ANNALS OF PHYSICS 150, 22–47 (1983)

Chiral Invariant Gross–Neveu Model: Classical versus Quantum

PIOTR GARBACZEWSKI*

School of Theoretical Physics, Dublin Institute for Advanced Studies, Dublin 4, Ireland

Received November 2, 1982

We present a unification of different and independently investigated aspects of the chiral invariant Gross–Neveu model. Special emphasis is placed on the relevance of classical (c-number, non Grassmann) spinor solutions of the G–N field equations for the construction, and thus understanding of the respective quantized Fermi model. To get an insight into the "quantum meaning of classical field theory" if specialized to the G–N case, we perform the path integral quantization procedure which first leads to the Fermi oscillator problem, and then, after appropriate generalizations, to the quantum Fermi G–N model. Path integrals are carried out with respect to c-number spinor paths only, and in fact no reference is necessary to the Grassmann algebra methods, which are conventionally used to integrate out fermions.

1. QUANTIZED MODEL, ITS LATTICIZATION AND ALL THAT

Let us consider the relativistic field theory model in $1+1$ dimensions describing the four-fermion interaction of the chiral ($\alpha = \pm 1$) N-component (colored, $a = 1, 2,\ldots, N$) field $\psi = \psi_{\alpha a}(x)$:

$$\begin{gathered} \mathscr{L} = \int dx\, (i\bar{\psi}\gamma^\mu\, \partial_\mu \psi + g[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2]), \\ [\psi_{\alpha a}(x), \psi^*_{\beta b}(y)]_+ = \delta_{\alpha\beta}\delta_{ab}\delta(x-y). \end{gathered} \tag{1.1}$$

It is known as the Nambu–Jona–Lasinio or the $SU(N)$ chiral invariant Gross–Neveu model, and its spectrum was found by using the Bethe Ansatz techniques in [1–4]. The Hamiltonian corresponding to (1.1),

$$H = \int dx\, (-i\psi^*_{+a}\, \partial_x \psi_{+a} + i\psi^*_{-a}\, \partial_x \psi_{-a} + 4g\psi^*_{+a}\psi^*_{-b}\psi_{+b}\psi_{-a}), \tag{1.2}$$

* Permanent address: Institute of Theoretical Physics, University of Wroclaw, Cybulskiego 36, 50-205 Wroclaw, Poland.

0003-4916/83 $7.50

solves the eigenvalue problem for vectors which are constructed from the Fock vacuum $|0\rangle$, $\psi_{a\alpha}(x)|0\rangle = 0 \ \forall \alpha, a, x$ as follows:

$$
\begin{aligned}
|k_1,..., k_n, \sigma_1,..., \sigma_n, F\rangle &= |k, \sigma, F\rangle \\
&= \int dx_1 \cdots \int dx_n \, \chi^{a_1 \cdots a_n}_{\alpha_1 \cdots \alpha_n}(x, k, \sigma, F) \\
&\times \psi^*_{\alpha_1 a_1}(x_1) \cdots \psi^*_{\alpha_n a_n}(x_n)\,|0\rangle \qquad (1.3) \\
\chi &= \sum_{\text{perm}}^{\pi} F_\pi \cdot \exp i \sum_{i=1}^{n} x_i k_{\pi(i)} \cdot \prod_{i=1}^{n} \delta_{\alpha_i \sigma_{\pi(i)}}.
\end{aligned}
$$

The coefficients $F_\pi = F(a_{\pi(1)},..., a_{\pi(n)})$ are supposed to describe the dependence of the wave function χ on isotopic color indices.

An explicit form of the eigenvectors can be found by making the apropriate regularizations of the problem: introducing an ultraviolet cutoff Λ and then solving the periodic problem on the finite space interval L. This route is equivalent to making the lattice approximation of the initially continuous model. Then, while having solved the lattice eigenvalue problem, one "fills the Dirac sea" and removes the cutoffs, thus arriving at physically relevant results in the continuum limit.

In this procedure, the latticization of (1.1) is rather indirect, since according to conventions one should reformulate (1.1) explicitly in terms of lattice fermions which is manifestly not the case in [1–5].

The lattice version of our eigenvalue problem comes through the formula

$$F_{id} = F(a_1,..., a_n) = P_{i,i+1} \cdot S_{i,i+1} \cdot F(a_1,..., a_{i+1}, a_i,..., a_n), \qquad (1.4)$$

where P_{ik} is the color exchange operator, while S_{ik} is the two-particle S-matrix, which acts on the color indices, while being explicitly dependent on chiralities only [4, 5]:

$$
\begin{aligned}
S_{kl} &= \begin{cases} [a(\lambda) + b(\lambda) P_{kl}] \exp i\lambda\varphi, & \lambda = \pm 2 \\ P_{kl}, & \lambda = 0, \end{cases} \\
a(\lambda) &= \frac{\lambda}{\lambda + ic}, \qquad a(\lambda) + b(\lambda) = 1, \qquad (1.5) \\
\varphi &= \operatorname{arctg} \frac{c}{2}, \qquad c = 4g(1 - g^2)^{-1}.
\end{aligned}
$$

The periodicity requirement results in the following eigenvalue problem:

$$
\begin{aligned}
\exp(ik_j L) \cdot F &= Z_j \cdot F, \qquad F = F_{id}, \quad \forall j = 1, 2,..., n, \\
Z_j &= S_{j+1,j} \cdots S_{n,j} S_{1,j} \cdots S_{j-1,j}
\end{aligned} \qquad (1.6)
$$

and both (1.4) and (1.5) have their relatives in the statistical physics investigations of [6–8] of many body problems in one space dimension with the repulsive delta function interaction [6, 7] and the generalized Heisenberg ferromagnet [8, 5].

Remark. The (semi) classical chiral invariant Gross–Nevey model is known to be a completely integrable system [9]. In the quantization of such systems (nonlinear Schroedinger, since Gordon, massive Thirring, Toda chain, *xyz* Heisenberg chain, etc.) as described in [10, 11] (see also [12, 13]), the starting point is the associated linear problem of the form $X\phi = 0$ which is then replaced by

$$\frac{\partial \phi}{\partial x} + Q(\lambda)\,\phi = 0, \tag{1.7}$$

where $X = X(\lambda)$ and λ is a spectral parameter. This linear problem allows us to introduce the so-called monodromy operator on the space interval $[-L, +L]$ such that

$$\phi(L) = T_L(\lambda)\,\phi(-L). \tag{1.8}$$

To determine $T_L(\lambda)$ for any specific nonlinear model to which (1.7) corresponds is the heart of the method. However, $T_L(\lambda)$ may be poorly definable in terms of quantum fields while on the continuum level. Therefore one makes an additional to the infrared regularization L, the ultraviolet one $\Lambda = L/\delta$. Then (1.7) is replaced by the discrete transfer problem

$$\phi_{n+1} = L_n(\lambda)\,\phi_n. \tag{1.9}$$

Appropriate L_n's were catalogued in [10] for the simplest models.

Warning: in the above we differentiate between "semiclassical models," which mean quantum mechanics of spinor systems, and "semiclassical quantization," which is related to the second quantization of the former first quantized model.

Let us consider the multi-component n particle quantum system. Its Hilbert space is $\mathscr{H}_n = h^{\otimes n}$, where $h = \bigoplus_{i=1}^{N} C_i$, $C_i = C\ \forall i$ is one dimensional. The Hamiltonian is

$$H = \varepsilon \sum_{k=1}^{n} P_{k,k+1}, \qquad \varepsilon = \pm 1, \tag{1.10}$$

provided we identify $n+1$ with 1. Here P_{kl} is the transposition operator interchanging the kth and lth factors in the tensor product $h^{\otimes n}$. The so defined generalized Heisenberg ferromagnet was investigated in [8] by means of the Bethe Ansatz. In [5] it was shown that the model obeys the quantum inverse scattering technique (complete integrability arising as a result). An algorithm to generate the eigenvectors and eigenvalues for (1.7) was completely described in [5].

The lattice transfer operator $L_n(\lambda)$ of (1.9) was introduced in [5] in the form

$$\begin{gathered} L_k(\lambda) = a(\lambda)\,I + b(\lambda)\,P_{k,k+1}, \\ a(\lambda) + b(\lambda) = 1, \qquad a(\lambda) = \lambda/(\lambda + i\varepsilon), \qquad \varepsilon = \pm 1. \end{gathered} \tag{1.11}$$

The transition operator $T_n(\lambda)=L_n(\lambda)\,L_{n-1}(\lambda)\cdots L_1(\lambda)$ has a trace $t(\lambda)=\mathrm{Sp}\,T_n(\lambda)$ with the property

$$i\left(\frac{d}{d\lambda}\right)\ln t(\lambda)\,t(0)^{-1}|_{\lambda=0}=\varepsilon\left(\sum_{k=0}^{n}P_{k,k+1}-n\right)=H \tag{1.12}$$

which in fact suggests a correction of the expression (1.10) for the Hamiltonian: otherwise the energy would diverge with $n\to\infty$.

