# Quantization of spinor fields

Piotr Garbaczewski

*Institute of Theoretical Physics, University of Wrocław, Cybulskiego 36, 50-205 Wrocław, Poland*
(Received 16 February 1977)

Influenced by Klauder's investigations on the same subject, we study the question of correspondence principle for Dirac fields, looking for its formulation without use of Grassman algebras. We prove that with each Fermi operator (the series with respect to asymptotic free fields): $\Omega(\psi,\bar\psi)$: one can associate the functional $\Omega^C(\psi^C, \bar\psi^C)$ with respect to classical spinor fields. Here the projector $1_F$ and the Hilbert (Fock) space $\mathcal{J}_F = 1_F \mathcal{J}_{F_B}$ are given such that the identity $1_F$: $\Omega^C(\psi^B, \bar\psi^B)$: $1_F \mathcal{J}_{F_F} = :\Omega(\psi, \bar\psi):\mathcal{J}_F$ defines the mediating boson level, where coherent state expectation values of operator expressions are in order: $\langle :\Omega^C(\psi^B, \bar\psi^B): \rangle = \Omega^C(\psi^C, \bar\psi^C)$. For proofs we employ functional differentiation (resp. integration) methods, especially in connection with the use of functional representations of the CCR and CAR algebras.

## 1. THE CORRESPONDENCE PRINCIPLE FOR SCALAR FIELDS

In the present paper we shall not go beyond the framework of the conventional quantum field theory, and all considerations are essentially based on its LSZ formulation.[1] The basic assumption here is that any operator quantity characterizing a given quantum system (scalar field) admits a decomposition into power series expansions with respect to normal ordered products of free asymptotic fields. With a given scalar quantum field

$$\phi(x) \underset{t\pm\infty}{\to} \varphi_{\substack{\text{in}\\ \text{out}}}(x) = \varphi(x),$$

we associate an algebra of all operators,

$$:F(\varphi): = \sum_n (f_n, :\varphi^n:) \tag{1.1}$$

where $(\cdot,\cdot)$ is a bilinear form, and the Schwartz nuclear theorem allows us to consider $(f_n, :\varphi^n:)$ in the form

$$\begin{aligned}(f_n, :\varphi^n:) &= \int dx_1 \cdots \int dx_n f_n(\mathbf{x}_n) :\varphi(x_1)\cdots\varphi(x_n):\\ \mathbf{x}_n &= (x_1, \ldots, x_n), \quad x_k \in M_4.\end{aligned} \tag{1.2}$$

In general there appears the highly nontrivial task of recovering conditions, necessary to impose on coefficient functions (distributions) $\{f_n\}$, to get proper algebraic properties on a suitably chosen domain. We do not bother with this question in the course of the paper. With the Fock representation of the CCR algebra (asymptotic condition) in mind, $\{a^*, a, \Omega_B\}_K$, $K = \mathcal{L}^2(\mathbb{R}^3)$, we introduce a coherent state domain for our operator algebra according to

$$\begin{aligned}&\mathcal{L}^2(\mathbb{R}^3) \ni \alpha, \quad (\alpha, \bar\alpha) = \int_{\mathbb{R}^3} dk\, \alpha(k)\bar\alpha(k) = \|\alpha\|^2,\\ &|\alpha) = \exp(-\|\alpha\|^2/2)\exp(\alpha, a^*)\Omega_B,\\ &(\alpha|a(k)|\alpha) = \langle a(k)\rangle = \alpha(k), \quad k \in \mathbb{R}^3.\end{aligned} \tag{1.3}$$

If $\alpha, \bar\alpha$ appear as classical (complex) Fourier amplitudes of $\overset{c}{\varphi}(x): \alpha, \bar\alpha \to a, a^* \Rightarrow \overset{c}{\varphi}(x) \to \varphi(x)$, we get

$$\begin{aligned}&(\alpha|\varphi(x)|\alpha) = \langle\varphi(x)\rangle = \overset{c}{\varphi}(x),\\ &(\alpha| :F(\varphi): |\alpha) = F(\overset{c}{\varphi}) = \sum_n (f_n, \overset{c}{\varphi}{}^n).\end{aligned} \tag{1.4}$$

The formula (1.4) establishes a *correspondence between the quantum and classical level of a given (scalar) field and the associated algebra*. All the algebraic manipulations appearing on the quantum level induce corresponding relations on the classical level, and therefore many essential questions as, e.g., estimates (concerning the convergence of operator series, criterions for joint multiplication) are transferred onto the classical level, where powerful analytic methods allowing to solve them are known (compare Ref. 2). In connection with those problems, it is extremely useful to employ so-called *functional representations* of the CCR algebra, arising in the theory of the functional power series.[2] Namely, let us assume that we have given the Hilbert space [Bargmann space $\mathcal{B}(\mathcal{L}^2(\mathbb{R}^3))$] of all functional power series $V(\bar\alpha)$:

$$\begin{aligned}V(\bar\alpha) &= \sum_n (1/\sqrt{n!})(v_n, \bar\alpha^n)\\ &= \sum_n (1/\sqrt{n!}) \int d\mathbf{k}_n\, v_n(\mathbf{k}_n)\,\bar\alpha(k_1)\cdots\bar\alpha(k_n),\end{aligned}$$

$$d\mathbf{k}_n = dk_1\cdots dk_n,$$

$$\begin{aligned}\|V\|^2 &= (\bar V, V) = \bar V\left(\frac{d}{d\bar\alpha}\right) V(\bar\alpha)_{|\bar\alpha=0} = \sum_n (\bar v_n, v_n)\\ &= \sum_n \|v_n\|^2 = \int \bar V(\gamma)\, V(\bar\gamma)\exp[-(\bar\gamma,\gamma)]\, d\,\frac{\gamma}{\sqrt{\pi}}\ .\end{aligned} \tag{1.5}$$

where $d/d\bar\alpha$ symbolizes the Gateaux derivative with respect to $\bar\alpha \in \mathcal{L}^2(\mathbb{R}^3)$, while $d(\gamma/\sqrt{\pi})$ the functional (Gaussian path) integration measure, compare Ref. 2 and J. Rzewuski's monograph.[3]

In $\mathcal{B}(\mathcal{L}^2(\mathbb{R}^3))$ we assume (Ref. 2) to have defined an algebra of double power series:

$$\begin{aligned}A(\bar\alpha,\alpha) &= \sum_{nm} \frac{1}{\sqrt{n!m!}}(a_{nm}, \bar\alpha^n\alpha^m) = \sum_{nm}\frac{1}{\sqrt{n!m!}} \int d\mathbf{k}_n \int d\mathbf{p}_m\\ &\quad\times a_{nm}(\mathbf{k}_n, \mathbf{p}_m)\,\bar\alpha(k_1)\cdots\bar\alpha(k_n)\,\alpha(p_1)\cdots\alpha(p_m),\end{aligned}$$

$$\begin{aligned}(AB)(\bar\alpha,\alpha) &= \sum_{nm}\frac{1}{\sqrt{n!m!}}\left(\sum_k (a_{nk}, b_{km}), \bar\alpha^n\alpha^m\right)\\ &= A\left(\bar\alpha, \frac{d}{d\bar\gamma}\right) B(\bar\gamma,\alpha)_{|\bar\gamma=0}\\ &= \int A(\bar\alpha,\gamma)\,B(\bar\gamma,\alpha)\exp[-(\bar\gamma,\gamma)]\, d\left(\frac{\gamma}{\sqrt{\pi}}\right),\end{aligned}$$

$$\begin{aligned}(AV)(\bar\alpha) &= V'(\bar\alpha) = \sum_n \frac{1}{\sqrt{n!}}\left(\sum_k (a_{nk}, v_k), \bar\alpha^n\right)\\ &= A\left(\bar\alpha, \frac{d}{d\bar\gamma}\right) V(\bar\gamma)_{|\bar\gamma=0} = \int A(\bar\alpha,\gamma)\, V(\bar\gamma)\\ &\quad\times\exp[-(\bar\gamma,\gamma)]\, d\left(\frac{\gamma}{\sqrt{\pi}}\right).\end{aligned} \tag{1.6}$$

 0022-2488/78/1903-0642$1.00  

The underlying Hilbert space and the algebra can be derived from much worse defined objects by applying suitable analytic restrictions (their study in the framework of double functional power series is given in Ref. 2).

In the course of the paper, we do not pretend to give highly correct meaning to the notion of functional (path) integrals (see, e.g., Ref. 4); all the definitions establishing a sufficient axiomatization of the formalism can be found in Rzewuski's monograph.[3]

*Theorem* 1 (*functional representation of the CCR*): Double power series $(\bar{\alpha}, f)\exp(\bar{\alpha}, \alpha) = a(f)^*(\bar{\alpha}, \alpha)$, $(\bar{f}, \alpha)\exp(\bar{\alpha}, \alpha) = a(f)(\bar{\alpha}, \alpha)$, $f \in \mathcal{L}^2(\mathbb{R}^3)$, play in $\mathcal{J}_B = \mathcal{B}(\mathcal{L}^2(\mathbb{R}^3))$ the role of generators $a(f)^*$, $a(f)$ respectively of the Fock representation of the CCR algebra with the vacuum vector $\Omega_B = 1$ (the whole set of complex numbers $\mathbb{C}$ spans in fact the vacuum sector).

*Proof*: Given in Refs. 2, 5; for further convenience we shall only quote

$$[a(f), a(g)^*]_-(\bar{\alpha}, \alpha) = (\bar{f}, g)\exp(\bar{\alpha}, \alpha) = (\bar{f}, g)\,\mathbb{1}_B(\bar{\alpha}, \alpha),$$
$$[a(f), a(g)]_-(\bar{\alpha}, \alpha) = 0, \quad (a(f)\Omega_B)(\bar{\alpha}) = 0. \tag{1.7}$$

As a corollary to Theorem 1, one can easily prove:

*Lemma* 1: For any $F(\overset{c}{\varphi}) = A(\bar{\alpha}, \alpha)$ (after suitable reordering of summations and integrations), the double power series $F(\overset{c}{\varphi})\exp(\bar{\alpha}, \alpha)$ play in $\mathcal{J}_B = \mathcal{B}(\mathcal{L}^2(\mathbb{R}^3))$ the role of the operator $:F(\varphi):$

$$:F(\varphi):(\bar{\alpha}, \alpha) = F(\overset{c}{\varphi})\exp(\bar{\alpha}, \alpha). \tag{1.8}$$

*Proof*: Immediate, by applying (1.6); see also Ref. 2. In consequence, in addition to the correspondence rule (1.4) we can formulate the quantization principle (1.8) allowing to reconstruct immediately the quantum object from a given classical object. Here (see Rzewuski's monograph) the algebraic structure on the quantum level induces a corresponding structure on the classical level:

$$:F_1(\varphi): \; :F_2(\varphi): \Rightarrow (:F_1(\varphi): \; :F_2(\varphi):)(\bar{\alpha}, \alpha)$$
$$= \exp(\bar{\alpha}, \alpha) \circ \{F_1(\overset{c}{\varphi})(*)F_2(\overset{c}{\varphi})\} = \exp(\bar{\alpha}, \alpha) F_{12}(\overset{c}{\varphi})$$
$$= :F_{12}(\varphi):(\bar{\alpha}, \alpha) \Rightarrow :F_{12}(\varphi):, \tag{1.9}$$

where

$$(*) = \exp\left(\frac{\overleftarrow{d}}{d\overset{c}{\varphi}} \Delta \frac{\overrightarrow{d}}{d\overset{c}{\varphi}}\right),$$
$$\frac{\overleftarrow{d}}{d\overset{c}{\varphi}} \Delta \frac{\overrightarrow{d}}{d\overset{c}{\varphi}} := \int \frac{\overleftarrow{d}}{d\overset{c}{\varphi}(x)} \Delta(x-y) \frac{\overrightarrow{d}}{d\overset{c}{\varphi}(y)}\, dx\, dy. \tag{1.10}$$

Arrows indicate the direction in which operators act, and $\Delta(x-y)$ is the Pauli–Jordan distribution.

