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On Quantum Solitons and Their Classical Relatives: Bethe Ansatz States in Soliton Sectors of the Sine–Gordon System

PIOTR GARBACZEWSKI*

Physics Department, Brown University, Providence, Rhode Island 02912

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Previously we have found that the semiclassical sine–Gordon/Thirring spectrum can be received in the absence of quantum solitons via the spin 1/2 approximation of the quantized sine–Gordon system on a lattice. Later on, we have recovered the Hilbert space of quantum soliton states for the sine–Gordon system. In the present paper we present a derivation of the Bethe Ansatz eigenstates for the generalized ice model in this soliton Hilbert space. We demonstrate that via "Wick rotation" of a fundamental parameter of the ice model one arrives at the Bethe Ansatz eigenstates of the quantum sine–Gordon system. The latter is a "local transition matrix" ancestor of the conventional sine–Gordon /Thirring model, as derived by Faddeev *et al.* within the quantum inverse-scattering method. Our result is essentially based on the $N < \infty$, $\Delta = 1$, $m \ll 1$ regime. Consequently, the spectrum received, though resembling the semiclassical one, does not coincide with it at all.

1. SINE–GORDON SOLITONS: CLASSICAL DERIVATION OF "COLLECTIVE" PARAMETERS

A. $\Phi_0(x, t) = \Phi_0 = 0$ trivially satisfies the famous sine–Gordon equation

$$\Box\Phi(x, t) = m^2 \sin \Phi(x, t), \qquad m > 0 \tag{1.1}$$

in $1 + 1$ dimensions. Let us rescale (1.1) to arrive at $m = 1$; then the N-soliton solution of (1.1), $N = 1, 2, \ldots$, can be generated from Φ_0 by a successive application of the so-called Bäcklund transformations:

$$B_a\colon \Phi_0(x, t) \to B_a(\Phi_0)(x, t) = \Phi_a(x, t) = 4 \tan^{-1} \exp \theta_a(x, t), \tag{1.2}$$

* Permanent address: Institute of Theoretical Physics, University, 50-205, Wroclaw, Poland

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where

$$\theta_a(x,t)=\gamma_a\cdot(x-v_a t),$$
$$v_a=\frac{a-a^{-1}}{a+a^{-1}}=\frac{a^2-1}{a^2+1},$$
$$\gamma_a=\frac{a+a^{-1}}{2}=(\operatorname{sgn} a)(1-v_a^2)^{-1/2}=\frac{a^2+1}{2a} \tag{1.3}$$

and

$$\Phi_N(x,t)=\Phi_{a_1\cdots a_N}(x,t)=(B_{a_1}\cdots B_{a_N})(\Phi_0)(x,t). \tag{1.4}$$

The Bäcklund mapping

$$B_a\colon \Phi_N=\Phi_{a_1\cdots a_N}\to\Phi_{N+1}=\Phi_{a_1\cdots a_N a} \tag{1.5}$$

is defined by [1].

$$\begin{aligned}\Phi^x_{N+1}&=\Phi^x_N+2a\sin\left(\frac{\Phi_N+\Phi_{N+1}}{2}\right)\\ \Phi^t_{N+1}&=-\Phi^t_N+2a^{-1}\sin\left(\frac{\Phi_{N+1}-\Phi_N}{2}\right)\end{aligned} \tag{1.6}$$

Here Φ^x, Φ^t denote space and time derivatives of Φ, respectively, and the parameter a is allowed to be complex.

In case of the sine–Gordon system, an analytic formula for $\Phi_N(x,t)$ is known [2, 3] in terms of the $N+2$ variables: $(a)=(a_1,\ldots,a_N), x, t$ plus N additional phase parameters (δ). There is a large freedom in the choice of $(\delta), x, t$ but there are severe limitations on the choice of (a) to arrive at sine–Gordon N-solitons. Namely, if specialized to the sine–Gordon system (1.1), Bäcklund transformations must satisfy the following identities [1, 3]:

$$a,b\in C,\qquad |a|,|b|\in(0,\infty),$$
$$B_a\cdot B_b=B_b\cdot B_a,\qquad a\neq b;\qquad B_a\cdot B_a=B_a^2\equiv 0;\qquad B_{-a}=B_a^{-1}. \tag{1.7}$$

Here we know [4] that if $a\in R$, then B_a realizes a canonical transformation π_a of one soliton into another. If $a\in C$, then a product mapping $B_aB_{a^*}=B_{a^*}B_a$ realizes a canonical transformation π_{a^*a}. The set of all π_a, π_{a^*a}, $|a|\in(0,\infty)$, if constrained by (1.6), (1.7), and (1.1), forms an Abelian group of canonical transformations of the sine–Gordon system, responsible for mappings between elements of its soliton sector.

B. Let us admit $m\neq 1$ in (1.1). This equation, upon power expansion of the sine reads

$$(\Box-m^2)\,\Phi(x,t)=m^2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k+1)!}\,\Phi(x,t)^{2k+1}, \tag{1.8}$$

i.e.,

$$\Lambda(\partial)\,\Phi(x,t)=J[\Phi](x,t),\qquad \Lambda(\partial)=\Box-m^2. \tag{1.9}$$

Let $\varphi(x,t)$ be a solution of $\Lambda(\partial)\,\varphi(x,t)=0$. Then, by making use of the Yang–Feldman relation, and performing an infinite sequence of successive iterations, we formally arrive at

$$\Phi(x,t)=\varphi(x,t)+\Lambda(\partial)^{-1}J[\varphi](x,t)\Rightarrow\Phi(x,t)=\Phi[\varphi](x,t) \tag{1.10}$$

for any (including solitons) solution of (1.1). However, the choice of Φ is restricted by a particular choice of φ. For example, if Φ is to be regarded as the 1-soliton, we should have [5. 6]

$$\varphi(x,t)=\varphi_a(x,t)=\exp[\theta_a(x,t)+\delta)=\exp[m\gamma_a x+\delta_a(t)],\qquad \delta_a(t)=\delta-m\gamma_a v_a t, \tag{1.11}$$

where δ is an arbitrary real phase (eventually modulo π to account for a change of sign of φ). Notice that $\theta_a(x,t)$ differs from the θ_a given by (1.3) by a multiplicative factor m (it equals 1 in (1.3)). For the general N-soliton solution Φ_N, which in the large-time asymptotics reads

$$\Phi_N\xrightarrow[|t|\to\infty]{}\sum_{i=1}^{n}\Phi_{a_i}+\sum_{j=1}^{m}\Phi_{a_ja_j^*}. \tag{1.12}$$

The respective free field φ should appear in the form [5, 6] (see also [7–9])

$$\varphi(x,t)=\varphi_{(a)}(x,t)=\sum_{i=1}^{N}\varphi_{a_i}(x,t) \tag{1.13}$$

with a restriction that if $a\in C$, then the parameter a^* appears in the sequence (a). In addition, neither a appears in the sequence (a) more than once, and the appearance of $-a$ is prohibited once a appears in (a). The breather ϕ_{a^*a} is characterized by

$$\begin{aligned}\theta_a(x,t)&=\frac{m\gamma_a}{|a|}\{a_R(x+v_a t)+ia_I(v_a x+t)\}\\&=\frac{m\gamma_a}{|a|}(a_R+ia_Iv_a)x+\frac{m\gamma_a}{|a|}(a_Rv+ia_I)t,\\a&=a_R+ia_I,\qquad \delta=\delta_R+i\delta_I,\\v_a&=\frac{|a|^2-1}{|a|^2+1},\qquad \gamma_a=(1-v_a^2)^{-1/2}.\end{aligned} \tag{1.14}$$