At this point we can come back to the latticized Gross–Neveu model, since due to (1.5) and (1.6) a discrete transfer problem (1.9) is provided by the two-particle S-matrix:

$$\hat{L}_k(\lambda)=S_{k,k+1}(\lambda) \tag{1.13}$$

with $S_{kl}(\lambda)$ given by (1.5). The operator $\hat{L}_n$ of (1.13) differs from that given by (1.11) through the factor $\exp i\lambda\varphi$ only, which is manifestly not influencing the trace operation for $\hat{T}_n(\lambda)$. Consequently (1.13) corresponds to

$$\hat{T}_n(\lambda)=\exp(i\lambda\cdot\varphi\cdot n)\cdot L_n(\lambda)\cdots L_1(\lambda) \tag{1.14}$$

provided we introduce $c=4g(1-g^2)^{-1}$ in the place of $\varepsilon=\pm 1$ appearing in (1.11). Recall that $\varphi=\operatorname{arctg} c/2$. Then by virtue of (1.12) we arrive at

$$\hat{t}(\lambda)=\mathrm{Sp}\,\hat{T}_n(\lambda)=\exp(in\lambda\varphi)\,\mathrm{Sp}\,T_n(\lambda)=\exp(in\lambda\varphi)\cdot t(\lambda) \tag{1.15}$$

which implies

$$i\left(\frac{d}{d\lambda}\right)\ln\hat{t}(\lambda)\,\hat{t}(0)^{-1}\Big|_{\lambda=0}=c\left(\sum_{k=1}^{n}P_{k,k+1}-n\right)-n\varphi=H-N\mu, \tag{1.16}$$

where N is the particle number operator in the linear chain, $\mu=c(1+\varphi/c)$, and H is given by (1.12).

Needless to say, because of (1.16) the periodicity conditions derived by Sutherland in [8] need only a minor modification to be converted into those of the chiral Gross–Neveu model while following the Bethe Ansatz route, [1–5].

Lattice operator quantities like those in (1.16) should in principle allow a reconstruction in terms of fundamental fields of the model. In fact in the quantum version of the inverse scattering scheme the lattice operators $L_k(\lambda)$ are defined in terms of the appropriate lattice fields [10]. In the chiral Gross–Neveu case, the field content of the model as provided by (1.1) and (1.3) needs a Fock representation of the CAR algebra. With the cutoffs L, Λ implicit we can define lattice fermions as follows:

$$\psi_{\alpha a}(k)=\frac{1}{\sqrt{\delta}}\int_{R^1}\chi_k(x)\,\psi_{\alpha a}(x)\,dx$$
$$\chi_k(x)=1,\qquad x\in\Delta_k;\ 0\ \text{otherwise}, \tag{1.17}$$

where Δ_k is the kth interval (site of length δ in the sequence covering $L = n \cdot \delta$). With the notation $\psi_{+a}(k) = b_a(k)$, $\psi_{-a}(k) = d_a(k)$ instead of (1.1) we have

$$[b_a(k), b^*_{a'}(k')]_+ = \delta_{aa'}, \qquad \delta_{kk'} = [d_a(k), d^*_{a'}(k')]_+, \tag{1.18}$$

all other anticommutators vanishing, and

$$b_a(k)\,|0\rangle = 0 = d_a(k)\,|0\rangle, \qquad \forall\, a = 1, 2, ..., N; \quad k = 1, ..., n. \tag{1.19}$$

To investigate the color exchange mappings let us consider the following operators:

$$\begin{aligned} n_{ik} &= \sum_{a=1}^{N} b^*_a(i)\, b_a(k), \qquad \tilde{n}_{ik} = \sum_{a=1}^{N} d^*_a(i)\, d_a(k), \\ \hat{n}_{ik} &= \sum_{i=1}^{N} \{d^*_a(i)\, b_a(k) + b^*_a(i)\, d_a(k)\}. \end{aligned} \tag{1.20}$$

It is easy to check that $(1 - n_{ki} n_{ik})$ is the color exchange operator between species of the same (positive) chirality:

$$\begin{aligned} &\left|\begin{matrix} 1 \cdots n \\ a_1 \cdots a_n \end{matrix}\right) = b^*_{a_1}(1) \cdots b^*_{a_n}(n)\,|0\rangle, \\ n_{ki} \cdot n_{ik} &\left(\begin{matrix} 1 \cdots n \\ a_1 \cdots a_n \end{matrix}\right) = -\left|\begin{matrix} 1 \cdots i \cdots k \cdots n \\ a_1 \cdots a_k \cdots a_i \cdots a_n \end{matrix}\right) + \left|\begin{matrix} 1 \cdots n \\ a_1 \cdots a_n \end{matrix}\right), \end{aligned} \tag{1.21}$$

while $(1 - \hat{n}_{ki} n_{ik})$ realizes the color exchange between species of different chirality with the objection, however, that its domain must consist of the chiral invariant vectors. Since such a domain is of general interest for ours, we can define the color exchange operator as

$$P_{ik} = 1 - n_{ki} n_{ik} - \tilde{n}_{ki} \tilde{n}_{ik} - \hat{n}_{ki} \hat{n}_{ik}. \tag{1.22}$$

If one inserts (1.22) into (1.16) one receives the lattice approximation of the chiral invariant Gross–Neveu model in terms of the original (latticized) field variables.

Notice that three distinct color exchange operations included in (1.22) refer to the cases $\lambda = \pm 2, 0$, respectively, of the two-particle scattering (1.5). The number operator N of (1.16) reads

$$N = \sum_{a=1}^{N} \sum_{i=1}^{n} \{b^*_a(i)\, b_a(i) + d^*_a(i)\, d_a(i)\}. \tag{1.23}$$

Remark. A latticization procedure for Fermi models needs some care, since an energy doubling problem may arise if the gradient term is incorrectly translated to the lattice language. With respect to Fermi models of some relevance it is known, for example, that the spin 1/2 xyz Heisenberg model is quite correct lattice approximation of the massive Thirring model (see e.g., [12]). For the (non-chiral) $SU(N)$ Gross–Neveu model two inequivalent lattice approximations were proposed [14, 15]. However, the approximation of [14], while approaching continuum, proves to be reliable for lowest excitations only, and one does not have full control of the limiting procedure. The construction of [15] involves the additional to Fermi, Bose degrees of freedom. Their contribution is shown to vanish in the continuum limit, then allowing one to reproduce the well-known semiclassical spectrum of the model. Nevertheless, this approximation is not purely fermionic.

In our chiral Gross–Neveu case we arrived at the lattice approximation which involves the original Fermi field degrees of freedom only. Let us mention that a reasonable lattice approximation in terms of purely fermionic variables may not exist for more sophisticated Fermi models, especially in $1+3$ dimensions. In such a case the lattice approximants should either involve the additional to Fermi, Bose degrees of freedom, or admit a purely Bose approximant. For some of the simplest examples it is now known that in the appropriate continuum/scaling limit the initially Bose model can suffer a "metamorphosis into fermion" according to the Bose → Fermi metamorphosis prescriptions investigated in [12, 13] and appearing to come into play on the more general footing, by virtue of [16].

2. "Bosonization" and the Classics–Quanta Relationship

The Bethe Ansatz solution of the eigenvalue problem for the chiral invariant Gross–Neveu model requires that basic quantities (observables, eigenvectors, etc.) are constructed in terms of the generators of a Fock representation of the CAR algebra, with the number $2N$ of internal degrees of freedom, N being the number of colors.

As proved in [17], such a representation can always be "bosonized," i.e., reconstructed in terms of the CCR algebra (Bose) generators with the number of internal degrees of freedom kept the same for bosons and fermions. On the other hand, one knows (see [18], also [19]) that the chiral Gross–Neveu fermion can be written in the canonical Bose form as

$$\psi_i(x) = K_i \left(\frac{m}{2\pi}\right)^{1/2} \left(\exp\frac{\pi}{4}\gamma^5\right) \cdot \exp\left\{-i\left(\frac{\pi}{N}\right)^{1/2} \left[\gamma^5\phi(x) + \int_x^{\infty} dz\, \dot{\phi}(z)\right.\right.$$
$$\left.\left. - i\sqrt{\pi}\left[\gamma^5\phi_i(x) + \int_x^{\infty} dz\, \dot{\phi}_i(z)\right]\right\}, \tag{2.1}$$
$$\sum_{i=1}^{N} \phi_i(x) = 0, \qquad i = 1, 2, ..., N,$$

where K_i is the Klein factor necessary to guarrantee the anticommutativity of ψ_i's at distinct space points. Since the formula (2.1) preserves its validity inthe interaction picture where free fields only enter the above expressions (compare the free field correspondences of [19]), we find that effectively the number of $2N$ (internal) Fermi degrees of freedom is described in terms of N (internal) Bose degrees of freedom. At first glance it is in manifest contradiction with our previous "bosonization" statement based on [17]. However, this is not the case. In [20] we have demonstrated that in the "bosonization" of Fermi models if one imposes the positivity requirement on the involved Bose Hamiltonian, then the number of the internal degrees if freedom is (effectively) diminished by one-half. This means that (2.1) is a correct expression on the physical domain. Recall, however, that the Bethe states (1.3) were by the construction outside of the physical domain.

If $a^*_{\alpha a}(k)$, $a_{\alpha a}(k)$, $\alpha = \pm$, $a = 1, \ldots, N$, are the CAR algebra generators for (1.1)–(1.3) the "bosonization" of [17] allows us to identify them in the CCR algebra, whose (Bose) generators we denote $A^*_{\alpha a}(k)$, $A_{\alpha a}(k)$. For any operator quantity of the chiral invariant Gross–Neveu model $\hat{F}$ we have then the expression (power series expansion in terms of normal products)

$$\hat{F} = F(a^*, a) = F(a^*, a)[A^*, A] \equiv F[A^*, A]. \tag{2.2}$$

Then for any element $\hat{F}$ of the Bose (CCR) algebra we can follow the standard coherent state expectation value argument in the tree (zero loop) approximation:

$$\begin{aligned} &A_{\alpha a}(k)\,|\varphi) = \varphi_{\alpha a}(k)\,|\varphi), \\ &(\varphi|\,{:}\hat{F}{:}\,|\varphi) = (\varphi|\,{:}F[A^*, A]{:}\,|\varphi) = (0|\,{:}F[A^* + \bar{\varphi}, A + \varphi]{:}\,|0) \\ &\qquad = F[\bar{\varphi}, \varphi] = F_{\mathrm{cl}}. \end{aligned} \tag{2.3}$$

It allows us to attribute the classical c-number images to the quantum objects. In this number, the procedure (2.3) allows us to map the quantum equations of motion into the corresponding classical (Euler) equations. For Fermi spinor systems, upon the "bosonization" we can thus follow the well known for Bose systems tree approximation routes (see [20–22]). Be aware, however, that we use the tree approximation concept for "bosonized" fermions without specifying the parameter which should be let equal zero (as the Planck constant h is used). The Planck constant is inappropriate in this respect.

