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THEORETICAL PHYSICS

Representations of the CAR Generated by Representations of the CCR. II

by

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Summary. An idea to construct a representation of the CAR algebra in a given Fock representation of the CCR algebra is extended to include pairs of representations determining arbitrary physically admissible free quantum fields. The arbitrariness concerns only internal (spin, charge) degree of freedom in the theory.

1. It was proved in [1] that one is able to construct a certain (Fock) representation of the CAR algebra in a given Fock representation of the CCR algebra. This result would at first seem unsatisfactory from the physical point of view: both representations are spanned over the same test function space, and therefore can describe the same number of internal degrees of freedom in the theory.

We attempt to finally overcome this difficulty (the case of spin 1/2 massive charged and spin 1 massive charged free quantum fields was considered independently in [2]). We show that in the representation generated by the Bose triple $\{a^*, a, \Omega\}_K$, K_N being an indefinite metric space K_J or a Hilbert space $\bigoplus_1^N \mathscr{L}^2(\boldsymbol{R}^3)$, we can construct representations of the CAR algebra generated by the triples $\{b^*, b, \Omega\}_{K}$, with $K' = \bigoplus_1^M \mathscr{L}^2(\boldsymbol{R}^3)$, N, M arbitrary positive integers.

2. Let $K = \bigoplus_1^4 \mathscr{L}^2(\boldsymbol{R}^3)$ be given with the sesquilinear form

$$(f, g) = \sum_{\mu\nu} g_{\mu\nu}(f_\mu, g_\nu) = \sum_\mu (f_\mu, g^\mu) \tag{1}$$

implemented by $(\cdot, \cdot)$, a bilinear form in $K_J \ni f, g$ and $g_{\mu\nu}$ being the metric tensor of Minkowski space.

The triple $\{a^*, a, \Omega\}_{K_J}$ is generating $U_B(K_J)$. a Fock representation of the CCR algebra over K_J:

$$\begin{aligned} &[a_\mu(k), a_\nu^*(p)]_- = g_{\mu\nu}\,\delta(k-p)\,\mathbf{1}_B, \\ &[a_\mu(k), a_\nu(p)]_- = 0, \\ &a_\mu(k)\,\Omega = 0, \qquad k, p \in \boldsymbol{R}^3. \end{aligned} \tag{2}$$

$U_B(K_J)$ is spanned by functional polynomials:

(3) $$\mathscr{F}(a^*, a) = \sum_{nm} \frac{1}{\sqrt{n!\,m!}} (\mathscr{F}_{nm}, \overset{*}{a}{}^n a^m) = \sum_{nm} \frac{1}{\sqrt{n!\,m!}} \int dk_n \int dp_m \times$$
$$\sum_{\mu_1,\dots,\mu_n} \sum_{\nu_1,\dots,m} \mathscr{F}_{nm}(\boldsymbol{\mu}_n, \boldsymbol{k}_n, \boldsymbol{\nu}_m, \boldsymbol{p}_m)\, a^*_{\mu_1}(k_1) \dots \overset{*}{a}_{\mu_n}(k_n) \cdot a_{\nu_1}(p_1) \dots a_{\nu_m}(p_m),$$

where

$$F_{nm} \in \bigoplus_1^{n+m} K_J, \quad d\boldsymbol{k}_n = dk_1, \dots, dk_n, \quad \boldsymbol{\mu}_n = (\mu_1, \dots, \mu_n), \quad \boldsymbol{k}_n = (k_1, \dots, k_n).$$

Acting on the Fock vacuum these polynomials span a linear manifold $U_B(K_J)\,\Omega$ consisting of the elements:

(4) $$\mathscr{F}(a^*, a)\,\Omega := F = \sum_k \frac{1}{\sqrt{k!}} (F_k, \overset{*}{a}{}^k)\,\Omega =$$
$$= \sum_k \frac{1}{\sqrt{k!}} \int d\boldsymbol{p}_k \cdot \sum_{\mu_1,\dots,\mu_k} \cdot F_k(\boldsymbol{\mu}_k, \boldsymbol{p}_k)\, |\mu_1, p_1, \dots, \mu_k p_k),$$

where $|\mu_1 p_1, \dots, \mu_k p_k)$ is the k-particle state vector. The indefinite metric structure of $U_B(K_J)\,\Omega$

(5) $$(\bar{F}, G) : \sum_k (\bar{F}_k, G_k)$$

can in many ways be equipped with the Hilbert space topology (see [5]). A completion of $U_B(K_J)\,\Omega$ with respect to any Hilbert space topology is denoted by $\mathscr{F}_B$ and called the representation space for the triple $\{a^*, a, \Omega\}_{K_J}$.