C. Let us concentrate on N-solitons with a real (a) parameterization. At this point it is worth recalling Hirota's [2] formula for N-soliton solutions of (1.1) at $m=1$:

$$\begin{aligned} &\tan(\tfrac{1}{4}\Phi) = g/f, \\ f &= \sum_{\mu=0,1}^{(e)} \exp\left[\sum_{i<j}^{N} B_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \mu_i(\gamma_i\xi_i + \delta_i)\right], \\ g &= \sum_{\mu=0,1}^{(o)} \exp\left[\sum_{i<j}^{N} B_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \mu_i(\gamma_i\xi_i + \delta_i)\right], \end{aligned} \tag{1.15}$$

where $\sum^{(e)}$ and $\sum^{(o)}$ denote summations over all possible combinations of $\mu_1 = 0, 1$, $\mu_2 = 0, 1, \ldots, \mu_N = 0, 1$ under the restrictions that $\sum_{i=1}^{N} \mu_i$ is even or odd, respecitvely. Here

$$\xi_i = x - v_i t, \qquad \gamma_i = (1 - v_i^2)^{-1/2} \operatorname{sgn} a_i, \tag{1.16}$$

$$\exp B_{ij} = \frac{(\gamma_i - \gamma_j)^2 - (\gamma_i v_i - \gamma_j v_j)^2}{(\gamma_i + \gamma_j)^2 - (\gamma_i v_i + \gamma_j v_j)^2}.$$

Because of (1.15) and (1.16) one is able to rewrite any N-soliton solution of (1.1) as an explicit function of N free fields φ_a and velocities v_i:

$$\Phi_N(x,t) = \Phi^N_{(a)}(\varphi_{a_1}, \ldots, \varphi_{a_N})(x,t) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \varphi^{n_1 \cdots n_N}_{v_1 \cdots v_N} \cdot \varphi^{n_1}_{a_1}(x,t) \cdots \varphi^{n_N}_{a_N}(x,t) \tag{1.17}$$

where the power series expansion of [5, 6] is adopted. One must be aware that the expansion coefficients of (1.17) do exhibit a manifest $(v) = (v_1, \ldots, v_N)$ dependence through $\exp B_{ij}$ of (1.16), and because of $v(a) = v(-a)$ are completely insensitive to reflection $a_i \to -a_i$.

In fact, at a fixed choice of the velocity sequence $(v) = (v_1, \ldots, v_N)$, (1.17) exhibits a manifest reflection $a_i \to -a$ and a displacement (translation) freedom of each $\varphi_a(x,t)$ entry.

Let us introduce the following notation:

$$\varphi_a(x,t) = \exp m\gamma_a(x - q_a), \tag{1.18}$$

where

$$\begin{aligned} q_a &= v_a(t - t_0) \Rightarrow \delta = \delta_a = m\gamma_a v_a t_0, \\ \dot{q}_a &= v_a \Rightarrow \gamma_a = (\operatorname{sgn} a)(1 - \dot{q}_a^2)^{-1/2}, \end{aligned} \tag{1.19}$$

i.e.,

$$\begin{aligned} \varphi_a(x,t) &= \exp m(\operatorname{sgn} a)(1 - \dot{q}_a^2)^{-1/2}(x - q_a) \\ &= \varphi[\operatorname{sgn} a, q_a, \dot{q}_a](x) \end{aligned} \tag{1.20}$$

and consequently

$$\Phi_N(x, t) = \Phi_N[\operatorname{sgn} a_1, q_1, \dot{q}_1, \ldots, \operatorname{sgn} a_N, q_N, \dot{q}_N]. \tag{1.21}$$

At a fixed choice of $(\dot{q}_1, \ldots, \dot{q}_N)$ which determines expansion coefficients of (1.17), we can freely vary both $(q) = (q_1, \ldots, q_N)$ and $(\operatorname{sgn} a) = (\operatorname{sgn} a_1, \ldots, \operatorname{sgn} a_N)$ variables, the continuous and discrete ones, respectively.

2. Analysis of Quantum Soliton Sectors

A. The quantized sine–Gordon field

$$\Lambda(\partial)\,\hat{\Phi}(x, t) = J[\hat{\Phi}](x, t), \tag{2.1}$$
$$\hat{\Phi}(x, t) = \hat{\varphi}_{in}(x, t) + \Lambda(\partial)^{-1} J[\hat{\varphi}_{in}](x, t)$$

can be found [5–10] in the form of the Haag series:

$$\begin{aligned}\hat{\Phi}(x, t) &= \Phi[\hat{\varphi}_{in}](x, t)\\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int d\sigma_1 \cdots \int d\sigma_n c(x, t, \sigma_1, \ldots, \sigma_n) :\hat{\varphi}_{in}(\sigma_1) \cdots \hat{\varphi}_{in}(\sigma_n):\end{aligned} \tag{2.2}$$

where $\hat{\varphi}_{in}(x, t)$ stands for a renormalized free-mass m neutral scalar Bose field (plane wave!) solution of

$$\Lambda(\partial)\,\hat{\varphi}_{in}(x, t) = 0; \tag{2.3}$$

σ_i is the space–time variable (x, t), and an operator product is normal ordered: annihilation operators to the right. Notice an essential contrast to the classical case, where the whole variety of non-plane-wave solutions of the sourceless equation was allowed. In the quantum case we consider a single plane-wave operator solution only. Its annihilation operator is given by

$$a(k) = \int dx \exp(-ikx)[\sqrt{k^2 + m^2}\,\hat{\varphi}_{in}(x, 0) + i\hat{\pi}_{in}(x, 0)] \tag{2.4}$$

where the initial data of the field satisfy the canonical commutation relations

$$[\hat{\varphi}_{in}(x, 0), \hat{\pi}_{in}(y, 0)]_- = i\delta(x - y) \tag{2.5}$$

and the Fock representation of the CCR is selected by demanding $|a(k)|0) = 0\,\forall k$.

By exploring (2.2) we can, in principle, follow the tree approximation scheme of [5–9]:

$$\begin{aligned}(0|\, :\Phi[\hat{\varphi}_{in} + \varphi]: (x, t)\,|0) &= \Phi[\varphi](x, t)\\ &= (\varphi|\, :\Phi[\hat{\varphi}_{in}]: (x, t)\,|\varphi)\end{aligned} \tag{2.6}$$

where each quantum solution $\hat{\varphi}_{in}$ is shifted (boson transformation) by a c-number solution, and $|\varphi)$ appears as a continuous generalization of the coherent product state [6], which formally reads

$$|\varphi) = U_\varphi |0) = \exp \left\{ i \int dx \, [f(x) \, \hat{\varphi}_{in}(x, 0) - g(x) \, \hat{\pi}_{in}(x, 0)] \right\}. \tag{2.7}$$

Here $f(x)$ and $g(x)$ are the real functions, which we relate to $\varphi(x)$ via complex amplitudes:

$$\alpha(k) = \int dx \exp(-ikx)[\sqrt{k^2 + m^2} f(x) + ig(x)]. \tag{2.8}$$

One must be aware that the classical amplitude $\alpha(k)$, in general, is not a square integrable function. Consequently, (2.7) has a formal meaning only, unless read in the direct product language of [6, 11] after putting the theory on a lattice (ultraviolet cutoff):

$$U_\varphi \equiv U_\varphi^{\otimes} = \prod_k^{\otimes} \{\exp(\alpha a^* - \bar{\alpha} a) f^0\}_k, \tag{2.9}$$

where f^0 is a Fock state for the Schrödinger representation of the CCR. The routine stationarity condition [12, 13] if applied to the coherent state expectation value of the sine–Gordon Hamiltonian in the tree approximation would imply $f(x) = 0$ and $g(x)$ to be the time-independent solution of the free-field equation $\Lambda(\partial) \, g(x) = 0$. Consequently,

$$\frac{1}{2\pi} \int \frac{dk}{\sqrt{k^2 + m^2}} \alpha(k) \, \bar{\alpha}(k) = \int dx \, g^2(x) \tag{2.10}$$

and the α-labeling of coherent states can be replaced by the g-labeling.