With (2.2), (2.3) and the "bosonization"/tree approximation recipe for studying fermions, we arrive at the problem first seriously discussed in [23, 24, 9] in connection with the family of Gross–Neveu-type models. Namely, that of the relevance of classical solutions of the nonlinear spinor field equations for the understanding of the related quantum Fermi systems (see, e.g., [25]).

As is well known, for boson fields the classical solutions of the field equations provide first (zero loop) approximations to the properties of the quantum system. For Fermi models the situation is not so clear, though some indications can be drawn from [9, 23–25]. However, the quantization-of-spinor-fields series [26, 20, 21] supplemented by [13] sheds a new light on this problem as summarized by (2.2), (2.3).

3. Classical Fields of the Chiral Invariant Gross–Neveu Model

Now we shall consider (1.1) as describing the (semi) classical system, which means that the entering fields are not operators but c-number (commuting ring) functions. We replace the coupling constant g of (1.1) by $g^2/2$, and notice that the model admits an equivalent description in terms of [23, 9]:

$$L = \bar{\psi}[i\gamma^{\mu}\partial_{\mu} - g(\sigma + i\pi\gamma_5)]\,\psi \tag{3.1}$$

provided the auxiliary fields σ, π satisfy the equations of motion:

$$\sigma = -g\bar{\psi}\psi, \qquad \pi = -ig\bar{\psi}\gamma_5\psi \Rightarrow L \to L' = L - \tfrac{1}{2}(\sigma^2 + \pi^2). \tag{3.2}$$

The Euler equations of motion for (3.1) describe N massless Dirac fields in $1 + 1$ dimensions in the external potential:

$$[i\gamma^{\mu}\partial_{\mu} - g(\sigma + i\pi\gamma_5)]\,\psi_i = 0, \qquad \forall\, i = 1, 2, \ldots, N, \tag{3.3}$$

to which the constraints (3.2) do apply.

A thorough study of the system (3.2), (3.3) is given in [9] by using the inverse scattering techniques, for the cases $N = 1$ which is the free field case, and the nontrivial $N = 2$ case.

At this point let us assume that spinors ϕ_1, ϕ_2 are solutions of the $N = 2$ system. Let us define the following sequence of spinors:

$$\begin{gathered} \psi_1 = b_1\phi_1, \qquad \psi_2 = b_2\phi_2, \ldots, \qquad \psi_{2k-1} = b_{2k-1}\phi_1, \qquad \psi_{2k} = b_{2k}\phi_2, \\ N = 2k, \qquad k = 1, 2, \ldots, n, \end{gathered} \tag{3.4}$$

where b_i's are real numbers. Observe that (3.4) reads

$$\sigma = -g\sum_{i=1}^{2k} \bar{\psi}_i\psi_i = -g\left[\left(\sum_{k=1}^{n} b_{2k-1}^2\right)\bar{\phi}_1\phi_1 + \left(\sum_{k=1}^{n} b_{2k}^2\right)\bar{\phi}_2\phi_2\right], \tag{3.5}$$

i.e., if we demand that real coefficients b_i satisfy

$$\sum_{k=1}^{n} b_{2k-1}^2 = 1 = \sum_{k=1}^{n} b_{2k}^2 \tag{3.6}$$

the solution of the $N = 2k$ problem with k an integer reduces to the $SU(2)$ one. For the $N = 2$ model, the upper and lower components of Gross–Neveu spinors are completely determined by Jost functions of the related linear problem (see (6.46) in [9]), and the explicit solutions for one soliton case were checked to lead to the formulas for σ, π which were previously obtained by Shei in [24]:

$$\begin{aligned} \sigma &= \frac{m}{g} - \frac{2k^2}{m}\,\frac{1}{1 + \exp[2k(x - x_0)]}, \\ \pi &= \frac{2k(m^2 - k^2)}{mg}\,\frac{1}{1 + \exp[2k(x - x_0)]}, \end{aligned} \tag{3.7}$$

with m, k being (arbitrary) integration constants, m being related to the boundary conditions at space infinity.

In the notation of [9] the formulas (3.7) follow from

$$\psi_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}, \qquad i = 1, 2, \ldots, N,$$
$$\phi = \sigma + i\pi = u^* \cdot v, \qquad \phi^* = \sigma - i\pi = v^* u, \tag{3.8}$$

so that

$$[i\partial\!\!\!/ - (\sigma + i\pi\gamma^5)]\, \psi_i = 0,$$
$$\sigma = \frac{1}{2} \sum_{i=1}^{N} \bar{\psi}_i \psi_i, \qquad \pi = \frac{i}{2} \sum_{i=1}^{N} \bar{\psi}_i \gamma^5 \psi_i, \tag{3.9}$$

where $\xi = \frac{1}{2}(t - x)$, $\eta = \frac{1}{2}(t + x)$, are replaced by

$$iu_{,\xi} = \phi^* v, \qquad iv_{,\eta} = \phi \cdot u. \tag{3.10}$$

By solving (3.10) one arrives at analytic expression for upper and lower components of the Gross–Neveu spinors in terms of the Jost functions of the related linear problem. These functions satisfy

$$\Psi = \Psi(\zeta, \eta, \xi) = \Psi(\zeta)_{|\zeta = \gamma/2} \xrightarrow[\eta \to -\infty]{} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-i\gamma\eta/2),$$
$$\Psi_1 \Psi_1^* + \Psi_2 \Psi_2^* = 1, \qquad \gamma = u^* u, \tag{3.11}$$

space-time dependence being suppressed, and lead to the following expressions for the Gross–Neveu spinors:

$$\psi_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \tag{3.12}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \sqrt{\gamma} \begin{pmatrix} \Psi_1\left(\frac{\gamma}{2}\right) \\ \Psi_2^*\left(\frac{\gamma}{2}\right) \end{pmatrix} \exp\left(-\frac{i}{2}\gamma\eta\right),$$
$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\sqrt{\gamma}} \begin{pmatrix} \Psi_1\left(\frac{\gamma}{2}\right) \bar{\Psi}_1^*(0) + \Psi_2\left(\frac{\gamma}{2}\right) \Psi_2^*(0) \\ -\Psi_1^*\left(\frac{\gamma}{2}\right) \Psi_2^*(0) + \Psi_2^*\left(\frac{\gamma}{2}\right) \Psi_1^*(0) \end{pmatrix} \exp\left(-\frac{i}{2}\gamma\eta\right), \tag{3.13}$$

so that

$$\sigma + i\pi = u^* v = \Psi_1(0),$$
$$\sigma - i\pi = v^* u = \Psi_1(0) + \Psi_1^*\left(\frac{\gamma}{2}\right)\Psi_1\left(\frac{\gamma}{2}\right)|\Psi_2(0) - \Psi_2^*(0)|, \tag{3.14}$$

the space-time dependence being suppressed again.

4. Quantum Meaning of Classical Field Theory: First Encounter

Let us recall that for the massive Thirring model, we have demonstrated in [13] that the (semi) classical model, which is known to be a completely integrable system, allows a consistent quantization in the quantum inverse scattering formalism. The resulting model is the Bose one, nevertheless, it is equivalent to the conventional Fermi one. We conjecture in [13] that the Bose MT should in principle allow a reconstruction as the reducible Fermi MT. A relationship of the latter with the (semi) classical MT was established in [21, 20] by using the "bosonization"/tree approximation recipe.

For the massive (partially broken chiral invariance) Gross–Neveu model [27, 23] there is no doubt that in the semiclassical quantization procedure, after integrating out fermions (see the DHN recipe of [28]), the stationary phase method, if applied to the effective action, leads to the solution

$$\psi_j = b_j \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \end{pmatrix} \exp(-iEt), \qquad \sum_{j=0}^{N} b_j^2 = 1,$$
$$R^2 = \frac{2(m-E)}{g^2}\frac{1}{\cosh^2\beta x\cdot(1+\alpha^2\tanh^2\beta x)},$$
$$\varphi = \tan^{-1}(\alpha\tanh\beta x), \tag{4.1}$$
$$\alpha = \left(\frac{m-E}{m+E}\right)^{1/2}, \qquad \beta = (m^2 - E^2)^{1/2},$$

with

$$-g\bar\psi\psi = \sigma = -\frac{2(m-E)}{g}\frac{1-\alpha^2\tanh^2\beta x}{\cosh^2\beta x\cdot(1+\alpha^2\tanh^2\beta x)},$$
$$-ig\bar\psi\gamma_5\psi = \pi = -\frac{4(m-E)\,\alpha}{g}\frac{\tanh\beta x}{\cosh^2\beta x\cdot(1+\alpha^2\tanh^2\beta x)^2}. \tag{4.2}$$

For the chiral invariant G–N model, we know that (3.7) is also a solution of the stationarity condition in the semiclassical quantization procedure of [24].

