REMARKS ON THE THEORY OF THE STERN-GERLACH EXPERIMENT (+)

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(I) The Stern-Gerlach experiment(1) is a text-book illustration of the basic ideas of quantum theory. However its theoretical explanations (2-4) do not seem to be adequate. Despite of the general agreement that it reflects essentially quantum features, and thus cannot to explained on purely classical grounds (see however(5,6)), even within the would-be-well established quantum framework there still remain both conceptual(7,8) and technical problems(9-11).

Among the recent publications on the subject(6-11) Ref. 10 only attempts to address the problem in full generality: One departs from the coupled set of the Heisenberg equation with the electromagnetic field solving the Maxwell equation. The latter feature is not the case in the standard (hence oversimplified) approaches to the problem.

From the conceptual point of view, we follow the statistical ensemble interpretation of the wave functions, with emphasis[12] on the fact that the only observational quantity which quantum mechanics needs to address is location.

Our goal is to analyse the propagation of the appropriately localized (Gaussian) wave packet, so that within the limitations of the Stern-Gerlach experiment the verifiable predictions are arrived at: about the location of geometric centers and intensities (probability distributions) of traces due to arise on the detecting photo-plate.

(II) We should study the Pauli equation for neutral massive (mass M) atoms with the magnetic moment, $\vec{\mu} = -\mu\vec{s}$ (for the electron $\mu = \mu_B \frac{2}{\hbar} = -\frac{e}{m_e c}$), where $\vec{s} = \frac{\hbar}{2}\vec{\sigma}$, $\vec{\sigma}$ denoting Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

The electric potential is equated zero, while the magnetic field is required to satisfy (as it should[10]) the equation

$$\operatorname{div} \vec{B} = 0$$

We choose:

$$\begin{aligned} \vec{A}(\vec{r}) &= (0, xzB_o(y), 0) \\ \vec{B}(\vec{r}) &= \nabla\times\vec{A}(\vec{r}) = (-x, 0, z)B_o(y) \end{aligned} \tag{3}$$

In the experiment the effective magnetic interaction is supposed to affect particles along the relatively small interval of length L in the y-direction. Along it we can take $B_o(y)$ to be a constant $\partial B_o/\partial y = 0$ which amounts to the gauge condition

$$\mathrm{div}\,\vec{A} = 0 \tag{4}$$

Hence the Pauli equation reads:

$$\left[-\frac{\hbar^2}{2M}\Delta + \frac{\mu B_o \hbar}{2}(z\sigma_z - x\sigma_x)\right]\Psi = i\hbar\partial_t\Psi \tag{5}$$

Its solution can be written in the form:

$$\Psi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int d^3p\, f(\vec{p})\left\{\exp\frac{i}{\hbar}\left[\vec{p}\vec{r} - \frac{\vec{p}^2}{2M}t + -\mu\vec{s}\vec{B}(\vec{r})t + \frac{\mu\hbar B_o}{4M}t^2(\sigma_z p_z - \sigma_x p_x) - \frac{(\mu B_o\hbar)^2}{12M}t^3\right]\right\}\chi \tag{6}$$

where $\chi = \binom{\alpha}{\beta}$ is a normalized spinor $|\alpha|^2 + |\beta|^2 = 1$. (6) is not very useful unless a specific choice of f(p) is made and the p-integrations performed (compare e.g. for example[7], which for arbitrary times may be analytically intractable.

To bypass this problem, we shall adopt the Heisenberg picture, where the time development of the dynamical variables $(\vec{r},\vec{p},\vec{\sigma})$ is given by:

$$\dot{r}_j = \frac{i}{\hbar}\left[H, r_j\right]_- = \frac{P_j}{M}$$

$$\dot{s}_j = \frac{i}{\hbar}\left[H, s_j\right]_- = \mu(\vec{s}\times\vec{B})_j \rightarrow \dot{s}_x = \mu B_o z S_y$$

$$\dot{s}_y = \mu B_o(z\sigma_x - x\sigma_z) \quad , \qquad \dot{s}_z = \mu B_o x s_y$$

$$\dot{p}_j = \frac{i}{\hbar}\left[H, p_j\right]_- = \mu \sum_k s_k \frac{\partial B_k}{\partial r_j} \rightarrow \dot{p}_x = \mu B_o s_x$$