The identity (1.9) recovers what is the relation between the quantum and (implied) classical multiplication rules. The situation appearing can be summarized in the following:

*Correspondence principle*: (*i*) *Correspondence rule*: $\{:F(\varphi):\} \to \{F(\overset{c}{\varphi})\}$:

$$(\alpha \mid :F(\varphi): \mid \alpha) = F(\varphi),$$
$$(\alpha \mid :F_1(\varphi): \; :F_2(\varphi): \mid \alpha) = F_1(\overset{c}{\varphi}) \exp\left(\frac{\overleftarrow{d}}{d\overset{c}{\varphi}} \Delta \frac{\overrightarrow{d}}{d\overset{c}{\varphi}}\right) F_2(\overset{c}{\varphi})$$
$$= F_{12}(\varphi). \tag{1.11}$$

(*ii*) *Quantization rule*: $\{F(\overset{c}{\varphi})\} \to \{:F(\varphi):\}$:

$$F(\overset{c}{\varphi})\exp(\bar{\alpha}, \alpha) = :F(\varphi):(\bar{\alpha}, \alpha) \Rightarrow :F(\varphi):,$$
$$F_{12}(\overset{c}{\varphi})\exp(\bar{\alpha}, \alpha) = (:F_1(\varphi): \; :F_2(\varphi):)(\bar{\alpha}, \alpha) \Rightarrow :F_{12}(\varphi):. \tag{1.12}$$

Commonly, the quantization is believed to be performed, if the Green's functions are given. For this purpose, one needs, however, the knowledge of the generating functional:

$$Z(\eta) = \frac{\int \exp\{i[S + \int dx\, \eta(x)\, \overset{c}{\varphi}(x)]\}\rho d(M\overset{c}{\varphi}/\sqrt{i\pi})}{\int \exp(iS)\, d(M\overset{c}{\varphi}/\sqrt{i\pi})} \tag{1.13}$$

where $S$ is the classical action, $M$ is an arbitrary linear operator, and $\overset{c}{\varphi}$ a quite arbitrary scalar field. The integration measure $d(M\overset{c}{\varphi}/\sqrt{i\pi})$ is defined according to Rzewuski's monograph[2] (Fresnel integral).

The two-point Green's function is then given by

$$G(x, y) = i\, \frac{d}{d\eta(x)}\, \frac{d}{d\eta(y)}\, Z(\eta)_{|\eta=0}. \tag{1.14}$$

In the above, $\eta$ is a suitable classical source function. It is useful to know that, in the free field case, $Z(\eta)$ reduces to

$$Z(\eta) = \exp[-(i/2)\int \eta(x)\, G(x, y)\, \eta(y)\, dx\, dy], \tag{1.15}$$

where $\Delta$ is one of the Green's functions of the $KG$ operator (the arbitrariness exists), usually chosen to be the causal function.

## 2. INTRODUCTION TO THE PROBLEM: FERMIONS

Pragmatists working in the domain of quantum field theory are strongly convinced (see, e.g., Coleman's opinion expressed in Ref. 6) that quite satisfactory (though even not fully correct) classical level for the algebra associated with any Fermi (Dirac, say) field is given in the framework of Grassman algebras, which are built of *c-number-like, but anticommuting objects*. This last property manifestly exhibits the Pauli exclusion principle, influencing the starting Fermion level. Investigations[3,7] have been going in this direction (especially because of the similarity of the formal scheme, allowing us to reproduce all the results in the manner analogous to this of Bose case). There was even founded a complete mathematical theory (Berezin's[3] monograph) of anticommuting numbers in functional-like differentiation and integration procedures.

Let us add that if in Theorem 1 we formally put elements of the anticommuting ring in place of square integrable functions, a Fock representation of the CAR would be obtained (see, e.g., Garbaczewski,[8] where a complete construction is given).

If we follow the Grassmanian way, the generating functional (the notion used here in rather ambiguous meaning) for the Green's functions of the Dirac field reads

$$Z(\eta,\bar{\eta}) = \frac{\int \exp\{i[S+\int [\bar{\eta}(x)\psi(x)+\eta(x)\bar{\psi}(x)]\,dx]\}\cdot d(M\psi/\sqrt{i\pi})}{\int \exp(iS)\,d(M\psi/\sqrt{i\pi})} \tag{2.1}$$

($M$ is an arbitrary linear operator). If electromagnetic interactions are taken into account (with the Faddeev–Popov measure $\delta\mu_A$; see Popov's monograph[2]), then

$$Z(\eta,\bar{\eta},\eta_\mu) = \frac{\int \exp\{i[S+\int(\bar{\eta}\psi+\bar{\psi}\eta+\eta_\mu A^\mu)\,dx]\}\,\delta\mu_A\,d(M/\sqrt{i\pi})}{\int \exp(iS)\,\delta\mu_A\,d(M\psi/\sqrt{i\pi})}, \tag{2.2}$$

where $\eta,\bar{\eta},\eta_\mu$ are "sources" of fields $\psi,\bar{\psi},A_\mu$ respectively. Notice that $\eta,\bar{\eta},\psi,\bar{\psi}$ belong to the Grassman algebra, and $\delta\mu_A$ integrates over classes (orbits with respect to the gauge group).

On the other hand it is perfectly well known that one can always construct the set of ($c$-valued!) functional power series with respect to free Dirac fields and equip this set with a suitable topology and algebraic structure. So, it is rather surprising that no reasonable correspondence with the (prospective) quantum level was found. Really, *the Pauli exclusion principle does not govern the considered classical level, in contrast to the Grassman approach.*

At this point we do not wish to tilt at windmills and advocate this pure $c$-number point of view, against the just-described Grassman tools (especially because these last are widely spread and quite convenient in explicit calculations). We wish, however, to prove that the reasonable *correspondence principle can be established between functional power series of Dirac spinors and operator series with respect to normal products of Dirac fermions.* This correspondence will be established in a correct and unambiguous way with no reference to Grassman methods.

Let us mention the two isolated attempts in this direction which are known to the author; see, e.g., Ref. 8. It was proved that $c$-number images of Fermion functionals do exist. Another possibility[8] was to construct a pure operator theory where functional-like differentiations and integrations would be carried out with respect to operators. In the free field case, the generating functional (2.2) reads

$$Z(\eta,\bar{\eta},\eta_\mu) = \exp[-i\int \bar{\eta}(x)G(x-y)\eta(y)\,dx\,dy - \tfrac{1}{2}\int \eta_\mu(x)D^{\rm tr}_{\mu\nu}(x-y)\eta_\nu(y)\,dx\,dy], \tag{2.3}$$

where

$$G(x-y) = (2\pi)^{-4}\int dp\,\exp[ip(x-y)]\cdot\frac{\hat{p}+m}{p^2-m^2+i0},$$
$$D^{\rm tr}_{\mu\nu}(x-y) = (2\pi)^{-4}\int dk\,\exp[ik(x-y)]\cdot\frac{-k^2\delta_{\mu\nu}+k_\mu k_\nu}{(k^2+i0)^2}. \tag{2.4}$$

The choice of the causal function is justified by the need for uniqueness of the expressions in exponents; the arbitrariness mentioned in connection with scalar case is thus removed. The integral received gives the photon part in the Lorentz gauge. For more details, see Ref. 3.

Let us add that the functional (Grassman level) definition of the two-point Green's function corresponding to the spinor field, by the use of (anticommuting) derivatives with respect to sources, reads

$$G_{\alpha\beta}(x,y) = -i\,\frac{d^2Z(\eta,\bar{\eta},\eta_\mu)}{d\bar{\eta}_\alpha(x)\,d\eta_\beta(y)}\Big|_{\eta=\bar{\eta}=0}. \tag{2.5}$$

## 3. INTERLUDE: BOSON EXPANSION METHOD IN THE QUANTUM THEORY OF FERMIONS

Realizing the program sketched in Sec. 2, we intend to close, by the present paper, the series,[5] developing the method of Boson expansions in application to Fermi systems. The first two papers of Ref. 8, of these series, include in fact an attempt to apply a $c$-number language in the functional formulation of the quantum theory of Fermi systems: So-called *functional representations* of the CAR algebra were invented there. The third paper of Ref. 5, of these series, generalizing results of the previous two onto the algebraic level, began a systematic study of the "*bosonization*" question (the term used by us as the shorthand version of the title of this section) from both mathematical and physical points of view.

*Theorem* 2 (*representation of the CAR*): Let us denote $\sigma_n(\mathbf{k}_n)=\sigma(k_1,\ldots,k_n)$, $k\in\mathbb{R}^3$, the Friedrichs–Klauder sign function[8,9] being a continuous generalization of the $n$-point Levi-Civitta tensor. Let $\{a^*,a,\Omega_B\}_{L^2(\mathbb{R}^3)}$ generate a Fock representation of the CCR algebra over the Hilbert space $L^2(\mathbb{R}^3)$. The underlying Fock space we denote $\mathcal{J}_B$. Then, the triple $\{b^*,b,\Omega_B\}_{L^2(\mathbb{R}^3)}$ with

$$(a^*,a)=\int dk\,a^*(k)\,a(k),$$
$$b(f)=:\exp[-(a^*,a)]\cdot\sum_{nm}(1/\sqrt{n!\,m!})\times\int d\mathbf{k}_n\int d\mathbf{p}_m\,f_{nm}(\mathbf{k}_n,\mathbf{p}_m)\times a^*(k_1)\cdots a^*(k_n)\,a(p_1)\cdots a(p_m):,$$
$$f_{nm}(\mathbf{k}_n,\mathbf{p}_m)=\sqrt{n+1}\,\delta_{m,1+n}\,\sigma_n(\mathbf{k}_n)\bar{f}(p_1)\,\sigma_{1+n}(\mathbf{p}_{1+n})\times\delta(k_1-p_2)\,\delta(k_2-p_3)\cdots\delta(k_n-p_{1+n}) \tag{3.1}$$

generates a Fock representation of the CAR algebra over $L^2(\mathbb{R}^3)$, whose (Fock) representation space $\mathcal{J}_F$ is selected from $\mathcal{J}_B$ due to projection properties of the operator unit $\mathbb{1}_F$:

$$[b(f),b(g)^*]_+=(\bar{f},g)\,\mathbb{1}_F, \tag{3.2}$$
$$\mathbb{1}_F=:\exp[-(a^*,a)]\cdot\sum_n\frac{1}{n!}\int d\mathbf{k}_n\,a^*(k_1)\cdots a^*(k_n)\,\sigma_n^2(\mathbf{k}_n)\times a(k_1)\cdots a(k_n):,$$
$$\mathcal{J}_F=\mathbb{1}_F\mathcal{J}_B,$$

*which implies the coincidence of the vacuum and one-particle sectors for the representations* (CCR and CAR respectively).

*Proof*: Details are given in Ref. 8 and in the first paper of Ref. 5. The only difference lies in that we use an explicit form $E_n(\mathbf{k}_n,\mathbf{p}_n)=\sigma_n(\mathbf{k}_n)\,\delta(k_1-p_1)\cdots\delta(k_n-p_n)$ of the integral kernel of the square root of the abstract projector $E_n^2$ appearing in the original derivation.