B. A classification of coherent states $|\varphi)$ and the related irreducibility sectors IDPS($|\varphi)$) of the CCR algebra (2.5) was given in [6] under a fundamental assumption that the boson transformation parameters $g(x)$ ($\alpha(k)$, respectively) allow the generation of classical sine-Gordon solitons from (2.1) and (2.6). The displacement freedom was taken into account in [6], but not the reflection one. Let us recall that the phase q is completely arbitrary in the exponent of the free field φ_a and that the time dependence is absorbed in phases. If to fix the sequence $(\dot{q})^N$ of N velocities, we have determined a concrete $(\dot{q})^N$th Hilbert space sector in the in-field (von Neumann's) Hilbert space H. Denote it by $H^N_{(\dot{q})}$. $H^N_{(\dot{q})}$ can still be decomposed into a direct sum (respectively integral) of pairwise orthogonal CCR algebra irreducibility sectors, each one being characterized by a fixed choice of the sequence $(\operatorname{sgn} a)^N$ and $(q)^N$, the former being discrete, $\operatorname{sgn} a = \pm 1$, and the latter being continuous, $q \in R$. A more precise notation should now replace this IDPS($|\varphi)$). Namely, a coherent state $|\varphi)$ is specified as follows:

$$\begin{gathered}|\varphi) = |\varphi)^N_{(\dot{q})} := |\underline{\operatorname{sgn} a}, \underline{q})^N_{(\dot{q})} = |\operatorname{sgn} a_1, q_1, \ldots, \operatorname{sgn} a_N, q_N)_{(\dot{q})}, \\ |\varphi) \in \mathrm{IDPS}^N_{(\dot{q})}(|\underline{\operatorname{sgn} a}, \underline{q})) \subset H^N_{(\dot{q})} \subset H^N \subset H_{\mathrm{sG}}\end{gathered} \tag{2.11}$$

where $H_{\mathrm{sG}} \subset H$ is the quantum soliton Hilbert space of [6]. Let us add that sectors $H^N_{(\dot{q})}$ and $H^N_{(\dot{q}')}$ are pairwise orthogonal in H_{sG} unless $(\dot{q}) \cap (\dot{q}') = (\dot{q}) = (\dot{q}')$. The integral (2.10) is obviously divergent; take $g(x) = \exp m\gamma x$ or $g(x) = \sum_{i=1}^N \exp m\gamma_i x$ as examples.

Once $(\dot{q})^N$ is fixed, then irrespective of the choice of $(\operatorname{sgn} a)^N$ and $(q)^N$ we still remain within the same Hilbert space sector $H^N_{(\dot{q})}$, which is then a carrier Hilbert space for the $(\dot{q})$th N-soliton operator. Hence, one can as well choose all $a_i \in R^+$ and $q_i = 0 \ \forall i = 1, \ldots, N$ to investigate the sectorial structure of the soliton Hilbert space in terms of $H^N_{(\dot{q})}$.

By virtue of (1.17), (1.21), and (2.2), the choice of $(\dot{q})^N$ fixes the expansion coefficients of the N-soliton quantum operator, and the notion of $\hat{\Phi}^N_{(\dot{q})}(x) = \Phi^N_{(\dot{q})}(\hat{\varphi}_{in})(x)$ should be introduced in place of $\hat{\Phi}_N(x) = \Phi_N(\hat{\varphi}_{in})(x)$ formerly defined in [6]. As far as the structure of the soliton Hilbert space is concerned, the explicit $(\dot{q})$ dependence of soliton operators can be disregarded. We however wish now to understand the operator structure of the model.

Now, $\hat{\phi}_N = \hat{\phi}^N_{(\dot{q})} = \phi^N_{(\dot{q})}(\hat{\varphi}_{in})$ implies that for any coherent soliton state $|\varphi) = |\varphi)^N_{(\dot{q})} = |\underline{\operatorname{sgn} a}, \underline{q})^N_{(\dot{q})} \in H^N_{(\dot{q})}$ the c-number field:

$$(\varphi| :\hat{\Phi}_N(x): |\varphi) = \Phi^N_{(\dot{q})}(x) = \Phi_N[\underline{\operatorname{sgn} a}, \underline{q}, \dot{\underline{q}}](x) \tag{2.12}$$

is the N-soliton solution of the sin–Gordon equation, (1.1).

Let us mention that no coinciding $\dot{q}$'s are allowed in the velocity sequence $(\dot{q})$. While constructing the coherent soliton states (2.6), (2.7) we were unable to guarantee the fulfillment of this "classical Pauli principle" [1], compare, e.g., (1.7). This defect was also inherent in [5] where we have made a priori choice of "correct" soliton states to complete the construction of H_{sG}. Quite the same problem is met, though not even mentioned, in the attempts of [15, 17] to construct quantum Bäcklund transformations for the sine–Gordon system.

C. Each 1-soliton velocity $\dot{q} = (a^2 - 1)/(a^2 + 1)$ gives rise to the (asymptotic) 1-soliton momentum:

$$k = \frac{8m\dot{q}}{\sqrt{1 - \dot{q}^2}} = 8m \frac{a^2 - 1}{2|a|} \tag{2.13}$$

which is a monotomically growing function of the positive argument $|a|$ such that $|a| \to 0 \Rightarrow k \to -\infty$, $|a| \to \infty \Rightarrow k \to +\infty$. Consequently, the $(\dot{q})^N$ labeling can be replaced by an explicit momentum $(k)^N$ labeling, with $k \in R$.

Each N-soliton and consequently a coherent soliton state exhibits some kink–

antikink content due to the occurrence of $(\operatorname{sgn} a)^N$ sequence. This content can be established as follows [3]: we assume the momenta $(k)^N$ to be ordered in the ascending order of magnitude $k_1 < k_2 < \cdots < k_N$. To k_1 we attach the number $s_1 = +1$, to k_2, $s_2 = -1$ and so on up to $s_N(-1)^{N+1}$. The number

$$R_i = (-1)^{i+1} \operatorname{sgn} a_i = s_i \operatorname{sgn} a_i \tag{2.14}$$

identifies the kink, $R = +1$, or antikink, $R = -1$, presence in the multilink coherent state.