$$\dot{p}_y = 0 \qquad \dot{p}_z = \mu B_o s_z$$

$$H = \frac{\vec{p}^2}{2M} - \mu \vec{s}\vec{B} \tag{7}$$

The essential simplification is achieved upon assuming that the substantial fraction of particles constituting the ensemble passes through the Stern-Gerlach magnet (i.e. along the distance L) in a relatively short time T<<1. In this short-passage-time approximation (see e.g. also[7]), the dynamics of the Heisenberg operators can be satisfactorily approximated by the leading contribution to the Taylor series:

$$t \in [0,T]$$

$$\vec{r}(t) = \vec{r}(0) + t\dot{\vec{r}}(0) + \frac{t^2}{2}\ddot{\vec{r}}(0) + \ldots \tag{8}$$

so that:

$$x(t) \cong x + \frac{t}{M} p_x + \frac{t^2}{2M} \mu B_o s_x \tag{9}$$

$$y(t) = y + \frac{t}{M} p_y$$

$$z(t) \quad z + \frac{t}{M} p_z + \frac{t^2}{2M} \mu B_o s_z$$

To pass to the Schrödinger picture dynamics, we shall follow [9,10] and invoke Kennard's propagator theorem. We denote the

2×2 matrix operators (9) by $\hat{x}(t)$, $\hat{y}(t)$, $\hat{z}(t)$ to avoid confusing them with the configuration variables x,y,z. The propagator for the problem comes from:

$$\hat{x}(-t)G(\vec{r},t;\vec{r}',0) = x'G(\vec{r},t;\vec{r}',0)$$

$$\hat{y}(-t)G(\vec{r},t;\vec{r}',0) = y'G(\vec{r},t;\vec{r}',0) \qquad (10)$$

$$\hat{z}(-t)G(\vec{r},t;\vec{r}',0) = z'G(\vec{r},t;\vec{r}',0)$$

i.e.

$$\left(x + i\hbar \frac{t}{M}\frac{\partial}{\partial x} + \frac{t^2}{2M}\frac{\mu B_o \hbar}{2}\sigma_x\right)G = x'G$$

$$\left(y + i\hbar \frac{t}{M}\frac{\partial}{\partial y}\right)G = y'G \qquad (11)$$

$$\left(z + i\hbar \frac{t}{M}\frac{\partial}{\partial z} + \frac{t^2}{2}\frac{\mu B_o \hbar}{2M}\sigma_z\right)G = z'G$$

so that:

$$G(\vec{r},t;\vec{r}',0) = \left(\frac{M}{i\hbar t}\right)^{3/2} \exp \frac{iM}{2\hbar t}$$

$$\left\{\left(x - x' - \frac{t^2}{2}\frac{\mu B_o \hbar}{2M}\sigma_x\right]^2 + (y-y')^2 +\right.$$

$$\left. + \left[z - z' - \frac{t^2}{2}\frac{\mu B_o \hbar}{2M}\sigma_z\right]^2\right\} \qquad (12)$$

which in the short passage-time-approximation reduces to:

$$G(\vec{r},t;\vec{r}',0) \cong \left(\frac{M}{i\hbar t}\right)^{3/2} \exp \frac{iM}{2\hbar t} (\vec{r} - \vec{r}')^2 \cdot (1 + i\gamma \vec{a}\vec{\sigma}) \tag{13}$$

$$\gamma = \frac{M}{2\hbar t}, \quad a_x = t^2 (x-x')\eta, \quad a_y = 0, \quad y_z = t^2 (z-z')\eta$$

$$\eta = \frac{\mu B_o \hbar}{2M}$$

This formula is sufficiently simple to enable us to perform the propagator on the spinor wave function.

(III) Let the initial wave packet, which enters the interaction area, be given by:

$$\Psi(\vec{r},0) = \phi(\vec{r},0)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$|\alpha|^2 + |\beta|^2 = 1 \tag{14}$$

where $\phi(\vec{r},0) = \rho^{1/2}(\vec{r},0)\exp\frac{i}{2}\psi(\vec{r},0)$ is the minimum uncertainty wave packet, which is centered about:

$$\langle\vec{r}\rangle = (0, y_o, 0)$$

$$\langle\vec{p}\rangle = (0, p_y, 0) \qquad p_y = Mv \tag{15}$$

Hence we deal with the localized wave, whose geometric center moves in the y-direction with the velocity v:

$$\phi(\vec{r},0) = \frac{1}{(\sigma\sqrt{2\pi})^{3/2}} \exp\left[-\frac{(\vec{r}-\langle\vec{r}\rangle)^2}{4\sigma^2}\right] \cdot \exp\frac{i}{\hbar}\langle\vec{p}\rangle\vec{r} \tag{16}$$

Here: $\delta r_i \delta p_i = \frac{\hbar}{2}$, $\delta x = \delta y = \delta z = \sigma$.

