb{1}_F^J(\underline{H}_B^J)^n\mathbb{1}_F^J|d),\end{aligned} \tag{3.16}$$

where $(d|\mathbb{1}_F^J\underline{H}_B^J\mathbb{1}_F^J|d)$ identifies the only part $H_F^J = \underline{H}_B^J(a^*\to\sigma^+, a\to\sigma^-)$ of

$$\underline{H}_F^J + (\mathbb{1}_B^J - \mathbb{1}_F^J)\underline{H}_B^J\mathbb{1}_F^J + \mathbb{1}_F^J\underline{H}_B^J(\mathbb{1}_B^J - \mathbb{1}_F^J) + (\mathbb{1}_B^J - \mathbb{1}_F^J)\underline{H}_B^J(\mathbb{1}_B^J - \mathbb{1}_F^J),$$

which while acting in d_λ^J will never produce any vector from beyond this domain. We easily find that $(d|\mathbb{1}_B^J\underline{H}_B^J\mathbb{1}_F^J|d) = (d|H_F^J|d)$, and analogously for all n, because

$$\underline{H}_B^J = H_F^J + (\mathbb{1}_B^J - \mathbb{1}_F^J)\underline{H}_B^J(\mathbb{1}_B^J - \mathbb{1}_F^J). \tag{3.17}$$

COMMENTS

1. For a particular example of ϕ_1^4 the λ dependence can be understood as the combined mass-coupling constant parametrization of the model. Just in the large mass -strong coupling regime, the Schwinger functions of ϕ_1^4 were shown to tend to suitably normalized Schwinger functions of the one-dimensional Ising model.[10-17] In this case the Ising limit of ϕ_1^4 arises via letting all higher energy levels of the (involved) anharmonic oscillator to become arbitrarily large if compared with the lowest two: The gap between these last can be kept insensitive to the limiting procedure. Obviously an assumption that ϕ_1^4 lives in the thermal bath is crucial in this point.

For ϕ_2^4 the λ dependence must give account of the (relative) weakening of the inbetween -sites couplings if compared with the single site contribution to the total energy. For weakly coupled chains the spin approximation (and spin $\frac{1}{2}$ in this number) arguments become reliable at nonzero finite temperatures; see, e.g., Ref. 15. For higher dimensions the "critical point" ($\lambda = \infty$ for ϕ_1^4) presumably corresponds to a true transition point for the system.

2. Quite similar (to ϕ_1^4) arguments were applied in Ref. 8 to get a deformation of the quantum pendulum into an elementary spin $\frac{1}{2}$. The basic goal was there a construction of the quantum analog of the classically arising angular momentum of plane pendulum (its average is nonzero for rotating motions). It concerns the still not finally resolved question of the relation classical-quantum for spin and Fermi systems; see, e.g., Ref. 1–8, but especially Ref. 22.

3. Though going a bit beyond the scope of the present paper, let us discuss shortly a fundamental question of the roots of the famous sine-Gordon –Thirring model equivalence[23-27]: spin-$\frac{1}{2}$ particles are here believed to arise in the original Bose theory. In fact, as shown in Ref. 26, the two-dimensional Coulomb gas of the charge $\pm q$ particles, at the inverse temperature β is equivalent to the sine-Gordon system with the Coleman's coupling constant given by $\beta_c = (4\pi\beta)^{1/2}q$. Then for $\beta q^2 \geqslant 2$ the dipole phase of the Coulomb system arises, while in the interval $0 \leqslant \beta q^2 < 2$ the system lives in the charged plasma phase. The value $\beta q^2 = 2$ corresponds to $\beta_c^2 = 8\pi$ which is the instability point in the Coleman's study of the sine-Gordon vacuum.[25-27]

Just below the critical threshold $\beta q^2 = 2$ the plasma phase of the Coulomb system (and of the equivalent to it, sine-Gordon system) can be rigorously described in terms of the Thirring model, while above $\beta q^2 = 2$ the dipole gas occurs (the ultraviolet divergences cause some people to refer to the "nonexistence" of the sine-Gordon model).

