BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. XXIII, No. 10, 1975

THEORETICAL PHYSICS

Spin-1 Vector Boson Structure of Free Spin-1/2 Quantum Field

by

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Presented by J. RZEWUSKI on May 16, 1975

Summary. Construction of a Fock representation of the CAR algebra in a given Fock representation of the CCR algebra is extended to include physically admissible quantum fields: an example of a free spin-1/2 quantum field constructed from a spin-1 vector boson quantum field is considered.

1. Let a triple $\{a^*, a, f_0\}_K$ generate a Fock representation of the CCR algebra over K (a separable complex Hilbert space) acting in $\mathscr{D}\subset\mathscr{H}$, $\mathscr{H}$ being another Hilbert space. Let — denote an involution in K and $(\cdot, \cdot)$ a bilinear form in $K\ni f, g$:

(1) $$[a(f), a(g)^*]_- = (\bar{f}, g)\,\mathbf{1}_B, \qquad a(f)f_0=0.$$

It was shown in [3] that each triple $\{a^*, a, f_0\}_K$ induces in $\mathscr{D}_F\subset\mathscr{D}$ (admitting an extension to the whole $H_F\subset\mathscr{H}$) the triple $\{b^*, b, f_0\}_K$ generating a Fock representation of the CAR alegbra over the same K:

(2) $$[b(f), b(g)^*]_+ = (\bar{f}, g)\,\mathbf{1}_F, \qquad b(f)f_0=0.$$

Numbers of internal degrees of freedom in the construction are the same for bosons and fermions $(K=\bigoplus_1^n \mathscr{L}^2(R^3))$, which seems rather unsatisfactory from the physical point of view. We attempt to overcome this difficulty for a concrete example of spin-1 massive charged and spin-1/2 massive free quantum fields.

2. Recall the standard notations:

(i) massive charged free vector field

(3) $$\mathscr{U}_\mu(x)=\mathscr{U}_\mu^+(x)+\mathscr{U}_\mu^-(x), \qquad \mathscr{U}_\mu^\pm(x)=\sum_{s=1}^{3}\int dp\, g_\mu^{s\,\pm}(p,x)\,d_s^\pm(p),$$
$$d_s^\pm(p)=\frac{1}{\sqrt{2}}(a_\mu^\pm+ia_{s+3}^\pm)(p), \qquad \overset{*}{d}{}_s^\pm(p)=\frac{1}{\sqrt{2}}(a_s^\pm-ia_{s+3}^\pm)(p),$$

*) Supported by the US National Science Foundation under Grant GF-41 959.

where

$$a^+ = a^*, \quad a^- = a,$$

(4) $$[a_s^*(k), a_t(p)]_- = \delta_{st}\,\delta(k-p)\,\mathbf{1}_B .$$

Hence we deal with the triple

$$\{a^*, a, f_0\}_K, \quad K = \bigoplus_1^6 \mathscr{L}^2(\mathbf{R}^3), \quad k, p \in \mathbf{R}^3.$$

(ii) complex massive spin-1/2 field

(5) $$\psi_\sigma(x) = \psi_\sigma^+(x) + \psi_\sigma^-(x), \quad \psi_\sigma^\pm(x) = \sum_{l=1}^{2} \int d\,p v_\sigma^{l\pm}(p, x)\, c_l^\pm(p),$$

with

$$c_l^\pm(p) = \frac{1}{\sqrt{2}}(b_l^\pm + b_{l+2}^\pm)(p), \quad \overset{*}{c}{}_l^\pm(p) = \frac{1}{\sqrt{2}}(b_l^\pm - i b_{l+2}^\pm)(p),$$

where:

$$b^+ = b^*, \quad b^- = b,$$

(7) $$[b_k^*(p), b_l(q)]_+ = \delta_{kl}\,\delta(p-q)\,\mathbf{1}_F .$$

Hence we deal with the triple

$$\{b^*, b, f_0\}_K, \quad K = \bigoplus_1^4 \mathscr{L}^2(\mathbf{R}^3).$$

3. To construct field (ii) in terms of field (*i*), we will induce the triple $\{\overset{*}{b}{}', b', f_0\}_{K'}$ from the triple $\{a^*, a, f_0\}_K$, where

$$K = \bigoplus_1^6 \mathscr{L}^2(\mathbf{R}^3) \quad \text{and} \quad K' \subseteq \bigoplus_1^4 \mathscr{L}^2(\mathbf{R}^3).$$

For this purpose let us introduce the following map $K \to K$, realized by the 6×6 matrix:

(8) $$D = \left[\begin{array}{ccc|ccc} a, & c, & \sqrt{1-a^2-c^2} & & & \\ b, & -\frac{abc}{1-a^2}, & -\frac{ab\sqrt{1-a^2-c^2}}{1-a^2} & & 0 & \\ 0, & 0, & 0 & & & \\ \hline & & & a, & c, & \sqrt{1-a^2-c^2} \\ & 0 & & b, & -\frac{abc}{1-a^2}, & -\frac{ab\sqrt{1-a^2-c^2}}{1-a^2} \\ & & & 0, & 0, & 0 \end{array}\right]$$

where a, b, c are real parameters restricted by the condition $\bar{D}^{kl} = D^{kl}$.