It means that upon an ordering $k_1 < \cdots < k_N$ of momenta, a sequence $(R)^N$ of topological charges [3] can be used instead of $(\operatorname{sgn} a)^N$:

$$\begin{aligned} \Phi_N(x) &= \Phi_N[\underline{R}, \underline{q}, \underline{k}](x), \\ |\varphi) &:= |\underline{R}, \underline{q})^N_{(k)}. \end{aligned} \tag{2.15}$$

The sine–Gordon Hamiltonian exhibits a translational symmetry, but neither the soliton fields nor the soliton coherent states are translationally invariant. At a fixed choice of $(k)^N$, $(R)^N$, any mapping of a sequence (q) into another (q') results in a new solution of the sine–Gordon equation and in a new coherent soliton state: $|(R), (q))^N_{(k)}$, $|(R), (q'))^N_{(k)}$ being orthogonal [6]. The conventional remedy in connection with these translation (in)variances is the introduction of (quantum) collective coordinates [7–9, 16, 17]. We shall, however, follow a slightly different route and eliminate the translation freedom by considering the general states in $H^N_{(k)}$, which are of the form

$$|\chi) = |\chi)^{(N)}_{(k)} = \sum_{(R)} \int_{R^1} dq_1 \cdots \int_{R^N} dq_N \chi^{R_1 \cdots R_N}(q_1, ..., q_N) \cdot |\underline{R}, \underline{q})^N_{(k)}. \tag{2.16}$$

The choice of such states as the state of interest for the quantum sine–Gordon system is motivated by the additional to translations of reflection freedom: if we replace a sequence $(R)^N$ in (2.15) by another $(R')^N$, we arrive at a different N-soliton solution of (1.1) at $(k)^N$ fixed and at a different coherent soliton state $|\underline{R}', \underline{q})^N_{(k)}$ which is orthogonal to $|\underline{R}, \underline{q})^N_{(k)}$. Notice that a reflection $R_i \to -R_i$ replaces a soliton by an antisoliton or, inversely, at a fixed momentum value k_i.

One should also notice that classical energy is the same for all possible choices of $(q)^N$, $(R)^N$, provided $(k)^N$ is untouched.

Remark. In the above, we do not consider solutions with breathers; let us, however, mention a classically arising "spin" sectorial structure of the set of sine–Gordon solitons. Namely, let us make use of the asymptotic formula (1.12). To each asymptotic field there corresponds a topological invariant

$$s_3(\Phi) = \frac{1}{4\pi} \int_{R^1} \frac{\partial \Phi}{\partial x} dx = \sum_{i=1}^{N} s_3(\Phi_i), \tag{2.17}$$

$s_3(\Phi)=1/2$ for the 1-soliton, $-1/2$ for the antisoliton, and 0 for a breather. Hence, for any $n+2m=N$ soliton solution the m-contribution to $s_3(\Phi)$ equals 0.

Let us denote

$$s(\Phi)=\sum_{i=1}^{N}|s_3(\Phi_i)| \tag{2.18}$$

and notice that the N-soliton solutions, N fixed, can be classified according to the $s(\Phi)$, $s_3(\Phi)$ labels:

$N=1$		$s=\frac{1}{2}$	$s_3=\pm\frac{1}{2}$			
$N=2$	$m=0$	$s=1$	$s_3=\pm 1, 0;$	$m=1$	$s=s_3$	$=0$
$N=3$	$m=0$	$s=\frac{3}{2}$	$s_3=\pm\frac{3}{2}, \pm\frac{1}{2};$	$m=1$	$s=\frac{1}{2},$	$s_3=\pm\frac{1}{2}$
$N=4$	$m=0$	$s=2$	$\lvert s_3\rvert\leqslant 2;$	$m=1$	$s=1, \lvert s_3\rvert\leqslant 1;$	$m=2, s=s_3=0$

and so on with the growth of N, where for each N, the specification of $s(\Phi)$ relies on the number of breathers involved.

In the considerations of Section 2, we have omitted breathers, hence $m=0\ \forall N$. Consequently, we can specify $s_3(\Phi)$ in a different way. Let n indicate the number of solitons, while $\tilde{n}$ is the number of antisolitons in the large-time asymptotics of Φ_N; then

$$N=n+\tilde{n}=2s, \qquad s_3=\tfrac{1}{2}(n-\tilde{n}). \tag{2.19}$$

3. Generalized Ice Model Eigenstates in H_{sG}

A. Let us consider a specialized version of (2.16), where summations with respect to $(R=\pm 1)^N$ are undone. Then

$$|\chi)=|\chi,\underline{R})^N_{(k)}=\int_{R^1}dq_1\cdots\int_{R^1}dq_N\chi(q_1,\ldots,q_N)\cdot|\underline{R},\underline{q})^N_{(k)}. \tag{3.1}$$

We admit these choices of χ only under which states (3.1) are normalized. It implies that at each fixed choice of χ, a nice orthonomality property can be observed in $H^N_{(k)}$:

$$(\chi,\underline{R}\,|\,\chi,\underline{R}')=\delta_{\underline{R}\underline{R}'}=\delta_{R_1R_1'}\cdots\delta_{R_NR_N'}. \tag{3.2}$$

For each choice of χ let us denote $h(\chi)$ a linear span of all $|\chi,\underline{R})^N_{(k)}$ in $H^N_{(k)}$. Recall that to introduce the $(R)^N$ parametrization, we have demanded an ordering $k_1<\cdots<k_N$ of the momentum set (k). Any fixed sequence $(R)^N$, say $(+1, +1, -1, +1,\ldots,-1)$, of topological invariants we call a configuration. To change a configuration of the state $|\chi,\underline{R})^N_{(k)}$ it suffices to make one or more reflections

$R_i \to -R_i$ of the topological invariants in the sequence $(R)^N$. With (3.2) in mind, we can introduce in $h(\chi)$ the following R-raising–lowering operators:

$$\begin{aligned} \sigma_i^+ &= \sum_{\text{conf}}^{(i)} |R_1,..., R_i = +1,..., R_N)(R_N,..., R_i = -1,..., R_1|, \\ \sigma_i^- &= \sum_{\text{conf}}^{(i)} |R_1,..., R_i = -1,..., R_N)(R_N,..., R_i = +1,..., R_1|. \end{aligned} \tag{3.3}$$

Here $\sum_{\text{conf}}^{(i)}$ means that we perform summation over all admissible configurations of $(R)^N$ under an assumption that R_i is left untouched. The simplified notation $|\underline{R}) = |\chi_1 \underline{R})_{(k)}^N$ was used in (3.3). A sequence $\{\sigma^+, \sigma^-\}^N$ of operators (3.3) satisfies the following commutation relations on $h(\chi) \in N_{(k)}^N$:

$$\begin{aligned} [\sigma_i^+, \sigma_j^+]_- &= 0 = [\sigma_j^-, \sigma_i^-]_-, \\ [\sigma_i^-, \sigma_j^+]_- &= 0, \qquad i \neq j, \end{aligned} \tag{3.4}$$

and

$$[\sigma_i^-, \sigma_i^+]_+ = \sum_{\text{conf}}^{(i)} |..., -1,...)(..., -1,...| + \sum_{\text{conf}}^{(i)} |..., +1,...)(..., +1,...| = 1_\chi, \tag{3.5}$$

where 1_χ is an identity on $h(\chi)$. We also have

$$\begin{aligned} (\sigma_i^+)^2 &= 0 = (\sigma_i^-)^2, \\ \sigma_i^+ |..., -1,...) &= |..., +1,...), \\ \sigma_i^- |..., +1,...) &= |..., -1,...), \end{aligned} \tag{3.6}$$

and, consequently,

$$\sigma_i^- |..., -1,...) = 0 = \sigma_i^+ |..., +1,...). \tag{3.7}$$

A sequence $\{\sigma^+, \sigma^-\}^N$ of Pauli operators determines a Lie algebra of the $SU(2)^N$ group in $h(\chi)$:

$$[\sigma_i^a, \sigma_j^b]_- = i\delta_{ij} \cdot \varepsilon_{abc}\sigma_j^c, \qquad a, b, c = 1, ,2, 3, \qquad i, j = 1, 2,..., N, \tag{3.8}$$

where $\vec{\sigma}_i$ is the spin-1/2 $SU(2)$ group generator related to $\sigma_i^\pm$ by

$$\begin{aligned} \sigma_i^1 = \frac{1}{\sqrt{2}}(\sigma_i^+ + \sigma_i^-), \qquad \sigma_i^2 &= \frac{i}{\sqrt{2}}(\sigma_i^+ - \sigma_i^-), \\ \sigma_i^3 = (-1/2) &+ \sigma_i^+\sigma_i^-. \end{aligned} \tag{3.9}$$