Notice that formally, we can investigate this phase transition at a fixed temperature $1/\beta$, by varying the charge value q only. Then a one-parameter family of the sine-Gordon systems arises and an analogy with consideration of the previous sections appears to be striking: Recall that Coleman conjectured in Ref. 25 that spectra of the sine-Gordon and Thirring model Hamiltonians do coincide in the interval indicated above. For further investigations on this subject and the spin $\frac{1}{2}$ xyz linear chain approximation of the sine-Gordon system, see, e.g., Ref. 28.

4. In connection with Lemma 2, it seems reasonable to give an explanation for when the finite energy bound E is of no matter (at least approximately). Namely let E be a fixed single site energy bound, $h_E = P_E h$. With $H = \Sigma_n E_n e_n \otimes \bar{e}_n$ (nondegenerate discrete spectrum) in mind, we denote $2s = n_E = \{\max n, (n|H|n) \leqslant E, |n) = e_n\}$. The triple $\mathbf{S}_E = P_E \mathbf{S} P_E$ with $\mathbf{S}$ given by the Holstein–Primakoff formula,[5]

$$S^+ = \sqrt{2s}\,a^*(1 - a^*a/2s)^{1/2},$$

$$S^- = \sqrt{2s}\,(1 - a^*a/2s)^{1/2}a, \quad S^3 = s - a^*a$$

defines an irreducible in h_E representation of the SU(2) Lie algebra corresponding to spin $s = n_E/2$. Here the following relations are immediate:

$$[S^+, S^-]_- = 2S^3, \quad [S^3, S^\pm]_- = \pm S^\pm,$$

and h naturally splits into the two orthogonal subspaces h_E, $(1 - P_E)h$ respectively, which are invariant under the action principle restrict all f_s in the above to belong to h_E.

Theorem 3: Let $\mathbb{1}_F$ be a projection on the linear span of $\{e_0,e_1\}$ in h. U_λ converges to $\mathbb{1}_F$ as $\lambda\to\infty$.

Proof: Observe that $\mathbb{1}_F = :\exp(-a^*a): + a^*:\exp(-a^*a):a$ so that $\mathbb{1}_F$ projects on $\lim_{\lambda\to\infty} f_s^\lambda = \Sigma_{k=0,1} f_s^k e_k = f_s^\infty$. Consequently

$$\lim_{\lambda\to\infty} |\lambda) = \mathbb{1}_F|0) = \mathbb{1}_F|\lambda)\ \ \forall\ \lambda\in[0,\infty). \tag{3.6}$$

Corollary: $U_\lambda^J = \Pi_{s\in J}\otimes U_\lambda^s$ converges to $\Pi_{s\in J}\otimes \mathbb{1}_F^s = \mathbb{1}_F^J$ in $\Pi_{s\in J}\otimes h_s$ as $\lambda\to\infty$.

Proof: Through an immediate calculation one can check that

$$\Big(\prod_{s\in J}\otimes\ \mathbb{1}_F^s\Big)\prod_{s\in J}\otimes f_s = \prod_{s\in J}\otimes f_s^\infty = \mathbb{1}_F^J\prod_{s\in J}\otimes f_s^\lambda$$
$$= \mathbb{1}_F^J\prod_{s\in J}\otimes f_s\,. \tag{3.7}$$

Now take into account the $(n+m)$-fold totally antisymmetric tensor $f_{r_1\cdots r_n s_1\cdots s_m}$ and denote $\epsilon_{r_1\cdots r_n}$, the n-fold Levi–Civita tensor. We define

$$f^c_{r_1\cdots r_n s_1\cdots s_m} = f_{r_1\cdots r_n s_1\cdots s_m}\cdot\epsilon_{r_1\cdots r_n}\cdot\epsilon_{s_1\cdots s_m}\,. \tag{3.8}$$

Assume now to have defined in $\Pi_{s\in J}\otimes h_s$ the bounded functions of Bose generators $\{a_s^*,a_s\}_{s\in J}$ which are of the form

$$:F^c(a^*,a): = \sum_{nm}(f^c_{nm},a^{*n}a^m)$$
$$= \sum_{nm}\sum_{\{r\}\in J}\sum_{\{s\}\in J} f^c_{r_1\cdots r_n s_1\cdots s_m}\times a^*_{r_1}\cdots a^*_{r_n}a_{s_1}\cdots a_{s_m}. \tag{3.9}$$