*Comments*: (i) The extension of Theorem 2 to an arbitrary number of internal degrees of freedom is nearly immediate, and, by the substitutions

$$\bar{f} \to \bar{f}_s, \quad a, a^* \to a_s, a_s^*,$$
$$\delta(k_l - p_{l+1}) \to \delta_{s_l, t_{l+1}} \cdot \delta(k_l - p_{l+1})$$
$$\int d\mathbf{k}_n \to \sum_{\{s\}} \int d\mathbf{k}_n, \quad \sigma_n(K_n) \to \sigma_n(S_{n1} K_n),$$
$$(a^*, a) \to (a^*, a) = \sum_s \int dk\, a_s^*(k)\, a_s(k), \tag{3.3}$$

we get the pair of Fock representations (CCR and CAR) spanned over $\oplus_1^N \mathcal{L}^2(\mathbb{R}^3) \ni f_s$, and hence with the number $N$ of internal degrees of freedom.

(ii) By virtue of this result and the Haag—LSZ conjecture (compare Sec. 1), we can associate with each quantum field theory (QFT) of the boson system (asymptotic free bosons) the corresponding QFT of the fermion system (asymptotic free fermions). In the relativistic theory when the number of space—time dimensions is equal to two, the above conclusion can be proved in many ways; compare, e.g., Ref. 10. If Minkowski space is taken into account, then because both fermions and bosons have same number of internal degrees of freedom, one of those systems should violate assumptions of the spin-statistics theorem. Hence if the former field is the physical one, the latter can appear as a *subsidiary* (ghost) entity, or conversely.

(iii) On the other hand, if relativistic restrictions can be abandoned, the whole variety of interesting correspondences can be studied. For example, if we consider the low temperature limit of the Heisenberg ferromagnet, it is well known that the free magnon gas (bosons) behaves like the Heisenberg crystal itself. And really we have proved[5] that if the ferromagnet Hamiltonian is $H$, then there exists the boson (magnons) lattice Hamiltonian $H_B$ and a projection $P_0$ in the boson Fock space $\mathcal{J}_B$ such that $H = P_0 H_B P_0$ and $P_0 \mathcal{J}_B = \mathcal{J}_0$ is the Hilbert space of spin states of the Heisenberg ferromagnet.

An analogous effect was observed in the macroscopic model of the atomic nuclei (four-fermion interaction) where atomic spectra in weak excitation limit look like those of quadrupole bosons. Here the underlying boson Hamiltonian $H_B$ includes a two-boson interaction term, where each Boson corresponds to the Cooper pair of (starting) fermions.

Suitable modification of Theorem 2 was also used by us to make a transition from boson to fermion variables in the ultralocal quantization attempt for sine—Gordon 1-solitons. (This was the model study of the quantization procedure, where *by starting from the classical level, through the subsidiary boson level, the final physical Fermion level is achieved*).

*Theorem* 3 (*functional representation of the CAR*): Double power series

$$b(f)(\bar{\alpha}, \alpha) = \sum_{nm} \frac{1}{\sqrt{n!\,m!}} \int d\mathbf{k}_n \int d\mathbf{p}_m \{\sqrt{n+1}\, \delta_{m,1+n} \sigma_n(\mathbf{k}_n)$$
$$\times \bar{f}(p_1)\, \sigma_{1+n}(\mathbf{p}_{1+n})\, \delta(k_1 - p_2)$$
$$\times \delta(k_2 - p_3) \cdots \delta(k_n - p_{1+n})\}$$
$$\times \bar{\alpha}(k_1) \cdots \bar{\alpha}(k_n)\, \alpha(p_1) \cdots \alpha(p_{1+n})$$
$$= \sum_{nm} \frac{1}{\sqrt{n!\,m!}} (f_{nm}, \bar{\alpha}^n \alpha^m)\, b(f)^*(\bar{\alpha}, \alpha) = b(\bar{f})(\alpha, \bar{\alpha}) \tag{3.4}$$

play in $\mathbb{1}_F \mathcal{J}_B = \mathcal{J}_F$ the role of generators $b(f)^*$, $b(f)$ respectively of the Fock representation of the CAR algebra:

$$[b(f), b(g)^*]_+(\bar{\alpha}, \alpha) = (\bar{f}, g)\, \mathbb{1}_F(\bar{\alpha}, \alpha),$$
$$\Omega_B = 1, \quad \mathbb{1}_F(\bar{\alpha}, \alpha) = \sum_n \frac{1}{n!} (\bar{\alpha}^n, \sigma_n^2 \alpha^n). \tag{3.5}$$

*Proof*: The above theorem is a corollary to Theorem 2, and can be proved by making use of Theorem 1 and calculating the functional representation of objects appearing in (3.1), (3.2). It is useful to recall the formula (1.8): $F(\bar{\alpha}, \alpha) \exp(\bar{\alpha}, \alpha) = :F(a^*, a):(\bar{\alpha}, \alpha)$.

The fermion subspace of the Bargmann space is here given by

$$V \in B(\mathcal{L}^2(\mathbb{R}^3)),$$
$$\mathbb{1}_F\left(\bar{\alpha}, \frac{d}{d\bar{\gamma}}\right) V(\bar{\gamma})_{|\bar{\gamma}=0}$$
$$= \int \mathbb{1}_F(\bar{\alpha}, \gamma)\, V(\bar{\gamma})\, d\left(\frac{\gamma}{\sqrt{\pi}}\right)$$
$$= \sum_n \frac{1}{n!}\left(\bar{\alpha}^n, \sigma_n^2 \frac{d^n}{d\bar{\gamma}^n}\right) \sum_n \frac{1}{\sqrt{m!}} (v_n, \bar{\gamma}^n)_{|\bar{\gamma}=0}$$
$$= \sum_n \frac{1}{\sqrt{n!}} (v_n \sigma_n^2, \bar{\alpha}^n), \tag{3.6}$$

and includes vectors received by the Fock construction from symmetric functions $(v_n \sigma_n^2)(\mathbf{k}_n)$, which vanish if any two of variables coincide.

In the Fock construction *there is no difference* between such functions and the antisymmetric functions:

$$\sigma_n(v_n \sigma_n^2)(\mathbf{k}_n) = (v_n \sigma_n)(\mathbf{k}_n),$$
$$(v_n \sigma_n, \sigma_n \bar{\alpha}^n) = (v_n \sigma_n^2, \bar{\alpha}^n). \tag{3.7}$$

Both kinds of them appear in the theory on an equal footing. In this connection compare also Refs. 5, 8, where the study of Hilbert spaces of symmetric and antisymmetric functions is given (together with suitable isometries between them).

## 4. PROJECTION THEOREMS

Let us consider an arbitrary operator:

$$:F(b^*, b): = \sum_{nm} \frac{1}{\sqrt{n!\,m!}} (f_{nm}, b^{*n} b^m), \tag{4.1}$$

whose generating triple $\{b^*, b, \Omega_B\}$ is associated with the starting Bose triple $\{a^*, a, \Omega_B\}$, $f_{nm}$ is a totally antisymmetric $(n+m)$-point function (distribution in general). We have:

*Lemma* 2(*boson expansions*):

$$:F(b^*, b): = :\exp[-(a^*, a)] \circ \sum_{nm} \frac{1}{\sqrt{n!\,m!}} \sum_k \frac{1}{k!}$$
$$\times (\sigma_{n+k} f_{n\underset{\sim}{m}} \sigma_{m+k}, a^{*\,k+n} a^{k+m}):, \tag{4.2}$$

where $\underset{\sim}{m}$ denotes the reversed order of variables:

$$f_{n\underset{\sim}{m}}(\mathbf{k}_n, \mathbf{p}_m) = f_{nm}(k_1, \ldots, k_n, p_m, p_{m-1}, \ldots, p_1).$$

*Proof:* Immediate by applying the functional tools. Here, the fermion analog of (1.8) can be easily derived (see Ref. 8):

$$:F(b^*, b):(\overline{\alpha}, \alpha) = \sum_{nm} \frac{1}{\sqrt{n!\,m!}} \sum_k \frac{1}{k!} \times (\sigma_{n+k} f_{n\underset{\sim}{m}} \sigma_{m+k}, \overline{\alpha}^{k+n} \alpha^{k+m}). \tag{4.3}$$

The only difference if compared with the original[8] formula lies in the use of the explicit form $\sigma_{n+k}$ of the operators $E_{n+k}$ ($\sigma_{n+k}$ is the *alternating* function).

One can also easily check the following identity:

$$:F(b^*, b): = :\exp[-(a^*, a)] \cdot \overset{c}{F}(a^*, a):$$
$$= \sum_n \frac{(-1)^n}{n!} (a^{*n}, :\overset{c}{F}(a^*, a): a^n),$$
$$:F(b^*, b):\Omega_B = \sum_{nm} \frac{1}{\sqrt{n!\,m!}} (\sigma_n f_{n\underset{\sim}{m}} \rho \sigma_m, a^{*n} a^m)\,\Omega_B: \tag{4.4}$$
$$= :\overset{c}{F}(a^*, a):\Omega_B,$$

suggesting the equivalence relation between $:F(b^*, b):$ and $:\overset{c}{F}(a^*, a):$, where $\overset{c}{f}_{nm} = \sigma_n f_{n\underset{\sim}{m}} \sigma_m$ is a symmetric function in groups of variables $(m)$ and $(n)$ respectively, but antisymmetric with respect to permutations from $(m)$ into $(n)$, and converselv.

In connection with (4.4) we have the following:

*Theorem* 4 (*projection theorem*): Let $\mathbb{1}_F$ be given by (3.2), $\mathcal{J}_F = \mathbb{1}_F \mathcal{J}_B$. The following identity

$$\mathbb{1}_F : \overset{c}{F}(a^*, a) : \mathbb{1}_F \mathcal{J}_F = :F(b^*, b): \mathcal{J}_F, \tag{4.5}$$

holds for all operators $:F(b^*, b):$ and $:\overset{c}{F}(a^*, a):$ related by (4.4).

*Proof*: The study of isometries between Hilbert spaces of symmetric and antisymmetric functions, performed in Ref. 8, results in the basic projection formula:

$$\mathbb{1}_F \mathcal{J}_B = \mathcal{J}_F \ni V = \sum_n \frac{1}{\sqrt{n!}} (\overset{s}{v}_n \sigma_n^2, a^{*n})\,\Omega_B$$
$$= \sum_n \frac{1}{\sqrt{n!}} (\overset{s}{v}_n \sigma_n, b^{*n})\,\Omega_B \tag{4.6}$$

so that $\mathbb{1}_F V = V \Rightarrow (\mathbb{1}_F V)(\overline{\alpha}) = V(\overline{\alpha})$. We denote $v_n = \overset{s}{v}_n \sigma_n$, where $\overset{s}{v}_n$ is the $n$-point, symmetric function and thus $v_n$ is antisymmetric. Here, for all $V \in \mathcal{J}_F$, (4.5) reduces to $\mathbb{1}_F : \overset{c}{F}(a^*, a) : V = :F(b^*, b): V$. [Note that (4.5) is an identity on the whole of $\mathcal{J}_B$]. By (1.8)

$$\mathbb{1}_F(\overline{\alpha}, \alpha) = \sum_n \frac{1}{n!} (\overline{\alpha}^n, \sigma_n^2 \alpha^n),$$
$$:\overset{c}{F}(a^*, a):(\overline{\alpha}, \alpha) = \exp(\overline{\alpha}, \alpha) \cdot \overset{c}{F}(\overline{\alpha}, \alpha)$$
$$= \exp(\overline{\alpha}, \alpha) \sum_{nm} \frac{1}{\sqrt{n!\,m!}} (\overset{c}{f}_{nm}, \overline{\alpha}^n \alpha^m) \tag{4.7}$$
$$= \sum_{nm} \frac{1}{\sqrt{n!\,m!}} \sum_k \frac{1}{k!} (\overline{\alpha}^{k+n} \overset{c}{f}_{nm}, \alpha^{k+m})$$

with

$$(\overline{\alpha}^{k+n}, \overset{c}{f}_{nm} \alpha^{k+m}) = \int d\mathbf{q}_k \int d\mathbf{p}_m \int d\mathbf{r}_m\, \overset{c}{f}_{nm}(\mathbf{p}_n, \mathbf{r}_m) \overline{\alpha}(p_1) \cdots \overline{\alpha}(p_n) \times \alpha(r_1) \cdots \alpha(r_m)\, \overline{\alpha}(q_1)\, \alpha(q_1) \cdots \overline{\alpha}(q_k)\, \alpha(q_k) \tag{4.8}$$

