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Boson Expansions of the Jordan-Wigner Representation

by

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Summary. We derive the Boson expansion for the Jordan–Wigner representation of the algebra of the canonical anticommutation relations (CAR).

In the many-body problems, people frequently employ the Jordan–Wigner construction of the representation of the CAR algebra, which is based on the use of an infinite family:

$$(1)\qquad \begin{aligned} &[\sigma_k^{\pm}, \sigma_l^{\pm}]_- = 0 \qquad \text{for} \quad k \neq l \\ &[\sigma_k^{+}, \sigma_l^{-}]_- = \mathbf{1}_F \qquad (\sigma_k^{+})^2 = 0 = (\sigma_k^{-})^2 \end{aligned}$$

of spin 1/2 raising and lowering operators. In practical applications, the above part-Boson, part-Fermion nature of operators stands for a difficulty in the theory, because no simple linear transformation between σ_k^{+}'s and σ_k^{-}'s such as would be required to diagonalize a quadratic form (the Hamiltonian), leaves these rules invariant.

There is no difficulty in transforming the rules (1) into a complete set of the CAR. The famous Jordan–Wigner trick is here all right. The Fermion creation and annihilation operators appear according to:

$$(2)\qquad \begin{aligned} c_k &= \exp\left(i\pi \sum_{j=1}^{k-1} \sigma_j^{+} \sigma_j^{-}\right) \cdot \sigma_k^{-} \\ c_k^{*} &= \exp\left(i\pi \sum_{j=1}^{k-1} \sigma_j^{+} \sigma_j^{-}\right) \cdot \sigma_k^{+}. \end{aligned}$$

Here $c_k^{*}\, c_k = \sigma_k^{+}\, \sigma_k^{-}$ and the inverse transformation reads:

$$(3)\qquad \begin{aligned} \sigma_k^{-} &= \exp\left(i\pi \sum_{j=1}^{k-1} c_j^{*}\, c_j\right) \cdot c_k \\ \sigma_k^{+} &= \exp\left(i\pi \sum_{j=1}^{k-1} c_j^{*}\, c_j\right) \cdot c_k^{*}. \end{aligned}$$

Another construction of the generators of the CAR based on the use of (1) is realized by:

$$b_j = \prod_{k=1}^{j-1} (\mathbf{1} - 2\sigma_k^+ \sigma_k^-) \cdot \sigma_j^-$$

(4)
$$b_j^* = \prod_{k=1}^{j-1} (\mathbf{1} - 2\sigma_k^+ \sigma_k^-) \cdot \sigma_j^+$$

$$[b_k, b_l^*]_+ \subseteq \delta_{kl} \mathbf{1}_F$$

$$[b_k, b_l]_+ = 0 = [b_k^*, b_l^*]_+ .$$

Both in the case (2) and (4) we deal with a Fock representation if $\sigma_k^- \Omega = 0$ for all $k = 1, 2, \dots$.

Let us now consider spin operators to be constructed in the Fock representation of the CCR algebra, according to:

(5)
$$\sigma_k^- = :\exp(-a_k^* a_k): a_k$$

$$\sigma_k^+ = a_k^* :\exp(-a_k^* a_k):$$

$$[a_k, a]_- = \delta_{kl} \mathbf{1}_B$$

$$[a_k, a_l]_- = 0 \qquad a_k \Omega_B = 0 \text{ for all } k .$$

Let us further introduce the following projectors, acting as operators in the Boson-Fock space F_B: $\mathbf{1}_F^k F_B = F_F^k$

(6)
$$\mathbf{1}_F^k = :\exp(-a_k^* a_k) \cdot (1 + a_k^* a_k):$$

$$\mathbf{1}_F^k - 2a_k^* :\exp(-a_k^* a_k): a_k = :\exp(-a_k^* a_k): - a_k^* :\exp(-a_k^* a_k): a_k$$

we get at once:

$$b_j = \prod_{k=1}^{j-1} [\mathbf{1}_F^k - 2a_k^* :\exp(-a_k^* a_k): a_k] :\exp(-a_j^* a_j): a_j =$$

(7)
$$= :\exp\left(-\sum_{k=1}^{j} a_k^* a_k\right) \cdot \prod_{k=1}^{j-1} (1 - a_k^* a_k): a_j$$

$$b_j^* = a_j^* :\exp\left(-\sum_{k=1}^{j} a_k^* a_k\right) \cdot \prod_{k=1}^{j-1} (1 - a_k^* a_k):$$

which gives a complete "bosonization" formula for the Jordan–Wigner representation of the CAR, applicable both in cases (2) and (4).

In this connection we refer to [3], where a global "bosonization" prescription for Fock representations of the CAR was given. Obviously, due to the boundeness of the operators (7) (the general property of the CAR) one can extend these operators to the form of [3]. From a physical point of view, the "bosonization" (7) corresponds to the so-called spin 1/2 approximation of the given starting Boson theory, when each single degree of freedom is either singly excited or not excited at all. Such a situation for the (initially) Boson system can happen only if a contact with a suitable low-temperature environment is provided. For more information on the applications of the Boson expansion methods in quantum theory see [5].

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П. Гарбачевски, **Бозонные розложения представления Йордана-Вигнера**

Содержание. Рассмотрены бозонные представления алгебр канонических антикоммутационных соотношений в конструкции Йордана, Вигнера.