Theorem 4: Let us denote by $\mathscr{U}_B^J$ a representation of the CCR algebra generated in $\Pi_{s\in J}\otimes h_s$ by $\{a_s^*,a_s\}_{s\in J}$. In the bicommutant $(\mathscr{U}_B^J)''$ of $\mathscr{U}_B^J$ there arises a representation of the CAR algebra associated with the cyclic vector of the former, so that the following identity holds true:

$$\lim_{\lambda\to\infty}(\lambda|:F^c(a^*,a):|\lambda)$$
$$=(0|:F(b^*,b):|0)$$
$$=(\lambda|:F(b^*,b):|\lambda)\quad\forall\lambda\in[0,\infty). \tag{3.10}$$

Here $:F(b^*,b): = \Sigma_{nm}(f_{nm},b^{*n}b^m)$ and on $\mathbb{1}_F^J\Pi_{s\in J}\otimes h_s$ it holds that $[b(f),b(g)^*]_+ = (f,\bar g)\mathbb{1}_F^J$.

Proof: Notice that in $\Pi_{s\in J}\otimes h_s$ we have satisfied

$$\mathbb{1}_F^J = \prod_{s\in J}\otimes\ \mathbb{1}_F^s$$
$$=\sum_n\frac{1}{n!}\sum_{\{r\}\in J} a^*_{r_1}\cdots a^*_{r_n}:$$
$$\times\exp\Big(-\sum_{s\in J}a_s^*a_s\Big):a_{r_1}\cdots a_{r_n}\cdot\epsilon_{r_1\cdots r_n}\,. \tag{3.11}$$

It is the operator unit of the CAR algebra arising in $(\mathscr{U}_B^J)''$ by virtue of Refs. 1–3. By Theorem 3 we have

$$\lim_{\lambda\to\infty}(\lambda|:F^c(a^*,a):|\lambda) = (0|\mathbb{1}_F^J:F^c(a^*,a):\mathbb{1}_F^J|0), \tag{3.12}$$

but by virtue of projection theorems of Ref. 3, we find that $\mathbb{1}_F^J:F^c(a^*,a):\mathbb{1}_F^J$ is just equivalent to $:F(b^*,b):$ on all vectors taken from $\mathbb{1}_F^J\Pi_{s\in J}\otimes h_s$ (it is a Fock space for the representation).

In the above construction a concept of the spin-$\frac12$ approximation explicitly appears due to the following theorem.

Theorem 5: Suppose, we have constructed in IDPS($|f,\lambda)$) a domain $\mathscr{D}_\lambda^J$ received by applying all possible polynomials $W_J(a^*,a)$ of variables $\{a_s^*,a_s\}_{s\in J}$ on the generating vector $|f,\lambda)$. Furthermore let : $F_J(a^*,a)$: be a bounded function of $\{a_s^*,a_s\}_{s\in J}$. For each vector $|\mathscr{D},\lambda)\in\mathscr{D}_\lambda^J$ there holds

$$\lim_{\lambda\to\infty}(\mathscr{D},\lambda|:F_J(a^*,a):|\mathscr{D},\lambda)$$
$$=(f|\mathbb{1}_F^J W_J(a^*,a)^*:F_J(a^*,a):W_J(a^*,a)\mathbb{1}_F^J|f)$$
$$=(f|\mathbb{1}_F^J:G_J(a^*,a):\mathbb{1}_F^J|f)$$
$$=(f|:G_J(\sigma^+,\sigma^-):|f), \tag{3.13}$$

where

$$|f) = \lim_{\lambda\to\infty}|f,\lambda),\quad [\sigma_s^-,\sigma_t^+]_- = 0,\quad s\neq t,$$
$$[\sigma_s^-,\sigma_s^+]_+ = \mathbb{1}_F^s\quad\forall s\in J,$$

and

$$G_J(a^*,a): = W_J^*F_JW_J$$
$$=\sum_{nm}\sum_{\{s\}}\sum_{\{t\}}G_{s_1\cdots s_n r_1\cdots r_m}a^*_{s_1}\cdots a^*_{s_n}a_{r_1}\cdots a_{r_m}$$

arises as a normal ordered expression for $W_J^*F_JW_J$ while $:G_J(\sigma^+,\sigma^-): = :G_J(a^*\to\sigma^+,a\to\sigma^-):$.