One can easily prove the following property of D:

$$DD^* = \left|\begin{array}{c|c} \begin{matrix}1, & 0, & 0\\ 0, & 1, & 0\\ 0, & 0, & 0\end{matrix} & 0 \\ \hline 0 & \begin{matrix}1, & 0, & 0\\ 0, & 1, & 0\\ 0, & 0, & 0\end{matrix}\end{array}\right| \tag{9}$$

D is of course not a projector.

Let K' denote a subspace of K consisting of the following elements:

(10) $K \ni \{a_1, a_2, a_3, a_4, a_5, a_6\}, \qquad K' \ni Da = \{a_1', a_2', 0, a_3', a_4', 0\} := a'.$

We immediately obtain

$$D\overset{*}{D} = \mathbf{1}_{K'}.$$

Let us remark in this place that neither finite matrix can simultaneously satisfy both identities:

$$DD^* = \mathbf{1}_{K'}, \qquad \overset{*}{D}D = \mathbf{1}_K. \tag{11}$$

Hence

$$D^* K' \subset K, \qquad \overset{*}{D}K \neq K.$$

Denote $DK' = K'' \subset K$. One easily finds that D and D^* establish an isomorphism between subspaces of $K: K' \leftrightarrow K''$.

4. Given $\{a^*, a, f_0\}_K$, $K = \bigoplus_1^6 \mathscr{L}^2(\boldsymbol{R}^3)$, we introduce the triple $\{\overset{*}{a}', a', f_0\}_K$ in the following way:

$$K \supset K'' \ni f'',$$

$$a(f'') = (a, \bar{f}'') = (a, \overline{D^* f'}) = (Da, \bar{f}') := (a', \bar{f}') \tag{12}$$

being valid in $\mathscr{D}$ for all $f' \in K'$, or equivalently:

$$a_k'(p) = (Da)_k(p) = \sum_{s=1}^{6} D^{ks} a_s(p). \tag{13}$$

This makes it possible to consider the number of four generators:

$$\{a_1', a_2', 0, a_3', a_4', 0\} \tag{14}$$

with the property

$$k = 1, 2, 3, 4, \qquad [a_k'(p), \overset{*}{a}_l{}'(q)]_- = \delta_{kl}\, \delta(p-q)\, \mathbf{1}_B \tag{15}$$

fulfilled in $\mathscr{D}$, while smeared with test functions from K'.

We have thus constructed a certain CCR algebra describing four internal degrees of freedom in the CCR algebra describing six internal degrees of freedom in the theory, and obtained the desired "less physical" generators constructed in terms of physically permitted generators.

5. Having the triple $\{a', a', f_0\}_{K'}$, through a standard construction given in [3] we obtain the triple $\{b', b', f_0\}_{K'}$, generating a Fock representation of the CAR algebra in the Fock representation of the CCR algebra (generated by $\{\overset{*}{a}', a', f_0\}_{K'}$).

Neglecting any detailed explanations (cf. [1, 3]) we can write:

$$(16)\qquad \begin{aligned} b'(f') &= :\exp\{-(a', a')\} \sum_n \frac{1}{n!} (f' \overset{*}{a}'^n E'_n, E'_{1+n} a'^{1+n}): \\ b'(f')^* &= :\exp\{-(a', a')\} \sum_n \frac{1}{n!} (a'^{1+n} E'_{1+n} f' E'_n a'^n):, \end{aligned}$$

where $f' \in K'$, E'_n is an operator acting in $\overset{n}{\underset{1}{\otimes}} K'$.

Let $f \in K'' \subset K$, and let E_n denote an operator in $\overset{n}{\underset{1}{\otimes}} K''$ constructed from E'_n with the help of $D, D*, D^*$. We can write:

$$(17)\qquad \begin{aligned} b'(f') &= :\exp\{-(\overset{*}{a}, a)_D\} \sum_n \frac{1}{n!} (\bar{f} a^n E_n, E_{1+n} a^{1+n}): \\ b'(f')^* &= :\exp\{-(\overset{*}{a}, a)_D\} \sum_n \frac{1}{n!} (a^{1+n} E_{1+n}, f_n E_n a^n):, \end{aligned}$$

where

$$(18)\qquad (\overset{*}{a}, a)_D = (\overset{*}{a}, \overset{*}{D} Da) = ((Da)^*, Da).$$

There are generators $\{b^{*\prime}, b', f_0\}_{K'}$ constructed explicitly in terms of generators $\{\overset{*}{a}, a, f_0\}_K$ with K restricted to K''. In virtue of [3], we have:

$$(19)\qquad [b'(f'), b'(g')^*]_+ = (\bar{f}', g')\, \mathbf{1}_F = (\bar{f}, \overset{*}{D} Dg)\, \mathbf{1}_F = (\bar{f}, g)_D\, \mathbf{1}_F,$$

or equivalently:

$$(20)\qquad [b'_k(p), b'_1(q)]_+ = \delta_{kl}\, \delta(p-q)\, \mathbf{1}_F.$$

6. The above considerations enable us to calculate explicitly commutators between $\mathscr{U}_\mu(x)$ and $\psi_\sigma(y)$. In the conventional formulation of quantum field theory these commutators are stated axiomatically (see [5]). The question of such mutual commmutation relations for free quantum fields will be discussed in a forthcoming paper.

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П. Гарбачевски, **Конструкция свободных фермионов со спином 1/2 с массовных бозонов со спином 1**

Содержание. Конструкция представления Фока канонических антикоммутационных соотношений в представлении Фока коанническиx коммутационных соотнишений расширена на случай свободных полей: фермионного со списом 1/2 и массивного векторного (спин 1).