Applying (1.6), we get at once

$$\mathbb{1}_F : \overset{c}{F}(a^*, a) : (\overline{\alpha}, \alpha) = \sum_{nm} \frac{1}{\sqrt{n!\,m!}} \sum_k \frac{1}{k!} (\overline{\alpha}^{k+n} \sigma_{k+n}^2 \overset{c}{f}_{nm}, \alpha^{k+m}), \tag{4.9}$$

while (4.3) can be written in complete analogy with (4.7):

$$:F(b^*, b):(\overline{\alpha}, \alpha) = \sum_{nm} \frac{1}{\sqrt{n!\,m!}} \sum_k \frac{1}{k!} (\overline{\alpha}^{k+n} \sigma_{k+n} f_{n\underset{\sim}{m}} \sigma_{k+m}, \alpha^{k+m}). \tag{4.10}$$

In consequence (with the use of the identity $v_n \sigma_n^2 = v_n$) we get

$$(:F(b^*, b): V)(\overline{\alpha}) = \sum_{nm} \frac{1}{\sqrt{n!\,m!}} \sum_k \frac{\sqrt{(k+m)!}}{k!} \times (\overline{\alpha}^{k+n}, \sigma_{k+n} f_{n\underset{\sim}{m}} v_{k+m}) \tag{4.11}$$

and

$$(\mathbb{1}_F : \overset{c}{F}(a^*, a) : V)(\overline{\alpha}) = \sum_{nm} \frac{1}{\sqrt{n!\,m!}} \sum_k \frac{\sqrt{(k+m)!}}{k!} \times (\overline{\alpha}^{k+n}, \sigma_{k+n}^2 \sigma_n f_{n\underset{\sim}{m}} \sigma_m \sigma_{k+m} v_{k+m}). \tag{4.12}$$

To make the comparison between (4.11) and (4.12) there is enough to restrict considerations to respective bilinear forms. The integrations symbolized by the sign $(\cdot, \cdot)$ induce a nonzero counterpart only from these functions which are *totally symmetric both in the group of* $(n+k)$ *and* $(m+k)$ *variables and vanish if any two of variables coincide.*

(i) $(\overline{\alpha}^{k+n}, \sigma_{k+n}^2 \sigma_n f_{n\underset{\sim}{m}} \sigma_m \sigma_{k+m} v_{k+m})$. The coefficient function integrated with $\overline{\alpha}^{k+n}$, due to the $(n) \overset{a}{\sim} (m)$ symmetry [the change of sign if the variable from the group $(n)$ is permuted with any from $(m)$], can be decomposed into a sum of irreducible parts with respect to the symmetry group. Denoting $\mathcal{S}(n, m)$ as the symmetrization operator, we indicate the term of interest in explicit fashion:

$$\mathcal{S}(n, m)[\sigma_n \sigma_m f_{n\underset{\sim}{m}}] = f_{n\underset{\sim}{m}}[\mathcal{A}(n, m)\, \sigma_n \sigma_m],$$
$$\sigma_n f_{n\underset{\sim}{m}} \sigma_m = \mathcal{S}(n, m)[\sigma_n \sigma_m \cdot f_{n\underset{\sim}{m}}] + \text{other decomposition terms.} \tag{4.13}$$

Here we have clearly emphasized $[\mathcal{A}(n, m)]$ the fact that symmetrization of the expression is achieved by the antisymmetrization of the product $\sigma_n \sigma_m$.

In this way we have explicitly disclosed the totally symmetric in $(k+n)$ and $(k+m)$ variables function, whose decomposition terms possessing another symmetry are annihilated by the bilinear form:

$$f_{n\underset{\sim}{m}} v_{k+m} = \{\sigma_{k+n}^2 \mathcal{A}(n, m)[\sigma_n \sigma_m] \cdot \sigma_{k+m}\} \cdot f_{n\underset{\sim}{m}} v_{k+m} + \text{other decomposition terms.} \tag{4.14}$$

(ii) $(\overline{\alpha}^{k+n}, \sigma_{k+n} f_{n\underset{\sim}{m}} v_{k+m})$. Repeating arguments of (i) we must select a totally symmetric in variables $(k+n)$ and $(k+m)$ decomposition term of the function $\sigma_{k+n} f_{n\underset{\sim}{m}} v_{k+m}$. This can be obviously done by making use of (4.14):

$$\sigma_{k+n} f_{n\underset{\sim}{m}} v_{k+m} = \{\sigma_{k+n} \mathcal{A}(n, m)[\sigma_n \sigma_m] \cdot \sigma_{k+m}\} \sigma_{k+n} f_{n\underset{\sim}{m}} v_{k+m} + \text{other decomposition terms.} \tag{4.15}$$

The above symmetry analysis clearly shows that though visually the forms (i) and (ii) are different, they clearly coincide by virtue of performed integrations. Hence (4.11), (4.12) coincide also. The theorem is proved.

To complete the above analysis, let us prove one more theorem, concerning the relations

$$\mathbb{1}_F a(f) \mathbb{1}_F \mathcal{J}_F = b(f) \mathcal{J}_F, \quad \mathbb{1}_F a(f)^* \mathbb{1}_F \mathcal{J}_F = b(f)^* \mathcal{J}_F, \tag{4.16}$$

which is the special example satisfying Theorem 4.

*Theorem* 5 (*projected representation*): Given the Bose triple $\{a^*, a, \Omega_B\}$ and the associated Fermi triple $\{b^*, b, \Omega_B\}$. The CAR hold on $\mathcal{J}_F$ for operators $\mathbb{1}_F a(f) \mathbb{1}_F$ and $\mathbb{1}_F a(f)^* \mathbb{1}_F$. The corresponding representation of the CAR is called the projected representation [notice that formal operator expressions received after normal ordering of $\mathbb{1}_F a(f) \mathbb{1}_F$, $\mathbb{1}_F a(f)^* \mathbb{1}_F$, respectively, are quite different from these for $b(f)$, $b(f)^*$].

*Proof*: We make use of (4.11), (4.12).

(i) $n=0$, $m=1$ implies

$$[b(f)V](\bar{\alpha}) = \sum_k \frac{\sqrt{k+1}}{k!} (\bar{\alpha}^k, \sigma_k \bar{f} v_{k+1}), \quad [\mathbb{1}_F a(f) V](\bar{\alpha}) = \sum_k \frac{\sqrt{k+1}}{k!} (\bar{\alpha}^k, \sigma_k^2 \bar{f} \sigma_{k+1} v_{k+1}). \tag{4.17}$$

Let us notice that $\sigma_k^2 \sigma_{k+1} = \sigma_{k+1}$, so that the second of our bilinear forms reads $(\bar{\alpha}^k \bar{f}, \sigma_{k+1} v_{k+1})$.

In the case of $(\bar{\alpha}^k \bar{f}, \sigma_k v_{k+1})$ we discover the antisymmetry (change of sign) for permutations $(k) \,\&\, (1)$ so that the only part of the symmetry group decomposition of the product $\bar{\alpha}^k \bar{f}$ which does not vanish while integrated with the former function reads $\sigma_{k+1} \sigma_k \bar{\alpha}^k \bar{f}$:

$$\bar{\alpha}^k f = \sigma_{k+1} \sigma_k \bar{\alpha}^k f + \text{other decomposition terms.}$$

But it means that

$$(\bar{\alpha}^k \bar{f}, \sigma_k v_{k+1}) = (\sigma_{k+1} \sigma_k \bar{\alpha}^k \bar{f}, \sigma_k v_{k+1}) = (\bar{\alpha}^k \bar{f}, \sigma_{k+1} \sigma_k^2 v_{k+1}) = (\bar{\alpha}^k \bar{f}, \sigma_{k+1} v_{k+1}),$$

which proves the coincidence of both expressions (4.17).

(ii) $n=1$, $m=0$ implies

$$[b(f)^* V](\bar{\alpha}) = \sum_k \frac{1}{\sqrt{k!}} (\bar{\alpha}^{k+1}, \sigma_{k+1} f v_k), \quad [\mathbb{1}_F a(f)^* V](\bar{\alpha}) = \sum_k \frac{1}{\sqrt{k!}} (\bar{\alpha}^{k+1}, \sigma_{k+1}^2 f \sigma_k v_k). \tag{4.18}$$

Here the function $\sigma_{k+1}^2 f \sigma_k v_k$ is $(k+1)$-symmetric and appears as a suitable symmetry group decomposition term of the function

$$\sigma_{k+1} f v_k = \{\sigma_{k+1} \sigma_k\} \sigma_{k+1} f v_k + \text{other decomposition terms,} \tag{4.19}$$

which is the only term not annihilated by the bilinear form. The coincidence of both expressions (4.18) is thus immediate. Because identities (4.17), (4.18) hold for all vectors $V \in \mathcal{J}_F$ there is obvious that denoting

$$V'(\bar{\alpha}) = [b(g)V](\bar{\alpha}) = (\mathbb{1}_F a(g) V)(\bar{\alpha}),$$

we get at once

$$[b(f)^* V'](\bar{\alpha}) = [b(f)^* b(g) V](\bar{\alpha}) = [\mathbb{1}_F a(f)^* \mathbb{1}_F a(g) V](\bar{\alpha}) \tag{4.20}$$

and, in an analogous way, with $V''(\bar{\alpha}) = [b(f)^* V](\bar{\alpha}) = (\mathbb{1}_F a(f)^* V)(\bar{\alpha})$, we get

$$[b(g) V''](\bar{\alpha}) = [b(g) b(f)^* V](\bar{\alpha}) = [\mathbb{1}_F a(g) \mathbb{1}_F a(f)^* V](\bar{\alpha}), \tag{4.21}$$

which by virtue of

$$[b(f)^*, b(g)]_+ = (\bar{f}, g) \mathbb{1}_F \tag{4.22}$$

trivially implies

$$[b(f)^*, b(g)]_+ \mathcal{J}_F = [\mathbb{1}_F a(f)^* \mathbb{1}_F, \mathbb{1}_F a(g) \mathbb{1}_F]_+ \mathcal{J}_F = (f, \bar{g}) \mathcal{J}_F, \tag{4.23}$$

proving Theorem 5.

## 5. DIRAC FIELD: THE CORRESPONDENCE RULE

To get Fock representation of the CAR, suitable for the description of a free Dirac field, we must start from the triples $\{a^*, a, \Omega_B\}_{\oplus_1^4 L^2(\mathbb{R}^3)}$ and $\{b^*, b, \Omega_B\}_{\oplus_1^4 L^2(\mathbb{R}^3)}$ exhibiting the number four of the internal degrees (two charge and two spin degrees) of freedom in the theory. All previous results hold without any change for these representations (see, e.g., Theorem 2 and comments following it).