For the $N=2$ (nonchiral) Gross–Neveu model, the complete integrability was proved in [9], and the fields occurring in the associated linear problem are simply related to the fundamental spinor fields (c-number ones) $\psi_i, \bar{\psi}_i, i=1, 2$.

The solution for the bilinear σ derived in [9], formula (5.18), is precisely the DHN solution of [28] obtained via the stationarity condition for the effective action in the semiclassical quantization of the model. A generalization of this observation to $N>2$ cases is immediate by methods analogous to (3.5), (3.6).

Coming back to the chiral invariant Gross–Neveu model, let us recall that the starting point for the semiclassical quantization procedure is the functional integral

$$\operatorname{tr} \exp(-iHT) = \int [d\psi][d\bar{\psi}][d\sigma][d\pi] \exp\left[i \int_0^T dt \int_{-\infty}^{+\infty} dx\, \mathscr{L}(\sigma, \pi, \psi, \bar{\psi})\right], \tag{4.3}$$

where $\mathscr{L}(\sigma, \pi, \psi, \bar{\psi})$ is given by

$$\mathscr{L} = \bar{\psi}[i\partial\!\!\!/ - g(\sigma + i\pi\gamma_5)]\, \psi - \tfrac{1}{2}(\sigma^2 + \pi^2) \tag{4.4}$$

and all fields σ, π, ψ, $\bar{\psi}$ are viewed as independent: σ, π belonging to the commuting ring, while ψ, $\bar{\psi}$ being elements of the Grassmann algebra and satisfying, in addition

$$\begin{aligned} \psi(t+T) &= -\psi(t), & \bar{\psi}(t+T) &= -\bar{\psi}(t), \\ \sigma(t+T) &= \sigma(t), & \pi(t+T) &= \pi(t). \end{aligned} \tag{4.5}$$

The integration with respect to ψ, $\bar{\psi}$ leads to

$$\begin{aligned} \operatorname{tr} \exp(-iHT) &= \int [d\sigma][d\pi]\, I_F[\sigma, \pi] \cdot \exp\left\{ -\frac{i}{2} \int_0^T \int_{-\infty}^{+\infty} dt\, dx\, [\sigma^2 + \pi^2] \right\} \\ &:= \int [d\sigma][d\pi] \exp iS_{\text{eff}}[\sigma, \pi], \end{aligned} \tag{4.6}$$

where the Fermi $[d\psi][d\bar{\psi}]$ integral gives

$$\begin{aligned} I_F[\sigma, \pi] &= \int [d\psi][d\bar{\psi}] \exp\left\{ i \int_0^T dt \int_{-\infty}^{+\infty} dx\, \bar{\psi}[i\partial\!\!\!/ - g(\sigma + i\pi\gamma_5)]\, \psi \right\} \\ &= \left\{ \exp\left(\frac{i}{2} \sum_k |\alpha_k|\right) \prod_k (1 + \exp[-i\,|\alpha_k|]) \right\}^N \\ \alpha_k &= i \ln \frac{\psi_k(x, t+T)}{\psi_k(x, t)}, \end{aligned} \tag{4.7}$$

N being the number of color degrees, $k = 1, 2, \ldots$.

In the above ψ_k is the (conventional c-number) classical solution of the Dirac equation

$$[i\partial\!\!\!/ - g(\sigma + i\pi\gamma_5)]\, \psi_k = 0, \qquad \forall\, k = 1, 2, \ldots. \tag{4.8}$$

Since the action appearing in (4.6) is divergent one must make a subtraction of the vacuum self-energy, and then to renormalize the $\sigma^2 + \pi^2$ term. After these steps, the stationarity condition for the effective action $\delta S/\delta\sigma = 0 = \delta S/\delta\pi$ is found to be satisfied by the time-independent solution $\sigma = \sigma(x)$ given by (3.7). This result seems to be rather striking since although we have in principle integrated out fermions, further basic steps do involve classical (c-number) spinor relatives of Fermi fields. This allows us to suspect that the Grassmann integration procedure in a hidden way gives account of the classical constraints (3.2).

5. The Fermi Oscillator Problem

The derivation of the formula (4.7) is based on the formalism described in Appendix A of [28], aiming at the calculation of tr exp($-iHT$) for quantized Fermi systems by means of the path integral-like formula. The starting point is a simple Fermi oscillator defined by the Lagrangian:

$$\hat{L} = ia^*(t)\,\dot{a}(t) - \omega a^*(t)\,a(t) = a^*(t)\left[i\frac{d}{dt} - \omega\right] a(t). \tag{5.1}$$

Since this system has two energy levels separated by the interval ω, we may introduce them as

$$\varepsilon_0 = E_0 - \frac{\omega}{2}, \qquad \varepsilon_1 = E_0 + \frac{\omega}{2}, \tag{5.2}$$

with E_0 being arbitrary. Then, obviously,

$$\operatorname{tr}\exp(-iHT) = e^{-iE_0T}(e^{i\omega/2} + e^{-i\omega/2}) = 2e^{-iE_0T}\cos\frac{\omega T}{2}. \tag{5.3}$$

According to the formal rules applicable to Grassmann algebra elements [30, 31], we can arrive at the same result:

$$\begin{aligned}\operatorname{tr}\exp(-iHT) &= \int [da][da^*]\exp\left(i\int_0^T L_F\,dt\right)\\ &= 2e^{-iE_0T}\frac{\det(i\,d/dt - \omega)}{\det(i\,d/dt)} = 2e^{-iE_0T}\prod_{n=-\infty}^{+\infty}\frac{\varepsilon_n(\omega)}{\varepsilon_n(0)},\end{aligned} \tag{5.4}$$

where one assumes the normalization

$$\operatorname{tr}\exp(-iHT)_{|\omega=0} = 2e^{-iE_0T} \tag{5.5}$$

and solves the eigenvalue problem

$$\left(i\frac{d}{dt} - \omega\right) f_n(t) = \varepsilon_n(\omega) f_n(t), \tag{5.6}$$

which for the antiperiodic boundary condition

$$f_n(t+T) = -f_n(t) \tag{5.7}$$

implies

$$\varepsilon_n(\omega) = -\frac{(2n+1)\pi}{T} - \omega, \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{5.8}$$

and this in turn yields

$$\prod_{n=-\infty}^{+\infty} \frac{\varepsilon_n(\omega)}{\varepsilon_n(0)} = \prod_{n=-\infty}^{+\infty} \left(1 + \frac{\omega T}{(2n+1)\pi}\right) = \cos\frac{\omega T}{2}. \tag{5.9}$$

It is worth noting [31] that if the integration (5.4) had been done with $a^*(t)$, $a(t)$ as the c-number complex functions, the answer so obtained would be exactly the inverse of (5.4): one must take care of imposing the antiperiodic boundary conditions in this case as well.

Remark. In any theory in which Fermi fields enter the action through bilinear terms (or factors) one has a workable procedure of integrating over the Fermi (Grassmann algebra) variables (see, e.g., [30–32]). Suppose we have an action

$$S = -\int dt \int dx\, \bar{\psi} A \psi, \tag{5.10}$$

where ψ, $\bar{\psi}$ are either commuting or anticommuting functions, and A is some linear operator. We assume to have two complete orthonormal sets of functions (c-number ones) $\{\psi_r\}$, $\{\bar{\psi}_r\}$ such that

$$\int dt \int dx\, \bar{\psi}_r \psi_s = \delta_{rs}. \tag{5.11}$$

Then, upon making the expansions

$$\psi = \sum_r a_r \psi_r, \tag{5.12}$$

$$\bar{\psi} = \sum_r a_r^* \bar{\psi}_r, \tag{5.13}$$

we may replace ψ, $\bar{\psi}$ by a, a^*, which are again either commuting or anticommuting objects. The path integration measure in

$$\int [d\psi][d\bar{\psi}] \exp(-S) \tag{5.14}$$

can be replaced by $\prod_r da_r\, da_r^*$, and if to assume that ψ_r's are eigenfunctions of the operator iA, we arrive at the following result:

$$I = \int [d\psi][d\bar{\psi}] \exp(-S) = \int [da][da^*] \exp\left(-\sum_r \lambda_r a_r^* a_r\right). \tag{5.15}$$

The final result now depends on whether we work with commuting or anticommuting fields. In the commuting case we get

$$I_B = \prod_r \lambda_r^{-1} = (\det A)^{-1}, \tag{5.16}$$

while in the anticommuting case the answer is

$$I_F = \prod_r \lambda_r = (\det A). \tag{5.17}$$

Except for this difference between Fermi and Bose cases, the additional ones may come into play because of the difference in general choices of the (anti) periodicity conditions.

Let us now come back to the Fermi oscillator problem, which is defined by using the CAR algebra generators

$$[a, a^*]_+ = 1_F, \qquad a\,|0) = 0, \qquad a^*\,|0) = |1), \qquad a^{*2} = 0 = a^2, \tag{5.18}$$

which determine the Fock representation. Such a representation has an apparent embedding in the Fock representation of the CCR algebra as follows: take generators of the CCR algebra (the Schroedinger representation is a good example),

$$[b, b^*]_- = 1_B, \qquad b\,|0) = 0, \qquad \frac{1}{\sqrt{n!}}\, b^{*n}\,|0) = |n), \tag{5.19}$$

and introduce the operators

$$a^* = b^* :\exp(-b^*b):, \qquad a = :\exp(-b^*b):\, b. \tag{5.20}$$

These operators, though defined in the whole representation (Hilbert) space for (5.19), act nontrivially (e.g., without annihilating any other vector than the Fock vacuum $|0)$) on a proper subspace h_F of h spanned by two vectors $|0)$ and $|1)$. We have

$$[a, a^*]_+ = :\exp(-b^*b): + b^* :\exp(-b^*b):\, b = 1_F, \tag{5.21}$$

where 1_F is a projection in $h: h_F = 1_F h$.