In the adopted approximation regime, an effect of the pure spinor rotation is:

$$(1 + i\gamma\vec{a}\vec{\sigma})\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \cong \begin{pmatrix}\alpha\\ \beta\end{pmatrix} +$$

$$\frac{iM}{2\hbar t}\left[t^2(x-x')\eta\sigma_x + t^2(z-z')\eta\sigma_z\right] =$$

$$\begin{pmatrix}\alpha\ \exp\frac{iM}{2\hbar t}\eta t^2\left[(z-z') + (x-x')\right]\\ \beta\ \exp\frac{iM}{2\hbar t}\eta t^2\left[-(z-z') + (x-x')\right]\end{pmatrix} +$$

$$+\frac{iM}{2\hbar t}\eta t^2(x-x')(\beta-\alpha)\begin{pmatrix}1\\ -1\end{pmatrix} \tag{17}$$

The pure matrix action of the propagator leads to:

$$G(\vec{r},t;\vec{r}',0)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \left(\frac{M}{i\hbar t}\right)^{3/2} \exp\frac{iM}{2\hbar t}(\vec{r}-\vec{r}')^2\cdot(1+i\gamma\vec{a}\vec{\sigma})\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \cong$$

$$\begin{pmatrix}\alpha\left(\frac{M}{i\hbar t}\right)^{3/2} \exp\frac{iM}{2\hbar t}\left[(x-x'+\eta t^2)^2+(y-y')^2+(z-z'+\eta\frac{t^2}{2})^2\right]\\ \beta\left(\frac{M}{i\hbar t}\right)^{3/2} \exp\frac{iM}{2\hbar t}\left[\left(x-x'+\eta\frac{t^2}{2}\right)^2+(y-y')^2+\left(z-z'-\eta\frac{t^2}{2}\right)^2\right]\end{pmatrix}$$

$$+\left(\frac{M}{i\hbar t}\right)^{3/2}\left[\exp\frac{iM}{2\hbar t}(\vec{r}-\vec{r}')^2\right]\left[\frac{iM}{2\hbar t}\eta t^2(x-x')(\beta-\alpha)\right]\begin{pmatrix}1\\ -1\end{pmatrix} \tag{18}$$

We are interested in the short passage time $t \leq \frac{L}{v} = T \ll 1$, where L is the distance in the y-direction, along which the moving atoms feel the magnetic shock.

By taking into account the formula

$$K(\vec{r},t,\vec{r}',t') = \frac{1}{(2\pi\hbar)^3}\int \exp\left\{\frac{i}{\hbar}\left[\frac{\vec{p}}{M}(\vec{r}-\vec{r}') - \right.\right.$$

$$\left.\left. - \frac{p^2}{2M}(t-t')\right]\right\} d^3p = \left[\frac{2\pi i\hbar}{M}(t-t')\right]^{-3/2} \exp\left[i\frac{M}{2\hbar}\frac{(\vec{r}-\vec{r}')^2}{t-t'}\right] \tag{19}$$

with its well known property

$$t \to t' \Rightarrow K(\vec{r},t;\vec{r}',t') \to \delta(\vec{r}-\vec{r}') \tag{20}$$

expressions of the form (19) for short times can be viewed as (T) approximations of the Dirac delta. In fact, for small but non-zero times, we can approximate (18) as follows:

$$(2\pi)^{3/2} G(\vec{r},T;\vec{r}',0)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \cong$$

$$\begin{pmatrix} \alpha\delta_{(T)}\left(x-x'+\eta\frac{T^2}{2}\right)\delta_{(T)}(y-y')\delta_{(T)}\left(z-z'+\eta\frac{T^2}{2}\right) \\ \beta\delta_{(T)}\left(x-x'+\eta\frac{T^2}{2}\right)\delta_{(T)}(y-y')\delta_{(T)}\left(z-z'-\eta\frac{T^2}{2}\right)\end{pmatrix} - \tag{21}$$

$$- \frac{\eta T^2}{2}\,\delta_{(T)}(y-y')\,\delta_{(T)}(z-z')\left[\frac{\partial}{\partial x'}\delta_{(T)}(x-x')\right]\begin{pmatrix}1\\ -1\end{pmatrix}(\beta-\alpha)$$