In the fourth paper of Ref. 8 we have analyzed the standard construction

$$b^+ = \frac{1}{\sqrt{2}} \begin{bmatrix} b_1^* + i b_3^* \\ b_2^* + i b_4^* \end{bmatrix}, \quad b^- = \frac{1}{\sqrt{2}} \begin{bmatrix} b_1 + i b_3 \\ b_2 + i b_4 \end{bmatrix},$$
$$b^{*+} = \frac{1}{\sqrt{2}} \begin{bmatrix} b_1^* - i b_3^* \\ b_2^* - i b_4^* \end{bmatrix}, \quad b^{*-} = \frac{1}{\sqrt{2}} \begin{bmatrix} b_1 - i b_3 \\ b_2 - i b_4 \end{bmatrix} \tag{5.1}$$

(the analogous formulas for boson operators), allowing us to get the quintets: $\{b^\pm, b^{*\pm}, \Omega_B\}_{\oplus_1^2 L^2(\mathbb{R}^3)}$, $\{a^\pm, a^{*\pm}, \Omega_B\}_{\oplus_1^2 L^2(\mathbb{R}^3)}$ with

$$[b^+(f), b^{*-}(g)]_+ = (\bar{f}, g) \mathbb{1}_F = [b^-(f), b^{*+}(g)]_+; \tag{5.2}$$

the other anticommutators vanish.

On the level of functional representations in the place of $\alpha, \bar{\alpha} \in \oplus_1^4 L^2(\mathbb{R}^3)$, we introduce the new Fourier amplitudes $\alpha, \alpha^*, \beta, \beta^* \oplus_1^2 L^2(\mathbb{R}^3)$, so that

$$
\begin{aligned}
&(\bar{\alpha}, \alpha) = (\alpha, \alpha^*) + (\beta, \beta^*) \quad \text{and} \quad f \in \oplus_1^2 L^2(\mathbb{R}^3),\\
&a^+(f)(\bar{\alpha}, \alpha) = (\alpha, \bar{f}) \exp[(\alpha, \alpha^*) + (\beta, \beta^*)],\\
&a^{*+}(f)(\bar{\alpha}, \alpha) = (\alpha^*, f) \exp[(\alpha, \alpha^*) + (\beta, \beta^*)],\\
&a^-(f)(\bar{\alpha}, \alpha) = (\beta, \bar{f}) \exp[(\alpha, \alpha^*) + (\beta, \beta^*)],\\
&a^{*-}(f)(\bar{\alpha}, \alpha) = (\beta^*, f) \exp[(\alpha, \alpha^*) + (\beta, \beta^*)].
\end{aligned} \tag{5.3}
$$

Functional differentiations with respect to $\alpha, \bar{\alpha}$ can be apparently translated to the language of $\alpha, \alpha^*, \beta, \beta^*$ according to : $k=1,2$, $(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}) := \alpha$,

$$\frac{d}{d\alpha_{1k}} = \frac{1}{\sqrt{2}} \left( \frac{d}{d\alpha_k} + \frac{d}{d\beta_k^*} \right), \quad \frac{d}{d\bar{\alpha}_{1k}} = \frac{1}{\sqrt{2}} \left( \frac{d}{d\beta_k} + \frac{d}{d\alpha_k^*} \right),$$

$$\frac{d}{d\alpha_{2k}} = \frac{i}{\sqrt{2}}\left(\frac{d}{d\alpha_k} - \frac{d}{d\beta_k^*}\right), \quad \frac{d}{d\bar{\alpha}_{2k}} = \frac{i}{\sqrt{2}}\left(\frac{d}{d\beta_k} - \frac{d}{d\alpha^*}\right). \quad (5.4)$$

Again, in close analogy to (1.8), any normal ordered in the $a^+, a^{*+}, a^-, a^{*-}$ operator expression,

$$:F(a^+, a^{*+}, a^-, a^{*-}): = \sum_{nmkl} \frac{1}{\sqrt{n!m!k!l!}} \times (f_{nmkl}, a^{+n}a^{*+m}a^{-k}a^{*-l}), \quad (5.5)$$

admits a straightforward functional representation:

$$:F(a^+, a^{*+}, a^-, a^{*-}):(\bar{\alpha}, \alpha) = F(\alpha, \alpha^*, \beta, \beta^*)\exp(\alpha, \alpha), \quad (5.6)$$

where classical Fourier amplitudes $\alpha, \alpha^*, \beta, \beta^*$ appear in the place of boson operators.

Let us extend the *Haag—LSZ expansion conjecture* to the case of the Dirac field algebra ($\psi, \bar{\psi}$ are asymptotic free Dirac fields):

$$:\Omega(\psi, \bar{\psi}): = \sum_{nm} \frac{1}{n!m!}(\omega_{nm}, :\psi^n\bar{\psi}^m:)$$
$$= \sum_{nm} \frac{1}{n!m!} \sum_{\sigma\tau} \int d\mathbf{x}_n \, \omega_{nm}^{\sigma\tau}(\mathbf{x}_n, \mathbf{y}_m): \psi_{\sigma_1}(x_1)$$
$$\times \cdots \psi_{\sigma_n}(x_n)\bar{\psi}_{\tau_n}(y_1)\cdots\bar{\psi}_{\tau_m}(y_m):. \quad (5.7)$$

$\sigma, \tau$ are bispinor indices and the overbar denotes Dirac conjugation of bispinors.

It was proved in Ref. 8 that by the use of functional representations of the CCR and CAR the operator $:\Omega(\psi, \bar{\psi}):$ admits a straightforward $c$-number image:

$$:\Omega(\psi, \bar{\psi}):(\bar{\alpha}, \alpha) = \sum_{nm} \frac{1}{\sqrt{n!m!}}(s_{nm}, \sigma_n\bar{\alpha}^n\sigma_m\alpha^m) = \overset{c}{S}(\bar{\alpha}, \alpha), \quad (5.8)$$

with a suitable (rather involved function of $\omega_{kl}$) coefficient function $s_{nm}^{\mu\nu}(\mathbf{k}_n, \mathbf{p}_m)$, $\mu, \nu = 1, 2, 3, 4$, denoting vector indices in $\oplus_1^4 \mathcal{L}^2(\mathbb{R}^3)$.

Unfortunately, this $c$-number image of $:\Omega(\psi, \bar{\psi}):$ cannot be related so simply as in scalar case, with functional power series of *classical fermion fields*.

This seems to be a disadvantage of (5.8) if we compare it to a canonical classical-like image being based on the use of Grassman algebras (see also the fourth paper of Ref. 8). In this last case, one can satisfactorily reproduce operator identities on the functional-like level (though not in the language of ordinary $c$-number functionals). We wish now to remove this difficulty, and to find the functional power series of classical spinor (Dirac) fields, being in the correspondence relation with the starting operator series $:\Omega(\psi, \bar{\psi}):$.

*Theorem* 6 (*the correspondence rule*): For each operator series $:\Omega(\psi, \bar{\psi})$ one can find the functional power series $\overset{c}{\Omega}(\overset{c}{\psi}, \overset{c}{\bar{\psi}})$ with respect to classical free Dirac fields $\overset{c}{\psi}, \overset{c}{\bar{\psi}}$, such that:

$$\text{(i)} :\overset{c}{\Omega}(\overset{B}{\psi}, \overset{B}{\bar{\psi}}):(\bar{\alpha}, \alpha) = \overset{c}{\Omega}(\overset{c}{\psi}, \overset{c}{\bar{\psi}})\exp(\bar{\alpha}, \alpha), \quad (5.9)$$

where $\overset{B}{\psi}, \overset{B}{\bar{\psi}}$ are the subsidiary Dirac fields obeying (the thus improper) bose statistics, and

$$\text{(ii)} :\Omega(\psi, \bar{\psi}): \mathcal{J}_F = \mathbb{1}_F : \overset{c}{\Omega}(\overset{B}{\psi}, \overset{B}{\bar{\psi}}): \mathbb{1}_F \mathcal{J}_F. \quad (5.10)$$

The set of all functionals $\overset{c}{\Omega}(\overset{c}{\psi}, \overset{c}{\bar{\psi}})$ may stand for an exact classical image of the former set of operators $:\Omega(\psi, \bar{\psi}):$ realized via the mediation of the subsidiary boson level.

*Proof*: $:\Omega(\psi, \bar{\psi}):$ can be written in the following form, manifestly exhibiting the normal ordering of operators (below, the total antisymmetry of $\omega_{nm}$ in $n+m$ variables is essential):

$$:\Omega(\psi, \bar{\psi}): = \sum_{nm} \frac{1}{n!m!}(\omega_{nm}, :(\psi^+ + \psi^-)^n(\bar{\psi}^+ + \bar{\psi}^-)^m:)$$
$$= \sum_{nm} \frac{1}{n!m!}\left(\omega_{nm}, :\sum_k^n \binom{n}{k} \psi^{+k}(\psi^-)^{n-k} \sum_l^m \times \binom{m}{l} \bar{\psi}^{+l}(\bar{\psi}^-)^{m-l}:\right)$$
$$= \sum_{nmkl} \frac{1}{n!m!k!l!} \sum_{\mu\nu\sigma\rho} \int d\mathbf{x}_n \int d\mathbf{y}_m \int d\mathbf{z}_k \int d\mathbf{u}_l$$
$$\times \omega_{nmkl}^{\mu\nu\rho\sigma}(\mathbf{x}_n, \mathbf{y}_m, \mathbf{z}_k, \mathbf{u}_l)\psi_{\mu_1}^+(x_1)\cdots\psi_{\mu_m}^+(x_m)$$
$$\times \bar{\psi}_{\nu_1}^+(y_1)\cdots\bar{\psi}_{\nu_m}^+(y_m)\psi_{\rho_1}^-(z_1)\cdots\psi_{\rho_k}^-(z_k)$$
$$\times \bar{\psi}_{\sigma_1}^-(u_1)\cdots\bar{\psi}_{\sigma_l}^-(u_l)$$
$$= \sum_{nmkl} \frac{1}{n!m!k!l!}(\omega_{n+k,m+l}, \psi^{+n}(\bar{\psi}^+)^m(\psi^-)^k(\bar{\psi}^-)^l), \quad (5.11)$$

where the operators $\psi^\pm, \bar{\psi}^\pm$ depend linearly (through Fourier transformations) on the Fermi operators $b^\pm, b^{*\pm}$ defined by (5.1):

$$[b_i^+(k), b_j^{*-}(p)]_+ = \delta_{ij}\delta(k-p)\,\mathbb{1}_F = [b_i^-(k), b_j^{*+}(p)]_+. \quad (5.12)$$

The other anticommutators vanish. Indices $i, j$ denote here helicity states $i, j = 1, 2$ in contrast to bispinor indices $\mu, \nu$. In (5.11) we have clearly distinguished two groups of operators: $\psi^{+n}(\bar{\psi}^+)^m$ and $\psi^{-k}(\bar{\psi}^-)^l$, which involve, by (5.1) the $(n+m)$-point product of $b^*$'s and $(k+l)$-point product of $b$'s respectively.