Notice that (3.6) provides us with mappings of a soliton into antisoliton or, conversely, at a fixed momentum value k_i.

Equivalently, an application of $\sigma_i^+\sigma_j^+$ to $|..., +1,..., -1,...)$ can be interpreted as a momentum exchange between a soliton and antisoliton in the multikink coherent state $|\chi, \underline{R})^{(N)}_{(k)}$.

B. Let $|\psi) \in h(\chi)$, then

$$|\psi) = \sum_{\text{conf}} \psi_{R_1 \cdots R_N} |R_1, ..., R_N) := \sum_{\alpha} \psi_\alpha |\alpha), \tag{3.10}$$

where α stands for a configuration of topological invariants. Let $\hat{T}$ be a linear operator in $h(\chi)$:

$$\hat{T} = \sum_{\alpha, \beta} T_{\alpha\beta} |\alpha)(\beta| \tag{3.11}$$

which if applied to $|\psi)$ gives rise to

$$\hat{T}|\psi) = \sum_{\beta} \left(\sum_{\alpha} T_{\beta\alpha} \psi_\alpha \right) |\beta) := \sum_{\beta} (T\psi)_\beta |\beta). \tag{3.12}$$

Here $(T\psi)_\alpha$ is the αth coordinate of the vector $\psi = (\psi_\alpha)$, to which a transfer matrix T is applied.

We shall now choose a realization [18] for $\hat{T}$ in terms of the previously introduced Pauli operators:

$$\hat{T} = \operatorname{tr}(L_1 L_2 \cdots L_N). \tag{3.13}$$

Where the trace is calculated for the product of N, 2×2 matrices with operator-valued matrix elements,

$$L_k = \begin{pmatrix} w_3 \sigma_k^3 + w_4 \sigma_k^4, & w_1 \sigma_k^1 - i w_2 \sigma_k^2 \\ w_1 \sigma_k^1 + i w_2 \sigma_k^2, & -w_3 \sigma_k^3 + w_4 \sigma_k^4 \end{pmatrix} \tag{3.14}$$

and the w_i's are real numbers, $\sigma_k^4 = [\sigma_k^+, \sigma_k^-]_+$.

$\hat{T}$ is known as the transfer operator of the symmetric eight-vertex Baxter model [19, 20]. It is well known that under the periodic boundary conditions, the spin-1/2 xyz model Hamiltonian commutes with the transfer operator $\hat{T}$ on $h(\chi)$:

$$\hat{H}_{xyz} = -\sum_{j=1}^{N} \sum_{a=1}^{3} J_a \sigma_j^a \sigma_{j+1}^a \tag{3.15}$$

with $\vec{J}$ being related to parameters $\{w\}_1^4$, but still exhibiting a 1-parameter freedom of choice [21]: $\vec{J} = \vec{J}(\zeta)$, $\zeta \in R^+$.

Because of $[\hat{T}, \hat{H}_{xyz}]_- = 0$ a solution of the spectral problem for $\hat{T}$ establishes this for $\hat{H}_{xyz}$ or inversely. A study of

$$\hat{T}|\psi) = \tau|\psi) \Rightarrow (T\psi)_\alpha = \tau\psi_\alpha \ \forall \alpha \tag{3.16}$$

can be found in [19, 20]. It is important to notice that the eigenvectors and eigenvalues of T are completely specified by solving the matrix problem (3.16), and hence do not rely on the specific choice of χ and momentum sequence (k). The N-dependence, however, remains of interest. Once a set of coordinates $(\psi_\alpha) = \psi$ is given, a Hilbert space vector $|\psi)$ is an eigenvector of $\hat{T}$ in $h(\chi)$. A complete solution of the matrix eigenvalue problem (3.16) was given in [19, 20]. Compare, e.g., formula (13) in [20]. The number 2^N of linearly independent eigenvectors $\{\psi^k\}_{k=1,\ldots,2^N}$, $\psi^k = (\psi^k_\alpha)$ can be established and the corresponding eigenvalues of H_{xyz} were derived in [19]; see also [22, 23]. For a fixed choice of $\{w\}_1^4$ and $\zeta \in R^+$, a solution is unique, and the spectrum consists of the "free" energy and the "bound state" series Δ_n, which may terminate at any integer not exceeding $2^N - 1$.

C. For the isotropic $\hat{H}_{xyz}$ problem, an operator $\vec{S} = \sum_{i=1}^N \vec{\sigma}_i$ is a generator of the symmetry group. The respective Casimir invariants are $\vec{S}^2$ and S_3. Because of the conservation law $\dot{\vec{S}} = 0$, the diagonalization problem for $\hat{H}_{xyz}$ resolves the simultaneous diagonalization problem for $\vec{S}^2$ and S_3 as well. Hence, together with an energy eigenvalue E we should be able to specify the s, s_3 eigenvalues of $SU(2)^N$ Casimir operators, for each given eigenvector $|\psi)$.

For the general xyz case, all $s = N/2$, $N/2 - 1,\ldots$ eigenvalues are allowed to occur. At this point we shall make a severe restriction by demanding

$$w_1 = w_2 = 0 \tag{3.17}$$

in (3.11), which convertes the general xyz problem into its specialized version known also as the symmetric six-vertex or the generalized ice model [18, 24–27]. One should observe that (3.17) excludes vertices 7 and 8 of the initial eight-vertex problem, see [18, Sect. D] or [21, Sect. 3]. Equation (3.17) if combined with the periodic (toroidal) boundary conditions implies that the transfer operator $\hat{T}$ can be written as a sum of terms each of which contains the same number of σ^+ as this of σ^- and thus does not change the number of down (or up) spin arrows in a state. Consequently, the eigenvectors of $\hat{T}$ have a particularly simple form:

$$|\psi) = \sum_{\text{perm}} \psi_{\pi(R)} \colon |\pi(R)), \tag{3.18}$$

where for a given initial configuration (R), we have $\pi(R) = (R_{\pi(1)},\ldots, R_{\pi(N)})$, i.e., a permutation of the sequence (R). Notice that at a fixed choice of the momentum sequence $k_1 < k_2 \cdots < k_N$ one finds

$$|\pi(R)) = |\pi(R))_{(k)} = |\underline{R})_{\pi(k)}, \tag{3.19}$$

i.e., (3.19) describes the allowed momentum exchanges among 1-soliton constitutents of the quantum N-soliton state $|\psi) = |\psi)_{(k)}$ at a fixed choice of labels n and $\tilde{n}$ of (2.19).