After composing (21) with the Gaussian wave packet, we arrive at:

$$\psi(\vec{r},T) = \int d^3r' G(\vec{r},T,\vec{r}',0)\,\Psi(\vec{r},0) \cong$$

$$(2\pi)^{3/2}\begin{pmatrix}\alpha\phi\left(x+\eta\frac{T^2}{2},y,z+\eta\frac{T^2}{2},0\right)\\ \beta\phi\left(x+\eta\frac{T^2}{2},y,z-\eta\frac{T^2}{2},0\right)\end{pmatrix} +$$

$$- (2\pi)^{3/2}(\beta-\alpha)\,\frac{x}{\sigma^2}\,\eta\frac{T^2}{2}\,\phi(x,y,z,0)\begin{pmatrix}1\\ -1\end{pmatrix} \tag{22}$$

In the above $\phi(x+\Delta,y,z\pm\Delta,0)$, $\Delta = \frac{\eta T^2}{2}$ and $\phi(\vec{r},0)$ include the same phase factor $\exp \frac{i}{\hbar} \langle p\rangle(y-y_o)$ with $\langle p_y\rangle = Mv$. On the other hand, the form of Δ-deflections which modify x and z variables in the above can be interpreted as arising due to the uniformly accelerated motion, whose effect must be a non-zero increment of the respective velocity (hence momentum) components, which refer to the motion of the geometric center of our wave packet:

$$\begin{aligned} \langle p_x\rangle = 0 \quad &\rightarrow \quad \langle p_x\rangle = M\eta T \\ \langle p_z\rangle = 0 \quad &\rightarrow \quad \langle p_z\rangle = \pm M\eta T \end{aligned} \tag{23}$$

Within the adopted approximation regime the necessary corrections can be made, so that the final approximate formula for the Gaussian wave packet which leaves the interaction area reads:

$$\psi(\vec{r},T) \cong (2\pi)^{3/2} \begin{pmatrix} \alpha\phi(x+\Delta,y,z+\Delta,0)\exp\left[-\frac{i}{\hbar}M\eta T(x+z)\right] \\ \beta\phi(x+\Delta,y,z-\Delta,0)\exp\left[\frac{i}{\hbar}M\eta T(z-x)\right] \end{pmatrix} - (2\pi)^{3/2}(\beta-\alpha)\frac{x}{\sigma^2}\eta\frac{T^2}{2}\phi(x,y,z,0)\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{24}$$

the phase factor $\exp\frac{i}{\hbar}Mv(y-y_o)$ being included in the ϕ's.

(IV) The whole experiment can be described by dividing the particle motion area into three pieces:

Free motion between the state preparing device (source injecting the atoms into the Stern-Gerlach magnet) and the magnet, magnetic shock area (magnet itself), free motion between the magnet and the detectors.

To get some numerical data from our analysis, let us assume that the over-all distance to be run by atoms is of the magnitude 1 m. Let the interaction area be of the length ca 10^{-2} m, and let the free motion before entering the magnet be realized along as small distance.

The average speed of particles we choose as follows:

$$v = 10^2 \text{ m} \qquad T \sim 10^{-4} \text{ s} \qquad t_{tot} \sim 10^{-2} \text{ s} \tag{25}$$

By t_{tot} we denote the over-all passage time (in average during the experiment), while T refers to the time spent under the magnetic influence.

After leaving the magnet particles display the free propagation pattern. For the minimum uncertainty state with the width σ and expectations $\langle\vec{r}\rangle = (x_o, y_o, z_o)$, $\langle\vec{p}\rangle = (p_x, p_y, p_z)$:

$$\phi_{\langle\vec{r}\rangle,\langle\vec{p}\rangle}(\vec{r},t) = \frac{1}{(2\pi)^{3/4}\left(\sigma + \frac{i\hbar}{2M\sigma}\right)^{3/2}} \exp\left\{- \frac{1}{\left(4\sigma^2 + \frac{2i\hbar t}{M}\right)}\right.$$

$$\left.\left[\left(x-x_o - \frac{p_x}{M}t\right)^2 + \left(y-y_o - \frac{p_y}{M}t\right)^2 + \left(z-z_o - \frac{p_z}{M}t\right)^2\right]\right\}$$

$$\exp\left\{\frac{i}{\hbar}\left[p_x\left(x-x_o - \frac{p_x}{M}t\right)\right] + p_y\left(y-y_o - \frac{p_y}{M}t\right) + p_z\left(z-z_o - \frac{p_z}{M}t\right)\right]$$