Let us notice that the Hamiltonian $\hat{h}_F$ for the Fermi oscillator in the representation (5.20) reads

$$\hat{h}_F = \omega a^* a = \omega b^* :\exp(-b^*b):\, b, \tag{5.22}$$

while this for the Bose oscillator has the form

$$\hat{h}_B = \omega b^* b, \qquad \hat{L}_B = b^* \left\{ i \frac{d}{dt} - \omega \right\} b. \tag{5.23}$$

Let furthermore $|\beta)$ be a coherent (Bose oscillator) state for the Fock representation of the CCR algebra

$$|\beta) = \exp(\beta b^* - \beta^* b) \, |0). \tag{5.24}$$

Any bounded operator

$$\hat{B} = \sum_{nm} K_{nm} b^{*n} b^m \tag{5.25}$$

has its normal symbol [32] given by

$$(\beta| \, \hat{B} \, |\beta) = B = \sum_{nm} K_{nm} \beta^{*n} \beta^m. \tag{5.26}$$

Then its functional representative (e.g., kernel) reads

$$\hat{B}[\beta^*, \beta] = B \cdot \exp \beta^* \beta. \tag{5.27}$$

The kernel of the infinitesimal operator $\hat{U}(\Delta t)$ is of main importance in the derivation [30] of the path integral formula for $\operatorname{tr} \exp(-iHT)$. We have

$$\begin{gathered} \hat{U}_B[\beta^*, \beta](\Delta t) = \exp(\beta^* \beta - i h_B^{\mathrm{cl}} \, \Delta t), \\ h_B^{\mathrm{cl}} = (\beta| \, \hat{h}_B \, |\beta) = \omega \beta^* \beta, \end{gathered} \tag{5.28}$$

so that the (formal continuum limit) path integral representation of $\operatorname{tr} \exp(-i\hat{h}_B T)$ reads [30]

$$\begin{aligned} I_B &= \operatorname{tr} \exp(-i\hat{h}_B T) \\ &= \int [d\beta][d\beta^*] \exp i \int_0^T \{ i\beta^*(t) \, \dot{\beta}(t) - \omega \beta^*(t) \, \beta(t) \} \, dt \\ &= \int [d\beta][d\beta^*] \exp i \int_0^T L_B(t) \, dt, \end{aligned} \tag{5.29}$$

with the accuracy to the normalization factor which reflects the choice of the boundary conditions.

Because the Fermi propagator $\hat{U}_F(\Delta t)$ can be represented in the Hilbert space of the Bose oscillator, we can follow step by step the just described route. Let us notice that

$$\begin{aligned} \hat{U}_F(\Delta t) &= \exp(-i\hat{h}_F \, \Delta t) \cong 1_F - i\hat{h}_F \, \Delta t \\ &= :\!\exp(-b^* b)\!: + \, b^* :\!\exp(-b^* b)\!: b - i\omega b^* :\!\exp(-b^* b)\!: b, \end{aligned} \tag{5.30}$$

so that the normal symbol for $\hat{U}_F(\Delta t)$ reads

$$(\beta|\, \hat{U}_F(\Delta t)\, |\beta) \cong \exp(-\beta^*\beta) + \beta^*\beta \exp(-\beta^*\beta) - i\omega \cdot \Delta t \cdot \beta^*\beta \cdot \exp(-\beta^*\beta), \tag{5.31}$$

and consequently the infinitesimal kernel is

$$\begin{aligned}\hat{U}_F[\beta^*, \beta](\Delta t) &\cong 1 + \beta^*\beta - i\omega\beta^*\beta\, \Delta t \\ &= (1 + \beta^*\beta)\left(1 - i\omega\, \Delta t \frac{\beta^*\beta}{1 + \beta^*\beta}\right) \\ &\cong (1 + \beta^*\beta) \cdot \exp\left(-i\omega\, \Delta t \frac{\beta^*\beta}{1 + \beta^*\beta}\right) \\ &= \exp \ln(1 + \beta^*\beta) \cdot \exp\left(-i\omega\, \Delta t \frac{\beta^*\beta}{1 + \beta^*\beta}\right),\end{aligned} \tag{5.32}$$

where $\ln(1 + \beta^*\beta)$ replaces the $\beta^*\beta$ term of (5.28), while $\omega(\beta^*\beta/1 + \beta^*\beta)$ appears instead of $\omega\beta^*\beta$.

The formal continuum expression for $\operatorname{tr} \exp(-i\hat{h}_F T)$ evaluated according to the Bose oscillator recipe of [30] reads

$$\begin{aligned}I_F = \operatorname{tr} \exp(-i\hat{h}_F T) &= \int [d\beta][d\beta^*] \exp i \int_0^T \frac{i\beta^*\dot{\beta} - \omega\beta^*\beta}{1 + \beta^*\beta}\, dt \\ &= \int [d\beta][d\beta^*] \exp i \int_0^T dt \frac{L_B(t)}{1 + \beta^*(t)\,\beta(t)} = \int [d\beta][d\beta^*] \exp iS_F(\beta^*, \beta).\end{aligned} \tag{5.33}$$

It is a c-number alternative for the usual Grassmann algebra path integral formula, which, though not of a comparable calculational simplicity, does involve integrations with respect to the conventional c-number paths only. Let us stress that the crucial difference between the Bose oscillator formula (5.29) and (5.33) lies in the appearence of the "damping" factor $1/(1 + \beta^*\beta)$ in the otherwise oscillator action. It, however, implies that the only set of paths $\beta^*(t)$, $\beta(t)$ which give comparable contributions to both I_F and I_B consists of

(1) solutions of the equation $(i\, d/dt - \omega)\, \beta = 0$,

(2) all paths constrained to obey the restriction $|\beta^*\beta| \ll 1$.

This is the sense in which we find it reasonable to talk about the relevance of the classical c-number problem for the construction of its quantized Fermi partner: the oscillator c-number problem does indeed manifestly contribute to the Fermi oscillator transition amplitudes.

It is not useless to mention that the stationarity demand for I_F if imposed in the form

$$\frac{\delta S_F(\beta^*, \beta)}{\delta\beta^*} = 0 \tag{5.34}$$

results in the equation

$$(1+\beta^*\beta)\left(i\frac{d}{dt}-\omega\right)\beta=\beta^*\beta\left(i\frac{d}{dt}-\omega\right), \tag{5.35}$$

which in turn implies the conventional oscillator equation of motion

$$\left(i\frac{d}{dt}-\omega\right)\beta=0. \tag{5.36}$$

6. Quantum Meaning of Classical Field Theory: Chiral Invariant Gross–Neveu Model

The above analysis of the Fermi oscillator problem opens a possible line of attack against anold problem of whether classical solutions to c-number spinor field equations have any relevance for the construction of their quantized Fermi parters, ([23, 24], see also [25, 16, 26, 13]).

Let us recall basic features of the oscillator example:

$$\begin{gathered} h_B\,|1)=h_F\,|1),\qquad h_B\,|0)=h_F\,|0),\qquad h_F=1_F h_B 1_F,\\ [h_B,1_F]_-=0,\qquad \mathscr{H}_F=1_F\mathscr{H}_B=L(|0),|1)), \end{gathered} \tag{6.1}$$

with 1_F given by (5.21).

Coming back to the Fermi quantized CGN problem, (1.1) and (1.2), let us observe that the eigenvalue equation for $H=H_F$ with eigenvectors of the form [1]

$$|F,\xi)=\int)dx_1\cdots\int dx_n\sum_{\{\alpha,a\}}F(x_1,\ldots,x_n,\alpha_n,\ldots,\alpha_n)\,\xi(a_n,\ldots,a_n)\prod_1^n\psi^*_{a_i\alpha_i}(x_i)\,|0) \tag{6.2}$$

is solved immediately if $F(x,\alpha)$ is an eigenfunction of the n-particle Hamiltonian

$$h=-i\sum_{j=1}^n\alpha_j\partial_j-4g\sum_{ij}\delta(x_i-x_j)\,P^{ij}[\tfrac{1}{2}(1-\alpha_i\alpha_j)], \tag{6.3}$$

P_{ij} being an operator interchanging chiralities α_i and α_j. The eigenvalue problem for h is received after applying H_F to $|F,\xi)$, commuting all the operators through the product of ψ^*'s to $|0)$, and then inegrating by parts the kinetic term. Now suppose that instead of the Fermi Hamiltonian (1.2) we consider the Bose Hamiltonian:

$$\begin{gathered} H_B=H_F(\psi^*\to\phi^*,\psi\to\phi),\\ [\phi_{\alpha a}(x),\phi^*_{\beta b}(y)]_-=\delta_{\alpha\beta}\delta_{ab}\delta(x-y),\\ [\phi_{\alpha a}(x),\phi_{\beta b}(y)]_-=0,\qquad \phi_{\alpha a}(x)\,|0)=0,\qquad \forall\alpha,a,x, \end{gathered} \tag{6.4}$$

i.e., the one with the Fermi functional form but Bose operators appearing instead of the Fermi ones.