The validity of Theorem 5 is here immediate (compare also Comments to Theorem 2) so that

$$b_{i_1}^+(k_1)\cdots b_{i_n}^+(k_n)b_{j_1}^{*+}(p_1)\cdots b_{j_m}^{*+}(p_m)b_{s_1}^-(q_1)\cdots b_{s_k}^-(q_k)$$
$$\times b_{t_1}^{*-}(r_1)\cdots b_{t_l}^{*-}(r_l)\mathcal{J}_F$$
$$\triangleq \sigma_{n+m}(\mathbf{k}_n, \mathbf{p}_m)\,\sigma_{\widetilde{k+l}}(\mathbf{q}_k, \mathbf{r}_l)\,\mathbb{1}_F a_{i_1}^*(k_1)\cdots a_{i_n}^*(k_n)$$
$$\times a_{j_1}^{*+}(p_1)\cdots a_{j_m}^{*+}(p_m)a_{s_1}^-(q_1)\cdots a_{s_k}^-(q_k)$$
$$\times a_{t_1}^{*-}(r_1)\cdots a_{t_l}^{*-}(r_l)\mathbb{1}_F\mathcal{J}_F, \quad (5.13)$$

where $\triangleq$ means that the identity holds true only if integrated from both sides over all variables, with the suitable (antisymmetric) $(n+m+k+l)$-point function, $\sigma_{\widetilde{k+l}}(\mathbf{q}_k, \mathbf{r}_l) = \sigma_{k+l}(r_l, \ldots, r_1, q_k, \ldots, q_1)$, i.e., the tilde reverses the order of variables.

The operators $a_j^\pm(k)$, $a_j^{*\pm}(k)$ stand here for operators of the *ideal, fictitious, subsidiary bosons*, constituting the mediating level in the transition from (5.9) to (5.10).

Here obviously the fermion Fock space $\mathcal{J}_F$ appears as a subspace $\mathbb{1}_F \mathcal{J}_B$ of the boson Fock space $\mathcal{J}_B$. They are representation spaces for triples $\{b^*, b, \Omega_B\}_{\oplus_1^4 \mathcal{L}^2(\mathbb{R}^3)}$, $\{a^*, a, \Omega_B\}_{\oplus_1^4 \mathcal{L}^2(\mathbb{R}^3)}$ respectively.

Let us now restrict consideration to the two-point product $\psi_\mu^+(x)\psi_\nu^+(y)$, where we immediately get

$$\psi_\mu^+(x)\psi_\nu^+(y)\,\mathcal{J}_F$$

$$\stackrel{a}{=} (1/2\pi)^3 \int dk(\sqrt{2\omega_k})^{-1} \int dp(\sqrt{2\omega_p})^{-1} \sum_{ij} v_\mu^{+i}(k) v_\nu^{+j}(p)$$

$$\times \exp[i(kx+py)] \cdot \sigma_2(k,p)\,\mathbb{1}_F\, a_i^+(k)\, a_j^+(p)\,\mathbb{1}_F \mathcal{J}_F. \qquad (5.14)$$

Here again $\stackrel{a}{=}$ means the validity of (5.14) only after smearing with an antisymmetric two-point test function. Here, by the use of four-dimensional Fourier transformations we can introduce the sign operator $\mathcal{E}_2$, with the integral kernel:

$$\mathcal{E}_2(x'-x, y'-y) = \frac{1}{(2\pi)^4} \int dq \int dr\, \sigma_2(q,r) \exp(-iqx - iry) \times \exp[i(qx'+ry')], \qquad (5.15)$$

where

$$x, y \in M^4, \quad q = (\mathbf{q}, q_0), \quad \sigma_2(q,r)\Big|_{\substack{q_0=\omega_k \\ p_0=\omega_p}} = \sigma_2(\mathbf{q}, \mathbf{r}), \quad \mathbf{q}, \mathbf{r} \in \mathbb{R}^3.$$

Now, (5.14) reads

$$\psi_\mu^+(x')\psi_\nu^+(y')\,\mathcal{J}_F$$

$$\stackrel{a}{=} \frac{1}{(2\pi)^4} \int dx \int dy\, \mathcal{E}_2(x'-x, y'-y) \frac{1}{(2\pi)^3} \int dk(\sqrt{2\omega_k})^{-1}$$

$$\times \int dp(\sqrt{2\omega_p})^{-1} \sum_{ij} v_\mu^{+i}(k) v_\nu^{+j}(p)$$

$$\times \exp[i(kx+py)]\,\mathbb{1}_F\, a_i^+(k)\, a_j^+(p)\,\mathbb{1}_F \mathcal{J}_F$$

$$\stackrel{a}{=} \frac{1}{(2\pi)^4} \int dx \int dy\, \mathcal{E}_2(x'-x, y'-y)$$

$$\times \mathbb{1}_F \overset{B}{\psi}_\mu(x) \overset{B}{\psi}_\nu(y)\,\mathbb{1}_F \mathcal{J}_F$$

$$\stackrel{a}{=} \mathbb{1}_F(\mathcal{E}_2 \overset{B}{\psi}_\mu \overset{B}{\psi}_\nu)(x', y')\,\mathbb{1}_F \mathcal{J}_F, \qquad (5.16)$$

where the superscript $B$ means that $\overset{B}{\psi}{}^\pm, \overset{B}{\overline{\psi}}{}^\pm$ appear as positive and negative frequency parts of fictitious (as violating the spin-statistics theorem) spinor fields in which Fermi operators $b^\pm, b^{*\pm}$ are replaced by boson operators $a^\pm, a^{*\pm}$ of the associated boson representation.

The generalization of (5.16) is obvious, leading thus to the identity:

$$\psi_{\mu_1}^+(x_1)\cdots\psi_{\mu_m}^+(x_m)\,\overline{\psi}_{\nu_1}^+(y_1)\cdots\overline{\psi}_{\nu_m}^+(y_m)$$

$$\times \psi_{\rho_1}^-(z_1)\cdots\psi_{\rho_k}^-(z_k)\,\overline{\psi}_{\sigma_1}^-(u_1)\cdots\psi_{\sigma_l}^-(u_l)\,\mathcal{J}_F$$

$$\stackrel{a}{=} \mathbb{1}_F(\mathcal{E}_{n+m} \overset{B}{\psi}{}_{\mu_1}^+ \cdots \overset{B}{\psi}{}_{\mu_m}^+ \overset{B}{\overline{\psi}}{}_{\nu_1}^+ \cdots \overset{B}{\overline{\psi}}{}_{\nu_m}^+ \overset{B}{\psi}{}_{\rho_1}^- \cdots \overset{B}{\psi}{}_{\rho_k}^- \overset{B}{\overline{\psi}}{}_{\sigma_1}^- \cdots$$

$$\times \overset{B}{\overline{\psi}}{}_{\sigma_l}^- \underset{\sim}{\mathcal{E}_{k+l}})(\mathbf{x}_n, \mathbf{y}_m, \mathbf{z}_k, \mathbf{u}_l)\,\mathbb{1}_F \mathcal{J}_F, \qquad (5.17)$$

where the undertilde means that the order of the $k+l$ variables is reversed, $(z_1, \ldots, z_k, u_1, \ldots, u_l) \rightarrow (u_l, \ldots, u_1, z_k, \ldots, z_1)$. By virtue of (5.17) we get at once the required equivalence formula (5.9):

$$:\Omega(\psi, \overline{\psi}):\mathcal{J}_F = \mathbb{1}_F :\overset{c}{\Omega}(\overset{B}{\psi}, \overset{B}{\overline{\psi}}): \mathbb{1}_F \mathcal{J}_F$$

$$= \sum_{nm} \frac{1}{n!\,m!} (\omega_{nm} \mathcal{E}_n \mathcal{E}_{\underset{\sim}{m}},\ \mathbb{1}_F : \overset{B}{\psi}{}^n \overset{B}{\overline{\psi}}{}^m : \mathbb{1}_F)\,\mathcal{J}_F. \qquad (5.18)$$

Here the notation $\overset{c}{\omega}_{nm} = \omega_{nm}\mathcal{E}_n\mathcal{E}_{\underset{\sim}{m}}$ is used. We have proved that, with each Fermi field algebra, one can associate a projection of the subsidiary (mediating) Bose field algebra, so that on $\mathcal{J}_F$ both algebras coincide. On the (not projected) boson level, we have trivially realized (5.10) as a consequence of (5.3)—(5.6), so that with each operator $:\Omega(\psi, \overline{\psi}):$ we have finally associated the functional

$$\overset{c}{\Omega}(\overset{c}{\psi}, \overset{c}{\overline{\psi}}) = :\overset{c}{\Omega}(\overset{B}{\psi}, \overset{B}{\overline{\psi}}):(\overline{\alpha}, \alpha) \cdot \exp[-(\overline{\alpha}, \alpha)], \qquad (5.19)$$

depending on classical spinor fields $\overset{c}{\psi}, \overset{c}{\overline{\psi}}$ differing from $\psi, \overline{\psi}$ by the replacement of operators $b^\pm, b^{*\pm}$ by classical amplitudes [see (5.3)] $\alpha, \beta, \alpha^*, \beta^*$ respectively. The theorem is proved.

*Comment*: (i) As a consequence of Theorem 6, there is enough to start from the set of functionals $\{\overset{c}{\Omega}(\overset{c}{\psi}, \overset{c}{\overline{\psi}})\}$ to get a functional representation $\{:\overset{c}{\Omega}(\overset{B}{\psi}, \overset{B}{\overline{\psi}}):(\overline{\alpha}, \alpha)\}$ of the set of boson operators $\{:\overset{c}{\Omega}(\overset{B}{\psi}, \overset{B}{\overline{\psi}}):\}$, whose projection $\{\mathbb{1}_F :\overset{c}{\Omega}(\overset{B}{\psi}, \overset{B}{\overline{\psi}}): \mathbb{1}_F\}$ on the Fock space $\mathcal{J}_F$ is equivalent to the pure Fermi set $\{:\Omega(\psi, \overline{\psi}):\}$. This sequence of steps allows to state the question of *quantization* of classical spinor fields.

(ii) Note that operators $:\Omega(\psi, \overline{\psi}):$ and $\mathbb{1}_F :\overset{c}{\Omega}(\overset{B}{\psi}, \overset{B}{\overline{\psi}}): \mathbb{1}_F$ have the same matrix elements if calculated between arbitrary states from $\mathcal{J}_B$:

$$(m\,|:\Omega(\psi, \overline{\psi}):|\,n) = (m\,|\,\mathbb{1}_F :\overset{c}{\Omega}(\overset{B}{\psi}, \overset{B}{\overline{\psi}}): \mathbb{1}_F\,|\,n),$$

where $|m)$, $|n) \in \mathcal{J}_B$.

## 6. THE QUESTION OF ALGEBRAIC STRUCTURE

As was emphasized in the discussion of scalar fields, the operator multiplication on the quantum level, via the correspondence rule, results in the multiplication (*) on the classical level; see, e.g., (1.9).

In the case of Dirac fields, the situation is not so obvious, because classical spinors by *no reason can account for the Pauli exclusion principle*. The appearance of it on the quantum level should involve serious restrictions on the classical level.