Consequently, the eigenstates (3.19) of $\hat{T}$, in addition to a fixed $N = 2s = n + \tilde{n}$ label, admit the $s_3 = n - \tilde{n}$ parametrization. It establishes a correspondence of the classical sectorial structure of the set of sine–Gordon solitons without breathers, (2.17)–(2.20) and the quantum soliton states.

Consequently

$$|\psi) = |\psi)^{n\tilde{n}}, \tag{3.20}$$

where n indicates the number of 1-solitons, while $\tilde{n}$ this of antisolitons in the soliton eigenstate $|\psi)$ of $\hat{T}$.

D. We can immediately construct the soliton eigenstates of interest by taking into account the ice-model solution given in [15, 18]. Notice that a state $|+\cdots+) = |\psi)^{n,0}$ is an eigenstate of $\hat{T}$, and analogously for $|-,\dots,-) = |\psi)^{0,\tilde{n}}$. If we take the all-spins-up eigenstate $|\psi)^{n,0}_{(k)}$ as a reference state $|0)$ in $h^N_{(k)}(\chi)$, then other eigenstates of $\hat{T}$ read

$$|\psi)^{N-1,1} = |p) = \sum_{j=1}^{N} \exp(ipj)\cdot\sigma_j^-\,|0), \qquad \exp(ipN) = 1, \tag{3.21}$$

$$\vdots$$

$$|\psi)^{N-m,m} = |p_1,\dots,p_m) = \sum_{1\leqslant j_1<\cdots<j_m\leqslant N} f(j_1,\dots,j_m)\cdot\sigma_{j_1}^-\cdots\sigma_{j_m}^-|0) \tag{3.21$'$}$$

$$\vdots$$

where, in case of (3.21′), the expansion coefficients exhibit a manifest (p)-dependence:

$$f(j_1,\dots,j_m) = \sum_{\pi\in S_m} a(\pi)\cdot\exp\left(i\sum_{r=1}^{m} p_{\pi(r)}\,j_r\right). \tag{3.22}$$

Summations are carried out with respect to permutations of the sequence $(1, 2,\dots, m)$ and we have $m!$ coefficients $a(\pi)$, each one corresponding to one permutation π.

The set $(p) = (p_j,\dots,p_m)$ of wave numbers is not arbitrary and should be restricted by a periodicity condition after establishing the set of appropriate $a(\pi)$. This has been done in [27]. With the accuracy up to an overall normalization constant, we have

$$a(\pi) = (-1)^{\pi}\exp\left[-\frac{i}{2}\sum_{j<k}\theta(p_{\pi(j)}, p_{\pi(k)})\right], \tag{3.23}$$

$$\exp[-i\theta(p,q)] = \frac{1+\exp i(p+q)-\exp(ip)}{1+\exp i(p+q)-\exp(iq)}$$

and the periodicity condition reads

$$\begin{aligned}\exp(ip_jN) &= -\prod_{j\neq k} B(p_j,p_k),\\ B(p,q) &= -\exp[-i\theta(p,q)],\end{aligned} \tag{3.24}$$

thus imposing restrictions on the admissible values of wave numbers p_j.

The generalized ice problem is an example of the completely integrable system (like the more general Baxter model), and its solution via the quantum inverse method [26] allows construction of all the eigenvectors of $\hat{T}$ by starting from the reference state $|0)$ with all spins-up and then applying appropriate pseudoparticle creation operators:

$$|p_1,\ldots,p_m) = \prod_{j=1}^{m} b(p_j)\,|0). \tag{3.25}$$

It completes the derivation of the $\hat{T}$ eigenstates in $h^N_{(k)}(\chi)$. Notice that amplitudes $f(j,\ldots,j_m)$ are completely independent of the choice of χ and of the soliton momentum sequence (k). The N dependence only matters for the construction of (3.23)–(3.25).

Recall that $|0)$ is the N-soliton reference state consisting of 1-solitons only: $|0) = |+,\ldots,+)^N_{(k)}$. In the above considerations antisolitons play the role of particles put into the soliton "sea." For example, $b(p)\,|0) = |\,p)$ is a single pseudoparticle state, while $|\,p_1,\ldots,p_m)$ is the m-pseudoparticle state received by "putting" 1 or m antisolitons into the $N-1$ and $N-m$ soliton "sea," respectively.

For the wave functions and pseudoparticle energy spectrum, the χ, (k) dependence is completely irrelevant; hence quite a universal (N, m), $m \leqslant N$, sectorial structure can be recovered in H_{sG}. At each choice of N, soliton states of the form (3.21), (3.25) can be thought of as elements of an equivalence class labeled by the respective $\hat{T}$ operator eigenvalues $\tau(p,\ldots,p_m) = \tau^N_m$. By varying N, this equivalence class structure can be extended to the whole of H_{sG}.

4. "Wick Rotation" of the Ice-Model Variables: Sine–Gordon Eigenstates in H_{sG}

A. Within the quantum inverse-scattering method, quite a variety of 1 + 1 dimensional models, in the "local transition matrix" formalism [19, 20], exhibits the same algebraic structure. In terms of the basic inverse-scattering (operator-valued) data, they can be viewed as *representations of the same operator algebra.*

The basic object of the theory, the transition (monodromy) matrix for the N-site chain is given by

$$T = T_N(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \qquad C(\lambda) = -B^*(\lambda),\, D(\lambda) = A^*(\lambda), \tag{4.1}$$

where in the original formulation of [25] one relates N to an interval L divided into smaller pieces with a regular spacing $\Delta: N = L/\Delta$. In the above, one usually takes $\lambda \in R$, otherwise $-B^*(\bar{\lambda})$. $A^*(\bar{\lambda})$ should be introduced in (4.1). The fundamental commutation relations for matrix elements of T read as follows:

$$\begin{gathered}[A(\lambda), A(\mu)] = 0 = [B(\lambda), B(\mu)]_- = [A(\lambda) + D(\lambda), A(\mu) + D(\mu)]_-, \\ B(\lambda)\, A(\mu) = b(\lambda, \mu)\, B(\mu)\, A(\lambda) + c(\lambda, \mu)\, A(\mu)\, B(\lambda), \\ B(\mu)\, D(\lambda) = b(\lambda, \mu)\, B(\lambda)\, D(\lambda)\, D(\mu) + c(\lambda, \mu)\, D(\lambda)\, B(\mu).\end{gathered} \tag{4.2}$$

The factors $b(\lambda, \mu)$ and $c(\lambda, \mu)$ are elements of another important ingredient of the inverse method, the R-matrix:

$$R = R(\lambda, \mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.3}$$

which for the Baxter model differs in that $R_{11} = R_{44} = a \neq 1$, $R_{14} = R_{41} = d \neq 0$, while for the nonlinear Schrodinger model, the Toda lattice, since–Gordon system, Heisenberg ferromagnet has the form (4.3).