Let us furthermore introduce the "bosonized" Fermi fields of the CGN model as follows [17]:

$$\psi_{\alpha a}(x)=\sum_n \frac{\sqrt{1+n}}{n!} \sum_{b_1\cdots b_n} \sum_{\beta_1\cdots\beta_n} \int dy_1 \cdots \int dy_n\, \sigma(y_1,\beta_1,b_1,\ldots,y_n,\beta_n,b_n)$$
$$\times\, \sigma(x,\alpha,a,y_1,\beta_1,b_1,\ldots,y_n,\beta_n,b_n)\cdot \phi^*_{\beta_1 b_1}(y_1)\cdots\phi^*_{\beta_n b_n}(y_n)$$
$$\times :\exp\left\{-\sum_{b=1}^{N}\sum_{\beta=1}^{2}\int \phi^*_{\beta b}(z)\,\phi_{\beta b}(z)\,dz\right\}: \phi_{\alpha a}(x)\,\phi_{\beta_1 b_1}(y_1)\cdots\phi_{\beta_n b_n}(y_n), \quad (6.5)$$

where an antisymmetric function

$$\sigma(x_1,\alpha_1,a_1,\ldots,x_n,\alpha_n,a_n)=\sigma^3(x_1,\alpha_1,a_1,\ldots,x_n,\alpha_n,a_n),$$
$$\sigma(\cdots x_i\alpha_i a_i\cdots x_j\alpha_j a_j\cdots)=-\sigma(\cdots x_j\alpha_j a_j\cdots x_i\alpha_i a_i\cdots), \quad (6.6)$$

we define as follows

$$\sigma=\sigma(x_1,\alpha_1 a_1,\ldots,x_n,\alpha_n,a_n)=\prod_{1\leqslant j<k\leqslant n} p_{jk}, \quad (6.7)$$
$$p_{jk}=\delta_{\alpha_j\alpha_k}\delta_{a_j a_k}[\Theta(x_j-x_k)-\Theta(x_k-x_j)]+\delta_{a_j a_k}\Theta(|\alpha_j-\alpha_k|)(-1)^{1+\Theta(x_j-x_k)}$$
$$+(1-\delta_{\alpha_j\alpha_k})(1-\delta_{a_j a_k})(-1)^{\Theta(x_j-x_k)}+\delta_{\alpha_j\alpha_k}\cdot\Theta(|a_j-a_k|)(-1)^{\Theta(x_j-x_k)}.$$

Note that

$$\sigma^{2n+1}=\sigma, \qquad \forall n=0,1,2,\ldots, \quad (6.8)$$

and that at $x_j=x_k$, σ is symmetric with respect to an interchange $\alpha_j\leftrightarrow\alpha_k$:

$$p_{jk}(x_j=x_k)=\delta_{a_j a_k}\Theta(|\alpha_j-\alpha_k|)-\delta_{\alpha_j\alpha_k}\Theta(|a_j-a_k|)-(1-\delta_{\alpha_j\alpha_k})(1-\delta_{a_j a_k}). \quad (6.9)$$

It is not difficult to check (see [17]) that

$$\psi^*_{\alpha_1 a_1}(x_1)\cdots\psi^*_{\alpha_n a_n}(x_n)\,|0)$$
$$=\sigma(x_1,\alpha_1,a_1,\ldots,x_n,\alpha_n,a_n)\,\phi^*_{\alpha_1 a_1}(x_1)\cdots\phi^*_{\alpha_n a_n}(x_n)\,|0). \quad (6.10)$$

Consequently the Fermi CGN eigenvectors can be rewritten by using Bose operators, and then H_B can be consistently applied to them. The procedure is exactly the same as in the Fermi case and we arrive at the eigenvalue problem for h (6.3) but with another wave function

$$F^a(x,\alpha)=F(x,\alpha)\cdot\sigma(x,\alpha,a) \quad (6.11)$$

appearing in place of the previous $F(x, \alpha)$. However because of

$$\partial_i \sigma = \partial_i \sigma^{2n+1} \equiv (2n+1)\, \sigma^{2n} \partial_i \sigma = (2n+1)\, \sigma^2 \partial_i \sigma \tag{6.12}$$

to be valid for all permutations of $\{(x, \alpha, a)\}$ and all $n = 1, 2, \ldots$, we arrive at

$$\partial_i \sigma \equiv 0 \tag{6.13}$$

and consequently

$$-i \sum_{j=1}^{n} \alpha_j \partial_j F^a(x, \alpha) = -i\sigma(x, \alpha, a) \sum_{j=1}^{n} \alpha_j \partial_j F(x, \alpha). \tag{6.14}$$

On the other hand because of the symmetry of σ upon interchange $\alpha_j \leftrightarrow \alpha_k$ at $x_j = x_k$ we arrive at

$$\sum_{ij} \delta(x_i - x_j)\, P^{ij}[\tfrac{1}{2}(1 - \alpha_i \alpha_j)]\, F^a(x, a)$$
$$= \sigma(x, \alpha, a) \sum_{ij} P^{ij}[\tfrac{1}{2}(1 - \alpha_i \alpha_j)]\, \delta(x_i - x_j)\, F(x, \alpha) \tag{6.15}$$

so that the eigenvectors of H_F are at the same time the eigenvectors of H_B.

We observe that

$$H_B\,|F, \xi) = H_F\,|F, \xi), \quad 1_F\,|F, \xi) = |F, \xi), \quad 1_F H_B 1_F = H_F, \quad [H_B, 1_F]_- = 0, \tag{6.16}$$

where 1_F is an operator unit of the "bosonized" Fermi algebra [17]

$$[\psi_{\alpha a}(x), \psi^*_{\beta b}(y)]_+ = \delta_{\alpha\beta} \delta_{ab} \delta(x - y)\, 1_F \tag{6.17}$$

$$1_F = \sum_n \frac{1}{n!} \sum_{\alpha_1 \cdots \alpha_n} \sum_{a_1 \cdots a_n} \int dx_1 \cdots \int dx_n \sigma^2(x_1 \alpha_1 a_1, \ldots, x_n \alpha_n a_n)$$
$$\times\, \phi^*_{\alpha_1 a_1}(x_1) \cdots \phi^*_{\alpha_n a_n}(x_n) :\exp\left\{- \sum_{\beta=1}^{2} \sum_{b=1}^{N} \int dz\, \phi^*_{\beta b}(z)\, \phi_{\beta b}(z)\right\}:$$
$$\times\, \phi_{\alpha_1 a_1}(x_1) \cdots \phi_{\alpha_n a_n}(x_n).$$

Hence for the chiral invariant Gross–Neveu model, we have proved the existence of the same Bose–Fermi interplay we had previously recovered for the oscillator problem; see (6.1). It, however, implies that for both (6.1) and (6.16)

$$\mathrm{tr}\, \exp(-iH_B t) = \mathrm{tr}\, \exp(-iH_F t) + \mathrm{tr}\, \exp[-i(1 - 1_F)\, H_F (1 - 1_F)\, t]; \tag{6.18}$$

i.e., the Fermi trace appears as a well defined contribution to the Bose trace, and the only problem (this one is perfectly resolved by using the purely formal trick of the Grassmann algebra reformulation) is to be able to extract tr_F from tr_B.

On the other hand, since $\operatorname{tr}\exp(-iH_B t)$ has quite a conventional c-number path integral representation

$$I_B = \operatorname{tr}\exp(-iH_B t) = \int [d\varphi^*][d\varphi] \exp(-iS),$$

$$S = \int_0^t dt \int dx \, (i\bar{\varphi}\not\partial\varphi + g[(\bar{\varphi}\varphi)^2 - (\bar{\varphi}\gamma_5\varphi)^2]) \tag{6.19}$$

$$= \int_0^t dt \int dx \, [i\varphi^*\dot{\varphi} - H(x)],$$

with the Hamiltonian density given by (1.2);

$$H(x) = \sum_a \{-i\varphi^*_{+a}\partial_x\varphi_{+a} + i\varphi^*_{-a}\partial_x\varphi_{-a}\} + 4g\sum_{ab}\varphi^*_{+a}\varphi^*_{-b}\varphi_{+b}\varphi_{-a} \tag{6.20}$$

and $\varphi^*\dot{\varphi} = \sum_{a=1}^N \varphi^*_a\dot{\varphi}_a$, it is not completely hopeless to look for a c-number path integral representation of $\operatorname{tr}\exp(-iH_F t)$. The easiest way is to follow the stationary phase approximation concept. Let us recall that its essence lies in replacing the path integral $\int [d\phi]\exp(-iS)$ by $\exp(-i\underline{S})$ with $\underline{S} = S(\phi, (\delta S/\delta\phi = 0)$. In our case the stationarity conditions

$$\frac{\delta S}{\delta\phi^*} = 0 = \frac{\delta S}{\delta\phi} \tag{6.21}$$

result in identifying ϕ^*, ϕ with c-number classical solutions of the CGN equations of motion.