(i) Let us recall that the set of operators $\{:\Omega(\psi, \overline{\psi}):\}$ is in the equivalence relation on $\mathcal{J}_F$ with the reduction $\{\mathbb{1}_F :\overset{c}{\Omega}(\overset{B}{\psi}, \overset{B}{\overline{\psi}}): \mathbb{1}_F\}$ of the set $\{:\overset{c}{\Omega}(\overset{B}{\psi}, \overset{B}{\overline{\psi}}):\}$ of operators belonging to the (subsidiary) boson field algebra. By the use of functional representation of the CCR, one can easily reproduce a corresponding (*) operation [see, e.g., (3.7)]:

$$[:\overset{c}{\Omega}_1(\overset{B}{\psi}, \overset{B}{\overline{\psi}}): \,:\overset{c}{\Omega}_2(\overset{B}{\psi}, \overset{B}{\overline{\psi}}):](\overline{\alpha}, \alpha)$$

$$= \{\overset{c}{\Omega}_1(\overset{c}{\psi}, \overset{c}{\psi})(*)\,\overset{c}{\Omega}_2(\psi_c, \overline{\psi}_c)\} \exp(\overline{\alpha}, \alpha), \qquad (6.1)$$

$$(*) = \exp\left(\frac{\overleftarrow{d}}{d\alpha}, \frac{\overrightarrow{d}}{d\overline{\alpha}}\right)$$
$$= \exp\left[\left(\frac{\overleftarrow{d}}{d\alpha^*}, \frac{\overrightarrow{d}}{d\alpha}\right) + \left(\frac{\overleftarrow{d}}{d\beta}, \frac{\overrightarrow{d}}{d\beta^*}\right)\right]$$
$$= \exp\left[-i\left(\frac{\overleftarrow{d}}{d\overset{c}{\psi}} G^+ \frac{\overrightarrow{d}}{d\overset{c}{\overline{\psi}}}\right) - i\left(\frac{\overleftarrow{d}}{d\overset{c}{\psi}} G^- \frac{\overrightarrow{d}}{d\overset{c}{\overline{\psi}}}\right)\right]. \tag{6.2}$$

We have exploited here the fact that

$$\overset{c}{\psi}_\sigma(x) = \overset{c}{\psi}_\sigma(x, \alpha, \beta)$$
$$= \int dk \sum_j \{v^+_{j\sigma}(k,x)\,\alpha_j(x) + v^-_{j\sigma}(k,x)\,\beta_j(k)\} \tag{6.3}$$

and integrals over products of $v$'s allow us to get Green's functions of the Dirac equation, $G^\pm$, respectively:

$$\left(\frac{\overleftarrow{d}}{d\overset{c}{\psi}} G^+ \frac{\overrightarrow{d}}{d\overset{c}{\overline{\psi}}}\right) = \sum_{\sigma\tau} \int dx \int dy \frac{\overleftarrow{d}}{d\overset{c}{\psi}_\sigma(x)} G^+_{\sigma\tau}(x-y) \frac{\overrightarrow{d}}{d\overset{c}{\overline{\psi}}_\tau(y)}; \tag{6.4}$$

arrows indicate the direction in which differential operators act. In formulas above $\overset{c}{\psi}, \overset{c}{\overline{\psi}}$ are classical functions (the commuting ring).

(ii) If we follow the Grassman methods[13] (anticommuting ring of spinors), formulas, nearly identical with (6.1)–(6.4) appear:

$$[:\Omega_1(\psi,\overline{\psi}):\ :\Omega_2(\psi,\overline{\psi}):](\overline{\alpha},\alpha) = \exp(\overline{\alpha},\alpha)\{\Omega_1(\psi,\overline{\psi})(*)\,\Omega_2(\psi,\overline{\psi})\}, \tag{6.5}$$

with (*) given by (6.2). However, here $\overline{\alpha}, \alpha$ belong to the Grassman algebra, so that we deal with the functional-like representation (see, e.g., Ref. 8) of the CAR, formally coinciding (*in form*) with the functional representation of the CCR. On the lhs of (6.5) $\psi, \overline{\psi}$ are Fermi fields, while on the rhs there are functions from the anticommuting ring. Obviously functional expansion coefficients $\omega_{nm}$ in (6.5) are totally antisymmetric, while in (6.1) we have dealt with $\mathcal{E}_n \mathcal{E}_m \omega_{nm}$. Obviously, (6.5) can be rewritten with the use of functional-like (measures on Grassman algebras) integrals. We prefer however the differential way, as significantly simpler and easier to work with (notice an analogy of functional power series with power series of complex variables).

(iii) The relations between expansion coefficients of operators $:\Omega_1(\psi,\overline{\psi}):$ and $:\Omega_2(\psi,\overline{\psi}):$ following from their multiplications are well reproduced by (6.5). One may, however, proceed along less formal, though unfortunately not so elegant here, $c$-number way of previous sections. Here

$$:\Omega_1(\psi,\overline{\psi}):\ :\Omega_2(\psi,\overline{\psi}):\ \mathcal{J}_F$$
$$= :\Omega_{12}(\psi,\overline{\psi}):\mathcal{J}_F = \mathbb{1}_F :\Omega_{12}(\psi,\overline{\psi}):\ \mathbb{1}_F\ \mathcal{J}_F$$
$$= \mathbb{1}_F : \overset{c}{\Omega}_1(\overset{B}{\psi}, \overset{B}{\overline{\psi}}):\ \mathbb{1}_F : \overset{c}{\Omega}_2(\overset{B}{\psi}, \overset{B}{\overline{\psi}}):\ \mathbb{1}_F\ \mathcal{J}_F \tag{6.6}$$

so that, by Theorem 6,

$$\overset{c}{\Omega}_{12}(\overset{c}{\psi}, \overset{c}{\overline{\psi}}) = [: \overset{c}{\Omega}_1(\overset{B}{\psi}, \overset{B}{\overline{\psi}}):\ \mathbb{1}_F : \overset{c}{\Omega}_2(\overset{B}{\psi}, \overset{B}{\overline{\psi}}):](\overline{\alpha},\alpha)\cdot\exp[-(\overline{\alpha},\alpha)] \tag{6.7}$$

would establish the required translation of quantum mutliplication rule (fermions) into the classical language. Because the representations of the CCR were defined with respect to primary Fourier amplitudes $\alpha, \overline{\alpha} \in \oplus_1^4 \mathcal{L}^2(\mathbb{R}^3)$, we indicate the possibility of suitable reordering of summations and integrations, writing

$$\overset{c}{\Omega}(\overset{c}{\psi}, \overset{c}{\overline{\psi}}) = F(\overline{\alpha},\alpha) = \sum_{nm} \frac{1}{\sqrt{n!\,m!}} (f_{nm}, \overline{\alpha}^n \alpha^m). \tag{6.8}$$

Now

$$\overset{c}{\Omega}_{12}(\overset{c}{\psi}, \overset{c}{\overline{\psi}}) = F_{12}(\overline{\alpha},\alpha)$$
$$= [:F_1(a^*,a):\mathbb{1}_F:F_2(a^*,a):](\overline{\alpha},\alpha)\exp[-(\overline{\alpha},\alpha)]$$
$$= F_1(\overline{\alpha},\alpha)(*)\,F_2(\overline{\alpha},\alpha), \tag{6.9}$$

where

$$(*) = \exp\left(\frac{\overleftarrow{d}}{d\alpha}, \frac{\overrightarrow{d}}{d\overline{\gamma}}\right) 1_F(\overline{\gamma},\gamma) \exp\left(\frac{\overleftarrow{d}}{d\gamma}, \frac{\overrightarrow{d}}{d\overline{\alpha}}\right)\Big|_{\substack{\gamma=\alpha\\ \overline{\gamma}=\overline{\alpha}}} \tag{6.10}$$

(after performing all differentiations one puts $\alpha=\gamma, \overline{\alpha}=\overline{\gamma}$) and

$$1_F(\overline{\gamma},\gamma) = \exp[-(\overline{\gamma},\gamma)]\cdot \mathbb{1}_F(\overline{\gamma},\gamma)$$
$$= \sum_n \frac{1}{n!} \sum_k \frac{(-1)^k}{k!} (\overline{\gamma}^{k+n}, \sigma_n^2 \gamma^{k+n}). \tag{6.11}$$

We have been not able to find any sensible representation of (6.9) in terms of pure $c$-number functions $\overset{c}{\psi}, \overset{c}{\overline{\psi}}$, and thus not in terms of amplitudes $\alpha, \beta \in \oplus_1^4 \mathcal{L}^2(\mathbb{R}^3)$. However, the formal rules (6.5) can be used as a *complementary tool*, satisfactorily reflecting relations between expansion coefficients, which follow from (6.9), and then allow us to define a $c$-number functional $\overset{c}{\Omega}_{12}(\overset{c}{\psi}, \overset{c}{\overline{\psi}})$ while starting from the Grassman functional $\Omega_{12}(\psi,\overline{\psi})$: $\omega^{12}_{nm} \to \mathcal{E}_n \mathcal{E}_m \omega^{12}_{nm}$.

## 7. QUANTIZATION OF DIRAC FIELD

In the case of the scalar field, having given an asymptotic free field $\overset{c}{\varphi}$, we could define sets of operators (functionals respectively) $:\Omega(\varphi):$, $\overset{c}{\Omega}(\overset{c}{\varphi})$.

In the case of the Dirac fields $\psi, \overline{\psi}$ we map $:\Omega(\psi,\overline{\psi}):$ onto a classical level through the mediation of the subsidiary boson level. However, this boson level itself allows us to consider its own classical map consisting from the set $\mathcal{S}$ of all functionals with respect to $\overset{c}{\psi}, \overset{c}{\overline{\psi}}$ whose expansion coefficients $\omega_{nm}$ are totally $(n+m)$-symmetric:

$$\overset{s}{\Omega}(\overset{c}{\psi}, \overset{c}{\overline{\psi}}) = \sum_{nm} \frac{1}{n!\,m!} (\overset{s}{\omega}_{nm}, \overset{c}{\psi}{}^n \overset{c}{\overline{\psi}}{}^m). \tag{7.1}$$

In the quantization attempts of any classical spinor field theory one starts from functionals $\overset{s}{\Omega}$ rather than from $\overset{c}{\Omega}$. At first we must have a reduction tool allowing us to transform $\mathcal{S}$ into the set $\mathcal{S}_0$ of functionals $\overset{c}{\Omega}$, which are the only ones of interest if it is required that the Fermi level be achieved.

*Lemma* 3: There exists the reduction operator $P_0$ on $\mathcal{S}$, such that

$$P_0 \mathcal{S} = \mathcal{S}_0.$$

*Proof*: We shall introduce into our considerations the following functional:

$$P_0(\overset{c}{\psi}, \overset{c}{\bar\psi}) = \sum_{nm} \frac{1}{n!m!} (\tilde{\mathcal{E}}_n \tilde{\mathcal{E}}_{\underset{\sim}{m}} \tilde{\mathcal{E}}_{n+m} \overset{c}{\psi}{}^n \overset{c}{\bar\psi}{}^m, \overset{c}{\psi}{}^n \overset{c}{\bar\psi}{}^m$$

$$= \sum_{nm} \frac{1}{n!m!} (\overset{c}{\psi}{}^n \overset{c}{\bar\psi}{}^m, \mathcal{E}_{n+m} \mathcal{E}_{\underset{\sim}{m}} \mathcal{E}_n \overset{c}{\psi}{}^n \overset{c}{\bar\psi}{}^m). \qquad (7.2)$$

If we consider $P_0$ as an operator in $\mathcal{S}$, acting according to the following (functional) rule,

$$(P_0 \overset{s}{\Omega})(\overset{c}{\psi}, \overset{c}{\bar\psi}) = \sum_{nm} \frac{1}{n!m!} \left( \overset{c}{\psi}{}^n \overset{c}{\bar\psi}{}^m, \mathcal{E}_{n+m} \mathcal{E}_{\underset{\sim}{m}} \mathcal{E}_n \frac{d^n}{d\overset{c}{\varphi}{}^n} \frac{d^m}{d\overset{c}{\bar\varphi}{}^m} \right)$$

$$\times \overset{s}{\Omega}(\overset{c}{\varphi}, \overset{c}{\bar\varphi})_{|\overset{c}{\varphi}=0=\overset{c}{\bar\varphi}}$$

$$= \sum_{nm} \frac{1}{n!m!} (\mathcal{E}_n \mathcal{E}_{\underset{\sim}{m}} \mathcal{E}_{n+m} \overset{s}{\omega}_{nm}, \overset{c}{\psi}{}^n \overset{c}{\bar\psi}{}^m) = \overset{c}{\Omega}(\overset{c}{\psi}, \overset{c}{\bar\psi}), \qquad (7.3)$$

where $\overset{c}{\omega}_{nm} = \mathcal{E}_{\underset{\sim}{m}} \mathcal{E}_n \mathcal{E}_{n+m} \overset{s}{\omega}_{nm} = \mathcal{E}_{\underset{\sim}{m}} \mathcal{E}_n \omega_{nm}$, and as possessing the expected symmetry properties, then $\omega_{nm}$ is totally $(n+m)$-antisymmetric. The lemma is proved. With this selection tool, we can formulate:

*Theorem 7 (quantization rule)*: Given the set $\mathcal{S}$ of functionals $\overset{s}{\Omega}(\overset{c}{\psi}, \overset{c}{\bar\psi})$, then $P_0 \mathcal{S} = \mathcal{S}_0$, if equipped with the algebraic structure (6.9), allows us the quantization map

$$\overset{c}{\Omega}(\overset{c}{\psi}, \overset{c}{\bar\psi}) \to : \overset{c}{\Omega}(\overset{B}{\psi}, \overset{B}{\bar\psi}) : \Rightarrow \mathbb{1}_F : \overset{c}{\Omega}(\overset{B}{\psi}, \overset{B}{\bar\psi}) : \mathbb{1}_F \mathcal{J}_F = : \Omega(\psi, \bar\psi) : \mathcal{J}_F, \qquad (7.4)$$

connecting with each element $\overset{c}{\Omega}$ of $\mathcal{S}_0$ the corresponding element $:\Omega(\psi, \bar\psi):$ of the Fermi field algebra. The converse map is realized by the correspondence rule of Theorem 6.