To specify a concrete model of interest, one must choose the appropriate matrix elements of (4.3). The representation of (4.2) is constructed in a Hilbert space

$$\mathscr{H}_N = \prod_{i=1}^{N}{}^{\otimes} h_i, \qquad h_i = h \ \forall i,$$

h being some Hilbert space, by selecting a state $\Omega_0 \in \mathscr{H}_N$ such that (we put $\Delta = 1$, !)

$$\begin{gathered} A(\lambda)\, \Omega_0 = \exp\{a(\lambda) N\}\, \Omega_0, \\ D(\lambda)\, \Omega_0 = \exp\{d(\lambda) N\}\, \Omega_0, \\ C(\lambda)\, \Omega_0 = 0, \qquad d(\lambda) = \overline{a(\bar{\lambda})}. \end{gathered} \tag{4.4}$$

Then, a Hilbert space vector:

$$|\lambda_1, ..., \lambda_n) = \prod_{i=1}^{n} B(\lambda_i)\, \Omega_0 \tag{4.5}$$

under a restriction (periodicity condition)

$$\exp\{[a(\lambda_k) - d(\lambda_k)] N\} = \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{c(\lambda_j, \lambda_k)}{c(\lambda_k, \lambda_j)}, \qquad k = 1, 2, ..., n \tag{4.6}$$

is an eigenvector of the operator $A(\lambda) + D(\lambda)$:

$$[A(\lambda) + D(\lambda)]\, |\lambda_1, ..., \lambda_n) = \Lambda(\lambda, \lambda_1, ..., \lambda_n)(\lambda_1, ..., \lambda_n), \tag{4.7}$$

where

$$\Lambda(\lambda, \lambda_1, \ldots, \lambda_n) = e^{a(\lambda)N} \cdot \prod_{j=1}^{n} \frac{1}{c(\lambda_j, \lambda)} + e^{d(\lambda)N} \prod_{j=1}^{n} \frac{1}{c(\lambda, \lambda_j)}. \tag{4.8}$$

To specify a concrete model of interest one must thus choose the appropriate matrix elements for R and the appropriate values (which in fact follow from the former choice) for $a(\lambda)$, $b(\lambda)$, $c(\lambda)$, and $d(\lambda)$.

B. The underlying specification of the generalized ice model reads as follows [25]:

$$\begin{aligned} b(\lambda, \mu) &= \frac{\sin(v - v')}{\sin(v - v' - \eta)}, \qquad v = \ln \lambda,\ v' = \ln \mu, \\ c(\lambda, \mu) &= \frac{-\sin 2\eta}{\sin(v - v' - 2\eta)} \end{aligned} \tag{4.9}$$

and in addition to the variable v, a parameter η appears, which is related to the Baxter's parametrization of $\{w\}_1^4$:

$$w_1 = w_2 = \rho \sin 2\eta, \qquad w_3 = \rho \sin \eta \cos v, \qquad w_4 = \rho \cos \eta \sin v \tag{4.10}$$

an overall normalization ρ being irrelevant. Then

$$\begin{aligned} \exp\{a(\lambda)N\} &= [\sin(\sigma + \eta)]^N, \\ \exp\{d(\lambda)N\} &= [\sin(v - \eta)]^N \end{aligned} \tag{4.11}$$

allows one to compute (4.6) and (4.8).

One should realize that the (v) parametrization is related to the (p) parametrization of the previous section via [18]:

$$\exp(ip) = \frac{\sin(v + \eta)}{\sin(v - \eta)} \tag{4.12}$$

so that we have in fact

$$\begin{aligned} |p_1, \ldots, p_m) &= \prod_{i=1}^{n} B(v_i)\, \Omega_0, \\ \exp(ip_k N) &= \left(\frac{\sin(v_k + \eta)}{\sin(v_k - \eta)}\right)^N \\ &= \prod_{\substack{j=1 \\ j \neq k}}^{n} \left(\frac{\sin(v_k - v_j + 2\eta)}{\sin(v_k - v_j - 2\eta)}\right). \end{aligned} \tag{4.13}$$

C. The quantum inverse-scattering data for the sine–Gordon model read [29]

$$b(\lambda, \mu) = \frac{\operatorname{sh}(v - v')}{\operatorname{sh}(v - v' + i\gamma)},$$

$$c(\lambda, \mu) = \frac{i \sin \gamma}{\operatorname{sh}(v - v' + i\gamma)}, \tag{4.14}$$

$$a(\lambda) = \frac{m^2}{8} \operatorname{ch}(2v - i\gamma), \qquad d(\lambda) = \overline{a(\bar{\lambda})},$$

where, consequently,

$$\begin{aligned} \exp(ip_k N) &= \exp\left\{ i \frac{m^2}{4} N \sin \gamma \operatorname{sh} 2v_k \right\} \\ &= \exp\{[a(\lambda_k) - d(\lambda_k)] N\} \\ &= \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{\operatorname{sh}(v_k - v_j - i\gamma)}{\operatorname{sh}(v_k - v_j + i\gamma)} \end{aligned} \tag{4.15}$$

and to be in agreement with [29], $m \ll 1$ must be assumed. Equation (4.15) determines the momentum variable by analogy to (4.13). Let us emphasize that we have determined the quantum sine–Gordon case of [29] in the regime $N < \infty$, $\Delta = 1$, which obviously contrasts with the Faddeev *et al.* Final goal of letting Δ go to 0 and $L = N\Delta$ to infinity (then the semiclassical sine–Gordon/massive Thirring spectrum can be recovered).

D. With respect to the eigenvalue formula (4.8) and the periodicity condition (4.6), both the ice model and the sine–Gordon model are related in a very simple way. Namely, it is enough to make a Wick rotation of the fundamental variables, e.g.,

$$v \to iv, \qquad v' \to iv', \qquad 2\eta = \gamma, \qquad \gamma \in (0, 2\pi) \tag{4.16}$$

to go from one one model to another, for then

$$\sin(v - v' - 2\eta) \to \sin[i(v - v' + i\gamma)] = i \operatorname{sh}(v - v' + i\gamma). \tag{4.17}$$

One point however must be clarified. Namely, upon the Wick rotation (4.10), the ice-model data (4.9) are transformed as follows:

$$\begin{aligned} \frac{\sin(v - v')}{\sin(v - v' - 2\eta)} &\to \frac{\operatorname{sh}(v - v')}{\operatorname{sh}(v - v' + i\gamma)}, \\ \frac{-\sin(2\eta)}{\sin(v - v' - 2\eta)} &\to \frac{i \sin \gamma}{\operatorname{sh}(v - v' + i\gamma)} \end{aligned} \tag{4.18}$$

and the emergence of the imaginary factor $(-i)$ in the context of (4.10) should be explained. For this particular purpose one must return to Baxter's model, from which both of the above-cited ice and sine-Gordon models can be derived.