Our problem is to investigate the role of such solutions with respect to I_F. Obviously (6.19) can be derived by starting from the Bose quantized CGN Hamiltonian, $H_B = H_C(\varphi^* \to \phi^*, \varphi \to \phi)$. Let us observe that

$$\begin{aligned} H_F &= 1_F H_B 1_F \\ &= \int dx \, \Big\{ \sum_a [-i1_F\phi^*_{+a}1_F\partial_x 1_F\phi_{+a}1_F + i1_F\phi^*_{-a}1_F\partial_x 1_F\phi_{-a}1_F] \\ &\quad + 4g\sum_{ab} 1_F\phi^*_{+a}1_F\phi^*_{-b}1_F\phi_{+b}1_F\phi_{-a}1_F \Big\}, \end{aligned} \tag{6.22}$$

where $1_F^2 = 1_F$ and quantities $1_F\phi^{\#}1_F$ are the continuum analogs of the Fermi oscillator spin 1/2 variables. An explicit form of 1_F (6.17) does not promise any explicit simple formula for $(1_F\phi^{\#}1_F)(x)$; let us, however, make a lattice analysis of the problem. If to take a linear chain of spins 1/2,

$$\begin{gathered} 1_F\phi_k^{\#}1_F = 1_F^k\phi_k^{\#}1_F^k = \sigma_k^{\#} \\ 1_F = \prod_k 1_F^k, \qquad 1_F^k = :\exp(-\phi_k^*\phi_k): + \phi_k^* :\exp(-\phi_k^*\phi_k): \phi_k, \end{gathered} \tag{6.23}$$

with

$$\phi_k^{\#} = \frac{1}{\sqrt{\delta}} \int dx\, \chi_k(x)\, \phi^{\#}(x) \cong \sqrt{\delta}\, \phi^{\#}(x), \qquad x \in \Delta_k,$$
$$[\phi(x), \phi^*(y)]_- = \delta(x-y), \qquad [\phi(x), \phi(y)]_- = 0, \tag{6.24}$$

then formally:

$$1_F^k \cong :\exp(-\delta\phi^*(x)\,\phi(x)): + \delta\phi^*(x) :\exp(-\delta\phi^*(x)\,\phi(x)): \phi(x) \qquad x \in \Delta_k,$$
$$\sigma_k^+ \cong \sqrt{\delta}\, \phi^*(x) :\exp(-\delta\phi^*(x)\,\phi(x)): = \sqrt{\delta}\, \sigma^*(x), \tag{6.25}$$
$$\sigma_k^- \cong \sqrt{\delta} :\exp(-\delta\phi^*(x)\,\phi(x)): \phi(x) = \sqrt{\delta}\, \sigma(x),$$

so that (formally as well)

$$1_F = \prod_k 1_F^k \cong \sum_n \frac{1}{n!} \sum_{k_1 \cdots k_n} \delta^n \cdot \sigma^2(x_1, \dots, x_n)\, \phi^*(x_1) \cdots \phi^*(x_n)$$
$$\times :\exp \left\{ -\sum_k \delta\phi^*(x_k)\, \phi(x_k) \right\} : \phi(x_1) \cdots \phi(x_n)$$
$$\xrightarrow[\delta \to 0]{} \sum_n \frac{1}{n!} \int dx_1 \cdots \int dx_n\, \sigma^2(x_1, \dots, x_n)\, \phi^*(x_1) \cdots \phi^*(x_n)$$
$$\times :\exp \left(-\int dz\, \phi^*(z)\, \phi(z) \right) : \phi(x_1) \cdots \phi(x_n) = 1_F, \qquad x_i \in \Delta_{k_i}. \tag{6.26}$$

Hence from a purely formal point of view, a continuum analog of (6.23) for a single internal degree of freedom reads

$$1_F \phi^{\#}(x)\, 1_F \equiv 1_F^x \phi^{\#}(x)\, 1_F^x = \sigma^{\#}(x),$$
$$1_F^x = :\exp(-dx\, \phi^*(x)\, \phi(x)): + dx\, \phi^*(x) :\exp(-dx\, \phi^*(x)\, \phi(x)): \phi(x),$$
$$\sigma^*(x) = \phi^*(x) :\exp(-dx\, \phi^*(x)\, \phi(x)), \tag{6.27}$$
$$\sigma(x) = :\exp(-dx\, \phi^*(x)\, \phi(x)): \phi(x).$$

If adopted to the *CGN* model we would have

$$1_F^x = \prod_{a=1}^{N} \prod_{\alpha=1}^{2} 1_F^{x\alpha a}, \qquad 1_F^{x\alpha a} = 1_F^x(\phi \to \phi_{\alpha a}), \tag{6.28}$$

and then

$$
\begin{aligned}
H_F &= 1_F H_B 1_F \\
&= \int dx \left\{ \sum_a \left[-i\sigma^*_{+a}(x)\, \partial_x \sigma_{+a}(x) + i\sigma^*_{-a}(x)\, \partial_x \sigma_{-a}(x) \right] \right. \\
&\quad \left. + 4g \sum_{ab} \sigma^*_{+a}(x)\, \sigma^*_{-b}(x)\, \sigma_{+b}(x)\, \sigma_{-a}(x) \right\} \\
&= \int dx\, H_F(x),
\end{aligned}
\tag{6.29}
$$

where to understand the meaning of the kinetic terms it is necessary to use the more rigorous lattice predecessor of (6.29):

$$
\begin{aligned}
H_F \to \sum_k &\left\{ \sum_a \left[-i\sigma^+_{+a}(k)\, \partial_\gamma \sigma^-_{+a}(k,\gamma) + i\sigma^+_{-a}(k)\, \partial_\gamma \sigma^-_{-a}(k,\gamma) \right]\Big|_{\gamma=0} \right. \\
&\quad \left. + \frac{4g}{\delta} \sum_{ab} \sigma^+_{+a}(k)\, \sigma^+_{-b}(k)\, \sigma_{+b}(k)\, \sigma_{-a}(k) \right\} \\
= \sum_k &\, H_F(k),
\end{aligned}
\tag{6.30}
$$

$$
\sigma_{\alpha a}(k) = \sigma(\phi \to \phi_{\alpha a})(k), \qquad \sigma_{\alpha a}(k,\gamma) = \sigma(\phi_{\alpha a}(k,\gamma)),
$$

$$
\phi_{\alpha a}(k,\gamma) = \frac{1}{\sqrt{\delta}} \int dx\, \chi_k(x)\, \phi(x+\gamma),
$$

which allows for the computation

$$
\begin{aligned}
&\sigma^+(k,\gamma)\, \partial_\gamma \sigma(k,\gamma)_{|\gamma=0} \\
&\quad = \phi^*(k) :\exp(-\phi^*(k)\, \phi(k)): \partial_\gamma [:\exp(-\phi^*(k,\gamma)\, \phi(k,\gamma)]: \phi(k,\gamma)]_{|\gamma=0} \\
&\quad = \phi^*(k) :\exp(-\phi^*(k)\, \phi(k)): \partial_\gamma \phi(k,\gamma)_{|\gamma=0} \\
&\qquad + \phi^*(k) :\exp(-\phi^*(k)\, \phi(k)): \{-[\partial_\gamma \phi^*(k,\gamma)]_{|\gamma=0} :\exp(-\phi^*(k)\, \phi(k)): \phi^2(k) \\
&\qquad - \phi^*(k) :\exp(-\phi^*(k)\, \phi(k)): [\partial_\gamma \phi(k,\gamma)]_{|\gamma=0}\, \phi(k)\} \\
&\quad \equiv \phi^*(k) :\exp(-\phi^*(k)\, \phi(k): \partial_\gamma \phi(k,\gamma)_{|\gamma=0} \\
&\quad \to \delta\phi^*(x) :\exp(-\delta\phi^*(x)\, \phi(x)): \partial_x \phi(x), \qquad x \in \Delta_k
\end{aligned}
\tag{6.31}
$$

so that due to the identification

$$
\sigma^+_{\alpha a}(x)\, \partial_x \sigma^-_{\alpha a}(x) \equiv \phi^*_{\alpha a}(x) :\exp(-dx\, \phi^*_{\alpha a}(x)\, \phi_{\alpha a}(x)): \partial_x \phi_{\alpha a}(x),
\tag{6.32}
$$

we arrive at the (formal again) identity

$$
\begin{aligned}
H_F &= 1_F H_B 1_F \\
&= \int dx \Big\{ \sum_a [-i\phi_{+a}(x) :\exp(-dx\, \phi^*_{+a}(x)\, \phi_{+a}(x)): \partial_x \phi_{+a}(x) \\
&\quad + i\phi^*_{-a}(x) :\exp(-dx\, \phi^*_{-a}(x)\, \phi_{-a}(x)): \partial_x \phi_{-a}(x)] \\
&\quad + 4g \sum_{ab} \phi^*_{+a}(x)\, \phi^*_{-b}(x) \\
&\quad \times :\exp\Big(-dx \Big[\sum_\alpha \phi^*_{\alpha a}(x)\, \phi_{\alpha a}(x) + \phi^*_{\alpha b}(x)\, \phi_{\alpha b}(x)\Big]\Big)\, \phi_{+b}(x)\, \phi_{-a}(x) \Big\} \\
&= \int dx\, H_F(x)
\end{aligned}
\tag{6.33}
$$