Proof: By Theorem 3 and Corollary, Theorem 5 immediately follows if neither of indices in the sequence $\{s\}$ or $\{t\}$ in the sum appears more than once. Then it is enough to notice that $\mathbb{1}_F^s a_s^*\mathbb{1}_F^s = \sigma_s^+$, $\mathbb{1}_F^s a_s\mathbb{1}_F^s = \sigma_s^-$ and $\mathbb{1}_F^s(a_s^*a_s)^k\mathbb{1}_F^s = (\sigma_s^+\sigma_s^-)^k$, $k\geqslant 1$.

If any index is repeated more than once (or at least once) in either $\{s\}$ or $\{t\}$, then the $\mathbb{1}_F^J(\cdots)\mathbb{1}_F^J$ "sandwiching" makes the corresponding term vanish. Consequently, due to $(\sigma_s^+)^2 = 0 = (\sigma_s^-)^2$ we can replace each $(a_s^*)^k$, $(a_t)^l$ by $(\sigma_s^+)^k$, $(\sigma_t^-)^l$, respectively.

Obviously Theorem 5 does not in any sense contradict Theorem 4. One knows that with a finite number ($J\ni s$) of spin-$\frac12$'s in hand, an application of the Jordan–Wigner formulas,[5] allows us to rewrite $:G_J(\sigma^+,\sigma^-):$ in terms of pure Fermi variables $:G_J(\sigma^+,\sigma^-): = :G_J^{\prime c}(b^*,b):$, where by Theorem 4, $:G_J^{\prime c}(b^*,b):$ has its corresponding $:G_J'(a^*,a):$.

Assume now a single site energy bound of Lemma 2 to be infinite, so that all finite energy levels at each site are allowed. We apply to $|f,\lambda)$ the set of all polynomials $\{W_J(a_\lambda^*,a_\lambda)\}$ with respect to $\{a_{s\lambda}^* = U_\lambda^J a_s^*(U_\lambda^J)^{-1}, a_{s\lambda} = U_\lambda^J a_s(U_\lambda^J)^{-1}\}_{s\in J}$ and denote $\{W_J(a_\lambda^*,a_\lambda)|f,\lambda)\} = d_\lambda^J \subset$ IDPS($|f,\lambda)$). Notice that in d_λ^J we have $[a_{s\lambda},a_{t\lambda}^*]_- \subset \delta_{st}\mathbb{1}_B$. By making use of d_λ^J we are able to assign to each bounded function $:F_J(a^*,a):$ its "Fermi limit."

Theorem 6: For each bounded normal ordered operator function $:F_J(a^*,a):$ and all vectors $|d,\lambda)\in d_\lambda^J$ it holds that

$$\lim_{\lambda\to\infty}(d,\lambda|:F_J(a^*,a):|d,\lambda) = (d|:F_J(\sigma^+,\sigma^-):|d), \tag{3.14}$$

of S. Consequently the $P_E(\cdot)P_E$ sandwiching selects an irreducible component of the SU(2) based on h_E.

Let us now choose one more energy bound $\Delta \ll E$ and $h_\Delta = P_\Delta h$ with $n_\Delta \ll 2s$. In h_Δ the following holds:

$$\sqrt{2s}\,T_\Delta^+ = P_\Delta S_E^+ P_\Delta \simeq P_\Delta \sqrt{2s}\,a^* P_\Delta,$$
$$\sqrt{2s}\,T_\Delta^- = P_\Delta S_E^- P_\Delta \simeq P_\Delta \sqrt{2s}\,a^* P_\Delta,$$

where the triple $T^\pm, J_z$ with $J_z = S_E^3$ generates the E(2) group Lie algebra in h. It so happens that due to $n_\Delta \ll s$ the factors $(1 - n_\Delta/s)^{1/2} \simeq 1$ can be neglected when applying S on h. Furthermore: $[J_z, T^\pm]_- = \pm T^\pm$, $[T^+, T^-]_- = 1_B$ so that under the additional (to P_E) P_Δ sandwiching, there is no essential difference between SU(2), i.e., h_E and E(2), i.e., h, provided $n_\Delta \ll 2s$.

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