*Proof*: Repeats in fact arguments of Theorem 6.

Theorems 6 and 7, combined together, form a *correspondence principle* for Dirac fields.

## 8. ON GENERATING FUNCTIONALS FOR THE GREEN'S FUNCTIONS

The commonly used functionals (2.1) are based on Grassman concepts. Let us consider the functional of the same form:

$$Z(\eta^c, \overset{c}{\bar\eta}) = \frac{\int \exp\{i[\overset{c}{S} + \int(\overset{c}{\eta}\overset{c}{\bar\psi} + \overset{c}{\bar\eta}\overset{c}{\psi})dx]\} \cdot d(M\overset{c}{\psi}/\sqrt{i\pi})}{\int \exp(i\overset{c}{S})\, d(M\overset{c}{\psi}/\sqrt{i\pi})}, \qquad (8.1)$$

with the only difference lying in the replacement of Grassman objects by corresponding $c$-numbers (commuting ring) $\overset{c}{\eta}, \overset{c}{\bar\eta}, \overset{c}{\psi}, \overset{c}{\bar\psi}, d(M\overset{c}{\psi}/\sqrt{i\pi})$. Functionals of the form $Z(\overset{c}{\eta}, \overset{c}{\bar\eta})$ play the role played in the previous section by $\overset{s}{\Omega}$.

Let us introduce the following *reduction of* $Z(\overset{c}{\eta}, \overset{c}{\bar\eta})$: $Z_0(\overset{c}{\eta}, \overset{c}{\bar\eta}) = (P_0 Z)(\overset{c}{\eta}, \overset{c}{\bar\eta})$

$$= \sum_{nm} \frac{1}{n!m!} \left( \tilde{\mathcal{E}}_n \tilde{\mathcal{E}}_{\underset{\sim}{m}} \tilde{\mathcal{E}}_{n+m} \overset{c}{\eta}{}^n \overset{c}{\bar\eta}{}^m, \frac{d^n}{d\overset{c}{\varphi}{}^n} \frac{d^m}{d\overset{c}{\bar\varphi}{}^m} \right)$$

$$\times Z(\overset{c}{\varphi}, \overset{c}{\bar\varphi})_{|\overset{c}{\varphi}=\overset{c}{\bar\varphi}=0}. \qquad (8.2)$$

For the general case, the reduction formula (8.2) does not look too attractive. Let us see, however, what happens in the free field case, when

$$Z(\overset{c}{\eta}, \overset{c}{\bar\eta}) = \exp[-i(\overset{c}{\bar\eta}, G\overset{c}{\eta})], \qquad (8.3)$$

where $G_{\sigma\tau}(x-y)$ is the Green's function of the Dirac equation. We have defined at once the two-point Green's function by

$$\left( \tilde{\mathcal{E}}_2 \frac{d}{d\overset{c}{\eta}} \frac{d}{d\overset{c}{\bar\eta}} \right)(x, y) \cdot Z_0(\overset{c}{\eta}, \overset{c}{\bar\eta})_{|\overset{c}{\eta}=\overset{c}{\bar\eta}=0}$$

$$= \left( \tilde{\mathcal{E}}_2 \frac{d}{d\overset{c}{\eta}} \frac{d}{d\overset{c}{\bar\eta}} \right)(x, y) \cdot \int dx'\, dy'$$

$$\times (\tilde{\mathcal{E}}_2 \overset{c}{\bar\eta}_\sigma \overset{c}{\eta}_\tau (x', y')\, G_{\sigma\tau}(x-y) = G^\rho_{\sigma\tau}(x-y) \qquad (8.4)$$

which allows us to consider the reduced (boson) generating functional (8.1) as a ($c$-number) generating functional for the Green's functions of the Dirac field.

*Note added in proof*: In the course of the paper the words "classical" and "quantum" concern the $c$-number and $q$-number levels respectively of the given theory, and have nothing to do with any $\hbar \to 0$ limit. The natural system of units $\hbar = c = 1$ is employed.

A complete operator formulation of steps (5.14)—(5.16), which should be more convincing for an unfamiliar reader, can be found in the Phys. Rep. C (1978) paper of Ref. 5.

Let us emphasize that by virtue of the projection theorems each Bose field, which obeys the Haag—LSZ expansion conjecture, has its corresponding fermion contents. It happens independently of whether the spin-statistics theorem holds or not, and makes less surprising the fact that in some Bose field theory models (as, e.g., the sine-Gordon one) fermions are allowed to appear.


[1]The general structure of the quantum field theory (QFT) including functional formulation of the LSZ approach with respect to scalar fields is studied in: H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento I, 206 (1955); VI, 319 (1957); R. Haag, Dansk. Mat. Fyz. Medd. **29**, 13 (1955); V. Glaser, H. Lehmann, and W. Zimmermann, Nuovo Cimento **VI**, 1122 (1957); N.N. Bogolubov, A.A. Logunov, and I.I. Todorov, *Axiomatic Quantum Field Theory*, in Russian (Nauka, Moscow, 1963): K. Hepp, in *Brandeis Summer Institute* 1965, edited by M. Chretien and S. Deser (Gordon and Breach, New York, 1965).

[2]For an introduction to the Hilbert spaces of functional power series and functional representations of the CCR, study of estimates, and convergence criteria, see: J. Rzewuski, Rep. Math. Phys. **1**, 1 (1970); **1**, 195 (1971); see also S. Schweber, J. Math. Phys. **3**, 831 (1962); V. Bargmann, Comm. Pure Appl. Math. **14**, 187 (1961).

[3]Most systematic studies of functional integration and differentiation methods in application to QFT have been performed in monographs (spinors treated in Grassman language): F.A. Berezin, *The Method of Second Quantization*, in Russian (Nauka, Moscow, 1965); J. Rzewuski, *Field Theory, Vol. II. Functional Formulation of the S-Matrix Theory* (Iliffe, London, PWN, Warsaw, 1969); V.N. Popov, *Path Integrals in Quantum Field Theory and Statistical Physics*, in Russian (Atomizdat, Moscow, 1976).

[4]The so-called mathematical theory of Feynman path integrals (with a few limitations) is covered by: C. DeWitt-Morette, Comm. Math. Phys. **28**, 47 (1972); **37**, 63 (1974); S. Albeverio and R. Høegh-Krohn, *Mathematical Theory of*

*Feynman Path Integrals*, Lecture Notes in Mathematics (Springer-Verlag, Berlin, 1976).

[5]For boson expansion methods, as developed by the present author, see: Comm. Math. Phys. **43**, 131 (1975), "Representations of the CAR generated by representations of the CCR. Fock case"; Bull. Acad. Polon. Sci. Ser. Astr. Phys. Math. **23**, 113 (1975), "Spin-1 vector Boson structure of free spin-1/2 quantum field"; *ibid.* 24, 201 (1976), "Representations of the CAR generated by representations of the CCR. II /isometries/"; *ibid*, "Boson expansions of Jordan-Wigner representation," Bull. Acad. Pol. Sci. **25**, 711 (1977); with Z. Popowicz, Rep. Math. Phys. **11**, 57 (1977), "Representations of the CAR generated by representations of the CCR. III, Non Fock extension"; Proc. of the 13th Karpacz School, Acta Univ. Wrat. 1976, "Bosonization of Fermions in QFT", "Bosonization of Fermions in Heisenberg ferromagnet," in *Theoretical Physics, Memorial Book on J. Rzewuski's 60th Birthday* (Wroclaw, 1976); "The method of Boson expansions in the q.t. of Fermions," Phys. Rept. C in print; with Z. Popowicz, "Ultralocal quantizational of Sine—Gordon 1-solitons," submitted for publication; Int. Journ. Theor. Phys. **115**, 809 (1977), "Remark on Kalnay theory of Fermions constructed from Bosons."

[6]For the voice of a pragmatist, see: S. Coleman, "Secret symmetry," in *Laws of Subnuclear Matter*, Erice Summer Institute, edited by A. Zichichi (Academic, New York, 1975).

[7]Extension of the LSZ methods in terms of functional integrals onto Dirac spinors by making use of Grassman algebra tools, is considered in P.T. Matthews and A. Salam, Nuovo Cimento X, 120 (1955); Yu. Novoshilov and A.V. Tulub, Usp. Fiz. Nauk 61, 53 (1957).

[8]For attempts to find a $c$-number formulation of the QT of fermions, without use of Grassman algebras, see: J.R. Klauder, Ann. Phys. (N.Y.) **11**, 123 (1960), "The action option and Feynman quantization of spinor fields in terms of ordinary $c$-numbers"; F. Rohrlich, in *Analytic Methods in Mathematical Physics*, edited by R.P. Gilbert and R.G. Newton (Gordon and Breach, New York, 1970)," The coherent state representation and q.f.t."; P. Garbaczewski and J. Rzewuski, Rep. Math. Phys. **6**, 431 (1974), "On generating functionals for antisymmetric functions and their application in q.f.t."; P. Garbaczewski, Rep. Math. Phys. **7**, 321 (1975), "Functional representations of the CAR."

[9]K.O. Friedrichs, *Mathematical Aspects of the Quantum Field Theory Fields* (Interscience, New York, 1953).

[10]For strongly selected number of investigations on fermion—boson correspondence, especially in connection with Thiring and sine—Gordon systems, see: S. Coleman, Phys. Rev. D **11**, 2088 (1975); S. Mandelstam, Phys. Rev. D **11**, 3026 (1975); A.K. Pogrebkov and V.N. Sushko, Teor. Mat. Fiz. **24**, 425 (1975); B. Schroer and J.A. Swieca, "Spin and statistics of quantum kinks," CERN preprint 1976); B. Schroer, Q.f.t. of kinks in two-dimensional space-time," Cargese lecture notes (1976); H. Neuberger, "Bosonization in field theory in two space-time dimensions," Tel-Aviv preprint (1976) (Grassman algebras involved in the study of correspondence between Hamiltonians).

[11]The influence of Grassman methods: for works on pseudomechanics, as an example, we suggest: F. Casalbuoni, Nuovo Cimento A **33**, 115 (1976), "On the quantization of systems with anticommuting variables."