Because of (see (6.30))

$$
\begin{aligned}
\hat{U}_F(\Delta t) &= \exp(-iH_F \Delta t) \cong 1_F - i\,\Delta t\, H_F \\
&\cong \exp\Big(-i \sum_k H_F(k)\, \Delta t\Big) \cong \prod_k \Big[1^k_F - i\,\Delta t\, H_F(k)\Big], \\
1^k_F &= \prod_{\alpha a} 1^{k\alpha a}_F,
\end{aligned}
\tag{6.34}
$$

the functional kernel of the lattice operator (6.34)

$$
U_F(\Delta t) = \prod_k (\beta|\, 1^k_F - i\,\Delta t\, H_F(k)\, |\beta) \exp \sum_{a=1}^{N} \sum_{\alpha=1}^{\alpha} \beta^*_{\alpha a}(k)\, \beta_{\alpha a}(k) \tag{6.35}
$$

due to

$$
(\beta|\, 1^k_F\, |\beta) \exp \sum_a \sum_\alpha \beta^*_{\alpha a}(k)\, \beta_{\alpha a}(k) = \prod_{\alpha=1}^{a} \prod_{a=1}^{N} [1 + \beta^*_{\alpha a}(k)\, \beta_{\alpha a}(k)], \tag{6.36}
$$

and

$$
\beta^{*\alpha a}(k) \cong \sqrt{\delta}\, \beta_{\alpha a}(x), \qquad x \in \Delta_k,
$$

allows for the previous (Fermi oscillator) procedure, so that for the lattice case we have

$$
\begin{aligned}
\mathrm{tr}_\delta \exp(-iH_F t) &= \int [d\beta^*][d\beta] \exp i \int_0^t dt \\
&\quad \times \sum_k \Big\{ \sum_{\alpha a} i\, \frac{\beta^*_{\alpha a}(k)\, \dot{\beta}_{\alpha a}(k)}{1 + \beta^*_{\alpha a}(k)\, \beta_{\alpha a}(k)} - \frac{(\beta|\, H_F(k)\, |\beta) \exp \sum_{\alpha a} \beta^*_{\alpha a}(k)\, \beta_{\alpha a}(k)}{\prod_{\alpha a} [1 + \beta^*_{\alpha a}(k)\, \beta_{\alpha a}(k)]} \Big\},
\end{aligned}
\tag{6.37}
$$

with

$$(\beta|\, H_F(k)\,|\beta) = \sum_a \Big[-i\beta^*_{+a}(k) \exp(-\beta^*_{+a}(k)\,\beta_{+a}(k))\,\partial_y \beta_{+a}(k, y)_{|x=0}$$
$$+ i\beta^*_{-a}(k) \exp(-\beta^*_{-a}(k)\,\beta_{-a}(k))\,\partial_y \beta_{-a}(k, y)_{|y=0}$$
$$+ \frac{4g}{\delta} \sum_{ab} \beta^*_{+a}(k)\,\beta^*_{-b}(k)$$
$$\times \exp\Big(-\sum_\alpha [\beta^*_{\alpha a}(k)\,\beta_{\alpha a}(k) + \beta^*_{\alpha b}(k)\,\beta_{\alpha b}(k)]\Big)\,\beta_{+b}(k)\,\beta_{-a}(k). \quad (6.38)$$

If now to approach the continuum, then $\beta^*_{\alpha a}(k)\ \beta_{\alpha a}(k)$ goes over to $\delta\beta^*_{\alpha a}(x)$ $\beta_{\alpha a}(x) \to dx\, \beta^*_{\alpha a}(x)\ \beta_{\alpha a}(x)$. Note that in the path integral (6.37) we should integrate with respect to all $\beta(k)$, $\bar{\beta}(k)$ from $-\infty$ to $+\infty$. If, however, to restrict considerations to these collections of $\{\beta^*_{\alpha a}(k)\}$ for which the corresponding spinor trajectories satisfy

$$\beta^*_{\alpha a}(x)\,\beta_{\alpha a}(x) \leqslant A < \infty, \quad (6.39)$$

A arbitrary, then the integrand in (6.37) simplifies and allows for a consistent continuum limit

$$\delta \to 0 \Rightarrow \mathrm{tr}_\delta \exp(-iH_F t)_{|\beta^*\beta \leqslant A} \to \mathrm{tr} \exp(-iH_F t)_{|\beta^*\beta \leqslant A}$$
$$= \int_{\beta^*\beta \leqslant A} [d\beta^*][d\beta] \exp i \int_0^t dt \int dx \Big\{ \sum_{\alpha a} i\beta^*_{\alpha a}(x)\,\dot{\beta}_{\alpha a}(x) - H_C(x) \Big\}, \quad (6.40)$$

with

$$H_C(x) = \sum_a [-i\beta^*_{+a}(x)\,\partial\beta_{+a}(x) + i\beta^*_{-a}(x)\,\partial\beta_{-a}(x)]$$
$$+ 4g \sum_{ab} \beta^*_{+a}(x)\,\beta^*_{-b}(x)\,\beta_{+b}(x)\,\beta_{-a}(x), \quad (6.41)$$

provided we replace $[1 + \delta\beta^*_{\alpha a}(x)\,\beta_{\alpha a}(x)]$ and $[\exp \delta\beta^*_{\alpha a}(x)\ \beta_{\alpha a}(x)]$ by 1 when $\delta \to 0$.

It demonstrates that spinor paths subject to the restriction (6.39) give exactly the same contribution to both Bose I_B and Fermi I_F traces for the *CGN* model. All the classical (*c*-number) solutions of the *CGN* field equations which satisfy (6.39) are "stationary points" of the action in both Bose and Fermi cases.

Let us, however, emphasize that all of the quantal difference between bosons and fermions comes from the irregular paths, i.e., those for which it is incorrect to replace $\exp \delta\beta^*\beta$ by 1. Let us consider a *c*-number spinor field of the *CGN* system denoted $\bar{\chi}, \chi$ $\sigma = -g\bar{\chi}\chi$, $\pi = -ig\bar{\chi}\gamma_5\chi$. The most straightforward first step to account for quantum fluctuations about the classical spinor path is to compute the (Bose) path integral

$$I_B(\sigma, \pi) = \int [d\psi^*][d\psi] \exp iS(\bar{\psi}, \psi, \sigma = \sigma(\bar{\chi}, \chi), \pi = \pi(\bar{\chi}, \chi)), \quad (6.42)$$

so that potentials are kept in their 0-loop (classical) order at a fixed choice of $\bar{\chi}, \chi$. The Fermi contribution $I_F(\sigma, \pi)$ to $I_B(\sigma, \pi)$ is immediately extracted by using the previous (Shei's) path integral on the Grassmann algebra (4.7), but it is actually no wonder that the stationarity points σ, π of the effective action ought to be related to the initial classical c-number *CGN* problem. Otherwise the c-number and Grassmann algebra path integration methods would not be reconciled, and this in turn would contradict our previous observation about the relationship of Bose and Fermi traces for the *CGN* model.

Acknowledgments

I would like to express my warm thanks to Professors L. O'Raifeartaigh and J. T. Lewis for making my visit in Dublin possible. I have benefited much from illuminating conversations with Professor L. O'Raifeartaigh and Dr. T. Murphy, to whom I am greatly indebted.

References

1. N. Andrei and J. H. Lowenstein, *Phys. Rev. Lett.* **43** (1979), 1698.
2. N. Andrei and J. H. Lowenstein, *Phys. Lett. B* **90** (1980), 106.
3. N. Andrei and J. H. Lowenstein, *Phys. Lett. B* **91** (1980), 401.
4. C. Destri and J. H. Lowenstein, *Nuclear Phys. B* **205** (FS) (1982), 369.
5. P. P. Kulish and N. J. Reshotikhin, *Soviet Phys. JETP* **80** (1981), 214.
6. C. N. Yang, *Phys. Rev. Lett.* **19** (1967), 1312.
7. B. Sutherland, *Phys. Rev. Lett.* **20** (1968), 98.
8. B. Sutherland, *Phys. Rev. B* **12** (1975), 3795.
9. A. Neveu and N. Papanicolau, *Comm. Math. Phys.* **58** (1978), 31.
10. L. D. Faddeev, Completely integrable models of field theory, LOMI preprint, Leningrad, 2–79.
11. E. K. Sklyanin, L. A. Tahtajan, and L. D. Faddeev, *Teor. Mat. Fiz.* **40** (1979), 194.
12. P. Garbaczewski, *J. Math. Phys.* **24** (1983), 651.
13. P. Garbaczewski, *J. Math. Phys.*, to appear.
14. J. Shigemitsu and S. Elitzur, *Phys. Rev. D* **14** (1976), 1988.
15. R. J. Gonsalves and S. S. Shei, *Nuclear Phys. B* **130** (1977), 221.
16. F. Wilczek, *Phys. Rev. Lett.* **49** (1982), 957; see also A. S. Goldhaber, *Phys. Rev. Lett.* **49** (1982), 905, and R. Sorkin, Particle statistics in three dimensions, *IAS* Princeton preprint, 1982.
17. P. Garbaczewski, *Comm. Math. Phys.* **43** (1975), 131.
18. R. Koberle, V. Kurak, and J. A. Swieca, *Phys. Rev. D* **20** (1979), 897.
19. M. B. Halpern, *Phys. Rev. D* **12** (1975), 1684.
20. P. Garbaczewski, *J. Math. Phys.* **24** (1983), 341.
21. P. Garbaczewski, *J. Math. Phys.* **23** (1982), 442.
22. D. G. Boulware and L. S. Brown, *Phys. Rev.* **172** (1968), 1628.
23. S. Y. Lee and A. Gavrielides, *Phys. Rev. D* **12** (1975), 3380.
24. S. S. Shei, *Phys. Rev. D* **14** (1976), 535.
25. R. Jackiw, *Rev. Modern Phys.* **49** (1977), 682.
26. P. Garbaczewski, *J. Math. Phys.* **19** (1978), 642.
27. S. Y. Lee, L. K. Kuo, and A. Gavrielides, *Phys. Rev. D* **12** (1975), 2249.
28. R. Dashen, B. Hasslacher, and A. Neveu, *Phys. Rev. D* **12** (1975), 2443.
29. L. D. Faddeev, *Teor. Mat. Fiz.* **1** (1969), 3.

30. A. A. Slavnov and L. D. Faddeev, "Introduction to Quantum Theory of Gauge Fields," Nauka, Moscow, 1978. [in Russian]
31. S. Coleman, The uses of instantons, Erice 1977, *in* "Proc. in Subnuclear Series," No. 15 (A. Zichichi, Ed.), Plenum, New York, 1979.
32. F. A. Berezin, "Methods of Second Quantization," Nauka, Moscow, 1965. [in Russian]

595/150/1-4