It means that the corresponding probability distribution

$\rho_{\langle\vec{r}\rangle,\langle\vec{p}\rangle}(\vec{r},t)$ spreads out:

$$\sigma(t) = \left(\sigma^2 + \frac{h^2 t^2}{4M^2\sigma^2}\right)^{1/2} \tag{27}$$

Recalling that the electron mass equals $m_e \sim 10^{-30}$ kg, and $\hbar \sim 10^{-34}$Js, we can estimate the spreading for the realistic experimental fit $M = 10^4 m_e$. Then:

$$\frac{\hbar^2 t^2}{4M^2\sigma^2} \simeq \frac{10^{-4}}{4}\frac{t^2}{10^8\sigma^2} = \frac{10^{-14}}{4\sigma^2} m^2$$

$$\sigma \sim 10^{-3} m \quad \Rightarrow \sigma(t) = 10^{-3} m(1 + 10^{-5})^{1/2} \tag{28}$$

i.e. in the realistic experiment the spreading effects can be disregarded.

Let us now estimate the deflections of the geometric centers of the Gaussians, which compose the final wave function after $t = t_{tot}$. First of all:

$$\eta = \frac{\mu B_o \hbar}{2M} = -\mu_B \frac{B_o}{M}, \quad \mu_B \sim -10^{-23} \frac{J}{tesla}$$

$$M = 10^4 m_e \quad \rightarrow \quad \eta \sim B_o 10^3 \frac{J}{kg\ tesla} \tag{29}$$

The magnetic shock deflection $\Delta = \eta \frac{T^2}{2}$ leads thus to:

$$T = 10^{-4} s \quad \rightarrow \quad \Delta_o = 10^{-5} B_o \frac{m^2}{tesla}$$

$$p = M\eta T, \quad v = \eta T \quad \rightarrow \Delta v = B_o 10^{-1} \frac{m^2}{tesla \cdot s} \tag{30}$$

$$\Delta v \cdot t = B_o 10^{-3} \frac{m^2}{tesla} = \Delta_{tot}$$

These data show that the choice

$$B_o = 1 \frac{tesla}{m} = 10^2 \frac{gauss}{cm} \tag{31}$$

gives the value $\Delta_{tot} = 10^{-3}m = \sigma_o$ for $t = 10^{-2}$ s, $T = 10^{-4}$ s, $M = 10^4\ m_e$.

Let us consider in more detail the probability distribution corresponding to the above inputs. Since the separation of energy centers for all Gaussians entering the outgoing wave packet after time t, is of the order of their width at least, the cross -terms arising in the evaluation of the probability density are negligible compared to the others. Consequently (Δ_{tot} is denoted Δ_t):

$$|\psi(\vec{r},t)|^2 \cong (2\pi)^3\ |\alpha|^2|\phi(x+\Delta_o+\Delta_t,y-vt,z+\Delta_o+\Delta_t,0)|^2$$

$$+\ (2\pi)^3|\beta|^2|\phi(x+\Delta_o+\Delta_t,y-vt,z-\Delta_o-\Delta_t)|^2 \qquad (32)$$

$$+\ (2\pi)^3|\beta-\alpha|^2\ \frac{x^2}{\sigma^4}\ \frac{\eta^2T^4}{2}\ |\phi(x,y-vt,z,0)|^2$$

Neglecting the time dependent correction to the width we arrive at:

$$|\psi(\vec{r},t)|^2 \cong \frac{1}{(2\pi)^{3/2}\sigma^3}\ \exp\left[-\ \frac{(y-y_o-vt)^2}{2\sigma^2}\right]$$

$$\left\{|\alpha|^2\ \exp\left[-\ \frac{(x+\Delta_o+\Delta_t)^2 + (z+\Delta_o+\Delta_t)^2}{2\sigma^2}\right] +\right.$$

$$+\ |\beta|^2\ \exp\left[-\ \frac{(x+\Delta_o+\Delta_t)^2 + (z-\Delta_o-\Delta_t)^2}{2\sigma^2}\right] +$$

$$\left.+\ |\beta-\alpha|^2\ \frac{(\eta T)^2T^2}{2\sigma^4}\ x^2\ \exp\left[-\ \frac{x^2+z^2}{2\sigma^2}\right]\right\}$$

We deal thus with a sum of the three Gaussian distributions centered respectively about the points $x_o = -(\Delta_o + \Delta_t)$,

$y = vt$, $z_o = \pm(\Delta_o + \Delta_t)$ and $(0, vt, 0)$.