E. Let us exploit an original Baxter's parametrization of basic, symmetric eight-vertex model quantities, in terms of the elliptic functions of a fixed elliptic modulus k [30, 31]:

$$
\begin{aligned}
&a = w_4 + w_3, \qquad b = w_4 - w_3, \qquad c = w_1 + w_2, \qquad d = w_1 - w_2,\\
&a = \rho \sin(v + \eta, k) = \theta(2\eta)\, \theta(v - \eta)\, H(v + \eta),\\
&b = \rho\, \mathrm{sn}(v - \eta, k) = \theta(2\eta)\, H)\, \theta(v + \eta),\\
&c = \rho\, \mathrm{sn}(2\eta, k) = H(2\eta)\, \theta(v - \eta)\, \theta(v + \eta),\\
&d = \rho^{-2} kabc = H(2\eta)\, H(v - \eta)\, H(v + \eta).
\end{aligned}
\tag{4.19}
$$

One knows [18] that by defining a constant $\rho_0 = k^{1/2}\rho$ and then letting k go to 0 while ρ goes to ∞, we can recover the ice-model parametrization (4.10). Then

$$
\begin{aligned}
&a = w_4 + w_3 \to \rho_0 \sin\eta \cos v + \rho_0 \cos\eta \sin v,\\
&b = w_4 - w_3 \to \rho_0 \sin\eta \cos v - \rho_0 \cos\eta \sin v,\\
&c = w_1 + w_2 \to 2\rho_0 \sin 2\eta,\\
&d = w_1 - w_2 \to 0.
\end{aligned}
\tag{4.20}
$$

In connection with the Wick rotation (4.16), let us note the following property of the elliptic function $\mathrm{sn}(u, k)$ under the so-called Jacobi's imaginary transformation [32]:

$$
\mathrm{sn}(iu, k) = i\, \mathrm{sc}(u, k_1), \qquad k + k_1 = 1, \tag{4.21}
$$

where the following behavior of the elliptic functions $\mathrm{sn}(u, k)$ and $\mathrm{sc}(u, k)$ is of interest to us:

$$
\lim_{k\to 0} \mathrm{sn}(u, k) = \sin u, \tag{4.22}
$$

$$
\lim_{k_1\to 1} \mathrm{sc}(u, k_1) = \lim_{k\to 0} \mathrm{sc}(u, 1 - k) = \mathrm{sh}\, u.
$$

By making use of (4.21) and (4.22) the relationship of the generalized ice and sine–Gordon models via the Baxter model is made clear. Namely, the limit $k \to 0$ *before* the Wick rotation of $v \in R$ was done leads us to the generalized ice model. On the other hand, the limit $k \to 0$ *after* making the Wick rotation recovers the sine–Gordon system. In terms of (4.19) these limits are especially clear:

$$\begin{aligned} a &\to \rho_0 \sin(v+\eta), \\ b &\to \rho_0 \sin(v-\eta), \\ c &\to \rho_0 \sin(2\eta), \\ d &\to 0, \end{aligned} \tag{4.23}$$

while after the transformation $v \to iv$, we arrive at

$$\begin{aligned} a &= \rho i \operatorname{sc}(v - \tfrac{1}{2}i\gamma, k_1), \\ b &= \rho i \operatorname{sc}(v + \tfrac{1}{2}i\gamma, k_1), \\ c &= \rho \operatorname{sn} \gamma, \\ d &= \rho^{-2} kabc \end{aligned} \tag{4.24}$$

so that $k_1 \to 1$ (or, equivalently $k \to 0$ in $k + k_1 = 1$) implies

$$\begin{aligned} a &\to i\rho_0 \operatorname{sh}(v - \tfrac{1}{2}i\gamma), \\ b &\to i\rho_0 \operatorname{sh}(v + \tfrac{1}{2}i\gamma), \\ c &\to \rho_0 \sin \gamma, \\ d &\to 0. \end{aligned} \tag{4.25}$$

Now an effect of the Wick rotation on (4.18) is

$$a(\lambda) = \frac{m^2}{8} \cos[2(v+\eta)] \to a(\lambda) = \frac{m^2}{8} \operatorname{ch}(2v - i\gamma) \tag{4.26}$$

which completes the ice–sine–Gordon model relationship, see however [33].

F. The original Baxter model parameters a, b, c, and d are introduced as Boltzmann weights at a fixed inverse temperature of the reservoir which keeps a system at thermal equilibrium. This applies to the ice model, whose properties have thus a *purely thermal* (*statistical*) *origin*. However, the "Wick rotation" (4.16) transforms a set of weights (4.23)

$$\begin{gathered} a = \exp(-\beta\varepsilon_1) = \rho_0 \sin(v+\eta), \qquad b = \exp(-\beta\varepsilon_2) = \rho_0 \sin(v-\eta), \\ c = \exp(-\beta\varepsilon_3) = \rho_0 \sin 2\eta, \qquad d = 0 \end{gathered} \tag{4.27}$$

into a new set of *complex quasiweights*, where c and d only are left unchanged, so that the meaning of β persists after applying (4.16). Let us assume that after "Wick rotation" ε_i becomes $\varepsilon_i' + i\varepsilon_i''$, $i = 1, 2$. As a consequence, we get

$$\begin{aligned} i\rho_0 \operatorname{sh}(v - i\eta) &= \exp(-\beta\varepsilon_1') \exp(-i\beta\varepsilon_1'') \\ i\rho_0 \operatorname{sh}(v + i\eta) &= \exp(-\beta\varepsilon_2') \exp(-i\beta\varepsilon_2''), \qquad \eta \in (0, \pi). \end{aligned} \tag{4.28}$$

Hence,

$$\begin{aligned} \rho_0 \operatorname{ch} v \sin \eta &= \exp(-\beta\varepsilon_1') \cdot \cos \beta\varepsilon_1'' = -\exp(-\beta\varepsilon_2') \cos \beta\varepsilon_2'', \\ -\rho_0 \operatorname{sh} v \cos \eta &= \exp(-\beta\varepsilon_1') \cdot \sin \beta\varepsilon_1'' = \exp(-\beta\varepsilon_2') \sin \beta\varepsilon_2'' \end{aligned} \tag{4.29}$$

which implies that

$$\operatorname{tg} \beta\varepsilon_2'' = \operatorname{tg} \beta \left(\frac{\pi}{\beta} - \varepsilon_1'' \right) = \operatorname{tg} v \cdot \operatorname{ctg} \eta. \tag{4.30}$$

Then solutions for ε_1', $i = 1, 2$ are available from

$$\exp(-\beta\varepsilon_1') = \frac{\rho_0 \operatorname{ch} v \sin \eta}{\cos \beta\varepsilon_1''}, \qquad \exp(-\beta\varepsilon_2') = -\frac{\rho_0 \operatorname{ch} v \cdot \sin \eta}{\cos \beta\varepsilon_2''}. \tag{4.31}$$

Notice that

$$\exp[-\beta(\varepsilon_1' - \varepsilon_2')] = -\frac{\cos \beta\varepsilon_2''}{\cos \beta\varepsilon_1''} \tag{4.32}$$

and that solutions ε_i^α, $\alpha = {}'$, $''$, $i = 1, 2$, do exhibit a manifest β, v, η dependence if v and η are considered as independent variables.

G. A main consequence of the above discussion is that all the results obtained by Faddeev *et al.* in [29, 30] for the sine-Gordon model in the $N < \infty$, $\Delta = 1$ regime, can be completely translated to the ice-model language and inversely. Except for the "Wick rotation," both models have completely identical operator structure. Consequently, *upon the Wick rotation, all the results of Section* 3 *do reproduce properties of the sine–Gordon system in the presence of solitons.* The underlying energy spectrum is related to the momentum exchange among solitons only.

Notice that the assumption $\Delta = 1$ prevents us from obtaining a continuum limit which is the next step in [29]: to approach the semiclassical sine–Gordon/Thirring model spectrum. In fact, we have shown in [21] that a semiclassical spectrum can be recovered in the so-called spin-1/2 approximation of the sine–Gordon system (real-time development problem at nonzero temperature), but then without the notion of soliton operators. In [6] the quantum soliton operators were introduced and coherent state domains for them were constructed. As a straightforward continuation of [6], the present observations emerge. We have received a quantum soliton spectrum, which though resembling the semiclassical sine–Gordon/massive Thirring one does not at all coincide with the latter.

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