3. Assume an orthonormal (but necessarily incomplete) system $\{h_i\}_{i=1,2,\dots}$ in K_J is given

(1) $$(\bar{h}_i, h_j) = \sum_\mu \int dp\, \bar{h}_i^\mu(p)\, h_{j\mu}(p) = \delta_{ij},$$

(An example can be found in [4]). With this system we can take into account an enumerab'e infinite set of Bose generators $\{a_i, a_i^*\}_{i=1,2,\dots}$

(2) $$a_i = a(h_i) = \sum_\mu \int dp\, a_\mu(p)\, \bar{h}_i^\mu(p).$$

These generators fulfill

(3) $$[a_i, a_j^*]_- = \delta_{ij}\, \mathbf{1}_B, \quad [a_i, a_j]_- = 0, \quad a_i\,\Omega = 0.$$

Now a Hilbert space $K = \bigoplus_1^N \mathscr{L}^2(\boldsymbol{R}^3)$ is given. It is spanned by an orthocomplete set $\{e_i\}_{i=1,2,\dots}$:

(4) $$\sum_{k=1}^N \int dp\, \bar{e}_i^k(p)\, e_j^k(p) = \delta_{ij}, \quad \sum_i \bar{e}_i^k(p)\, e_i^l(q) = \delta_{kl}\, \delta(p-q).$$

With the help of (3.2) and (3.3), we can introduce in $U_B(K_J)$ a new Bose triple $\{a^*, a, \Omega\}_K$ generating a Fock representation of the CCR algebra over K and again defined in $\mathscr{D}\subset\mathscr{F}_B$. Here, by definition:

$$a_k(p)=\sum_i a_i \cdot e_i^k(p) \tag{5}$$

and operator series are convergent in terms of the coherent state expectation value:

$$\begin{gathered} z\in K, \quad (z\,|a_k(p)|\,z)=z_k(p)=\sum_i z_i\cdot e_i^k(p), \\ |z)=\exp\left(-\frac{1}{2}\|z\|^2\right)\cdot \exp(z,a^*)\,\Omega, \\ \|z\|^2=\sum_i |z_i|^2<\infty, \quad (z,a^*):=\sum_i z_i\,a_i^*. \end{gathered} \tag{6}$$

In the new triple, we have trivially fulfilled

$$\begin{gathered} [a_k(p), a_l^*(q)]_-=\delta_{kl}\,\delta(p-q)\,\mathbf{1}_B, \\ [a_k(p), a_l(q)]_-=0, \\ a_k(p)\,\Omega=0, \quad \forall\, k,p. \end{gathered} \tag{7}$$

Polynomials in a_k, a_k^* are, of course, through (3.5) and (2.4) polynomials in a_μ, a_ν^*, and hence — elements of $U_B(K_J)$. A representation space for the new triple is therefore included in F_B. Changing N in $K=\bigoplus_1^N \mathscr{L}^2(\boldsymbol{R}^3)$, we can obtain the whole family of representations of the CCR algebra, expressed by indefinite metric representation and allowing to consider an arbitrary needed (choose proper N) number of internal degrees of freedom in the theory.

In an analogous way, starting from any $\{a^*, a, \Omega\}_K$, $K=\bigoplus_1^N \mathscr{L}^2(\boldsymbol{R}^3)$, one can define in $U_B(K)$ the triple $\{a^*, a, \Omega\}_{K'}$, with $K'=\bigoplus_1^M \mathscr{L}^2(\boldsymbol{R}^3)$ and $N\neq M$.

4. In the construction of the representation of the CAR algebra in a given Fock representation of the CCR algebra given in [1] we considered the case of scalar generators (lack of spin and charge).