Because of the presence of x as the multiplicative factor in the third term it may give significant contributions only in the neighbourhood of the maximum of the function

$$f(x) = x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)$$
$$f'(x) = 0 \rightarrow x = \pm\sqrt{2}\,\sigma \qquad (34)$$

It means that the third term contributes about points $(\pm\sqrt{2}\,\sigma, vt, 0)$. Whether this contribution is sizable against the other two, depends on $|\alpha-\beta|^2$ and on the magnitude of the coefficient, which with our fits equals

$$\frac{(\eta T)^2 T^2}{2\sigma^4} \sim \frac{10^2}{2}\frac{1}{m^2}\,, \qquad f(\pm\sqrt{2}\sigma) = \frac{2\sigma^2}{e} \qquad (35)$$

thus implying

$$\frac{(\eta T)^2 T^2}{2\sigma^4}\, x^2 \exp\left(-\frac{x^2}{4\sigma^2}\right) \le \frac{1}{e}\, 10^{-4} \qquad (36)$$

so that the contribution from the third term is at least by the factor 10^{-4} smaller than the remaining two.

(V) As long as the spreading effects are negligible it is not meaningless to to exploit the classical concepts in the description of the quantum phenomena. Our previous analysis shows that the spreading does not matter for the final outcome of the experiment.

Accordingly the application of the Ehrenfest theorem should imply

$$\frac{d}{dt}\langle x\rangle = \frac{1}{M}\langle p\rangle$$

$$\frac{d}{dt}\langle p\rangle = \langle -\frac{\partial V}{\partial x}\rangle \sim -\frac{d}{d\langle x\rangle} V(\langle x\rangle) \qquad (37)$$

Because the spin part of the Hamiltonian is matrix-valued, the situation is slightly more involved, than (37) would suggest. To evaluate the expectation value (at t=0) we should use spinor wave functions, where the $\vec{r}$-dependence comes through the coherent (minimum uncertainty) state.
Accordingly we have:

$$\hat{V} = H_{spin} = -\mu\vec{s}\vec{B} = \frac{\mu\hbar B_o}{2}(z\sigma_z - x\sigma_x) = \hat{V}(\vec{r})$$

$$\hat{\vec{F}} = -\nabla\hat{V} = \frac{\mu\hbar B_o}{2}(\sigma_x, 0, -\sigma_z) \qquad (38)$$

$$\int\bar{\phi}(\vec{r},0)\hat{V}\phi(\vec{r},0)d^3r = \frac{\mu\hbar B_o}{2}(\langle z\rangle\sigma_z - \langle x\rangle\sigma_x) = \hat{V}(\langle\vec{r}\rangle)$$

$$\int\bar{\phi}(\vec{r},0)\vec{r}\phi(\vec{r},0)d^3r = \langle\vec{r}\rangle$$

and thus:

$$\hat{F}_j = -\frac{\partial}{\partial\langle r_j\rangle}\hat{V}(\langle\vec{r}\rangle) = -\frac{\partial}{\partial r}\hat{V}(\vec{r}) \qquad j\equiv x,y,z \qquad (39)$$

$$\langle\hat{F}_x\rangle = \frac{\mu\hbar B_o}{2}\langle\sigma_x\rangle$$

$$\langle\vec{F}_z\rangle = -\frac{\mu\hbar B_o}{2}\langle\sigma_z\rangle$$

In particular:

$$\langle\sigma_j\rangle = (\alpha,\beta)\,\sigma_j\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \rightarrow$$

$$\langle\hat{F}_x\rangle = \mu\hbar B_o \,\mathrm{Re}(\alpha\beta) \qquad (40)$$

$$\langle\hat{F}_z\rangle = -\frac{\mu\hbar B_o}{2}\left(|\alpha|^2 - |\beta|^2\right)$$

to be compared with the actual force exerted on the centroids of the wave packets (26):

$$F_x = M\eta = \frac{\mu B_o\hbar}{2} \rightarrow \qquad \Delta = \frac{\eta T^2}{2}$$

$$F_z = \pm M\eta = \pm\frac{\mu B_o\hbar}{2} \rightarrow \Delta = \pm\frac{\eta T^2}{2} \qquad (41)$$

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