Let us reproduce the formula for the annihilating generator in the Fermi triple $\{b^*, b, \Omega\}_K$ generated by the Bose triple $\{a^*, a, \Omega\}_K$ in a certain proper subspace of the Bose representation space $K=\bigoplus_1^N \mathscr{L}^2(\boldsymbol{R}^3)\ni f$ (cf. [1]):

$$\begin{aligned} b(f)=:\exp\{-(a^*,a)\}\cdot\sum_{nm}\frac{1}{\sqrt{n!\,m!}}\int d\boldsymbol{p}_n\int d\boldsymbol{q}_m\times \\ \times\sum_{kl}^N f_{nm}^{kl}(\boldsymbol{p}_n,\boldsymbol{q}_m)\cdot a_{k_1}^*(p)\ldots a_{k_n}^*(p_n)\,a_{l_1}(q_1)\ldots a_{l_m}(q_m): \end{aligned} \tag{1}$$

where

$$(a^*,a)=\sum_k^N \int dp\, a_k^*(p)\,a_k(p) \tag{2}$$

and

$$f^{kl}_{nm}(p_n, q_m) = \sqrt{n+1}\,\delta_{m,1+n}\int dp_n'\int dr \sum_{k'}\sum_{l'} E^{kk'}_n(p_n, p_n')\cdot f_{l'}(r)\,E^{l'k'l}_{1+n}(r, p_n', q_{1+n}).$$

Here an integral kernel $E^{kk'}_n(p_n, p_n')$ represents a certain operator E_n in $\overset{W}{\underset{1}{\oplus}} K$, whose properties are specified exactly in [1]. $k=(k_1, \ldots, k_n)$ represents a sequence of indices.

Making use of (3.2) and (3.4) for $N=4$ in $K=\overset{N}{\underset{1}{\oplus}} \mathscr{L}^2(R^3)$ we can explicitly obtain a representation of the CAR algebra corresponding to spin 1/2 massive charge Fermi field constructed in the electromagnetic free field representation of the CCR algebra.

In this place let us mention that it is possible to calculate explicitly commutators between generators taken from appropriate Fermi and Bose representations. Through an easy (though tedious) calculation for spin 1/2 and spin 1 (photons) defining generators one can prove that the strong mutual locality condition for free quantum fields constructed in terms of such generators rather cannot be achieved. Detailed calculations can be found in [3].

5. In view of the previous results, which make it possible to construct an arbitrary Fock representation of the CAR algebra in any Fock representation of the CCR algebra, the question arises of the seemingly trivial notion of normal ordering for polynomials in generators taken from the Bose and Fermi algebras. It is enough to consider scalar representations $K=\mathscr{L}^2(R^3)$.

In the literature, the symbol $:\cdot:$ is understood to order creation operators left to annihilation ones, not specifying what kind of statistics these operators obey. We attempt to show that the specification is necessary to avoid confusion.

The Bose triple $\{a^*, a, \Omega\}_K$ acting in $\mathscr{D}\subset\mathscr{F}_B$ and the induced Fermi triple $\{b^*, b, \Omega\}_K$ are given. According to the conventional definitions, we introduce:

(1) $$:a(f)\,a(g)^*:_B = a(f)\,a(g)^* - (\bar{f}, g)\,\mathbf{1}_B = a(g)^*\,a(f),$$

(2) $$:b(f)\,b(g)^*:_F = b(f)\,b(g)^* - (\bar{f}, g)\,\mathbf{1}_F = -b(g)^*\,b(f).$$

On the other hand, $b(f)\,b(g)^*$ can be expressed by the formula (see in this connection [1, 2])

(3) $$b(f)\,b(g)^* = (\bar{f}, g)\,\mathbf{1}_F - :\exp\{-(a^*, a)\}\,f(a^*, a):_B,$$

where

(4) $$:\exp\{-(a^*, a)\}\,f(a^*, a):_B = b(g)^*\,b(f).$$

Concluding

(5) $$:b(f)\,b(g)^*:_B = (\bar{f}, g)\,\mathbf{1}_F - :\exp\{-(a^*, a)\}\,f(a^*, a):_B = b(f)\,b(g)^*.$$

Comparing this fact with (5.2) one can see that it is necessary to distinguish between the normal ordering for the Bose and the Fermi generators in our formalism.

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П. Гарбачевски, **Представления CAR генерированы в представленниях CCR.**

Содержание. Конструкция представления алгебры CAR в фоковском представлении алгебры CCR обобщена на случай пар представлений определяющих довольные физически допустимы свободные квантовые поля. Свобода связана здесь с внутренними (спин, заряд) степенями свободы теории.