Some Aspects of the Boson-Fermion (In-) Equivalence

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The problem of whether fermions and bosons, on the level of quantum field theory, are related or unrelated is not an empty question. Different relationships are and were established, but we shall confine our attention to two basic routes in this context (beware that the supersymmetry or superstring enthusiasts would certainly advocate at least one more route as equally or even more, basic). The key word of route 1 we associate with the fermion pairs to bosons mapping idea [1], which although originating about 1932–1935, later on via Skyrme, Streater and Wilde, Freundlich, Coleman has finally led to so called boson-fermion reciprocity = duality = correspondence = equivalence concept (massive Thirring model versus sine-Gordon in 1+1).

We shall restrict further considerations to the route 2, in which each fermion degree of freedom is mapped into its boson analogue. It was initiated in the year 1974, although it is closely related to much earlier Klauder's, Girardeau's and Yang's investigations published in the years 1960–1967, see e.g. [1,2]. It pertains to the so called boson and fermion Fock space unification (the non-Fock extension exists [2]), and allows for: (1) another realization of the boson-fermion equivalence which includes no limitation on space-time dimension, (2) studying the semiclassical and classical features of Fermi models in a consistent way like e.g. the problem of relating the fermion, boson and classical (c-number commuting function ring) versions of such models like the massive Thirring or chiral invariant Gross-Neveu, (3) uncovering the quantum meaning of classical field theory for fermion systems, which includes the answer to the question: what for are the classical (non-linear) spinor fields? In fact, the status of route 2 as closed by the complement [3] to the basic paper [4] can be verbalized as follows: via the Fock construction the common Fock space for bosons and fermions can be introduced which implies that all local fermion field theory models have boson equivalents (which violate the weak local commutativity condition for space dimension three). However, it does not yet allow for the unrestricted boson-fermion equivalence for field theory models: not all boson models admit a pure fermion reconstruction.

To have this claim justified one should first realize that the analysis of Fock representations of the CCR and CAR algebra, implies that fermions are born by bosons in the (Hilbert) representation space.

H. Mitter et al. (eds.), *Recent Developments in Mathematical Physics*

By F we denote the Hilbert space of sequences of n-point, Lebesgue square integrable functions:

$$F = \overset{\infty}{\underset{n=0}{+}} K^{\times n} \quad , \qquad K = \overset{M}{\underset{1}{+}} L_i^2 \ (R^N) \tag{1}$$

Once a Fock representation of the CCR algebra over K is given, it automatically induces [4] a Fock representation of the CAR algebra in the boson Fock space (n-point functions are symmetric) which acts irreducibly on the following (proper) subspace of $F = F_B$:

$$\overset{1}{F}_B = \overset{\infty}{\underset{n=0}{+}} E_n^2 \ S_n \ K^{\times n} \tag{2}$$

Here S_n is the symmetrization operator in $K^{\times n}$, while E_n^2 is a projection:

$$E_n(A_n K^{\times n}) = E_n^2(S_n K^{\times n}) \subset S_n K^{\times n} \tag{3}$$

such that its square root E_n converts antisymmetric n-point functions into their symmetric images, which although symmetric do reflect the Pauli principle. We shall illustrate our general statements [3] by specifying $K = L^2(R^1)$ and choosing a specific realization of E_n in terms of the integral kernel:

$$E_n(s_1,\dots,s_n;t_1,\dots,t_n) = \sigma(s_1,\dots,s_n)\delta(s_1-t_1) \ \dots \ \delta(s_n-t_n)$$

$$\sigma(s_1,\dots,s_n) = \begin{cases} (-1)^\pi & s_i \neq s_j \\ 0 & s_i = s_j \end{cases} \tag{4}$$

Then generators of the CAR algebra can be explicitly constructed in terms of canonical (CCR algebra) generators for bosons [4,2]:

$$a(s) = \sum_{m=0}^{\infty} \frac{\sqrt{m+1}}{m!} \int ds_1 \ \dots \int ds_m \sigma(s,s_1,\dots,s_m)$$

$$b^*(s_1) \ \dots \ b^*(s_m) \ : \exp(-\int dt \ b^*(t)b(t)) : \ b(s)b(s_1) \ \dots \ b(s_n)$$

$$[a(s),a(t)^*]_+ = \delta(s-t)$$

$$[a(s),a(t)]_+ = 0 \qquad\qquad a(s)\Psi_0 = 0 \quad \forall s \tag{5}$$

so that the respective boson and fermion Fock vectors read:

$$F : a(f_1)^* \ \dots \ a(f_n)^*\Psi_0 = \int ds_1 \ \dots \int ds_n \ f_1(s_1) \ \dots \ f_n(s_n)$$

$$\sigma(s_1,\dots,s_n) b^*(s_1) \ \dots \ b^*(s_n)\Psi_0 \tag{6}$$

$$B : b(f_1)^* \ldots b(f_n)\Psi_0 = \int ds_1 \ldots \int ds_n \, f_1(s_1) \ldots f_n(s_n)$$

$$b^*(s_1) \ldots b^*(s_n)\Psi_0$$

which proves that via the Fock construction ($F \to H$) boson and fermion canonical algebras can be represented on a common (boson!) domain.

In particular it is possible [5] to demonstrate the following relationships (we refer to the two-point functions):

$$F : a(f_1)^* a(f_2)^* \Psi_0 = \int_{s_1 < s_2} ds_1 ds_2 \, \det(f_i(s_j)) b^*(s_1) b^*(s_2) \Psi_0$$

$$B : b(f_1)^* b(f_2)^* \Psi_0 = \int_{s_1 < s_2} ds_1 ds_2 \, \mathrm{per}(f_i(s_j)) a^*(s_1) a^*(s_2) \Psi_0 \qquad (7)$$

which most clearly exemplifies what is meant by the boson and fermion Fock space unification. Beware that an essential ingredient in passing from (6) to (7) is that contributions from sets of Lebesgue measure zero on R^2 (i.e. $s_1 = s_2$) were omitted. A straightforward consequence of the above construction is that: each fermion model can be equivalently rewritten as the boson one.

At this point let us addres the problem of whether the reverse statement would hold true. The answer is negative [3] as the paradigm example of the nonlinear Schrödinger model with a repulsive coupling in 1+1, does explicitly shows. Namely, we have:

$$H = -\frac{1}{2}\int \phi^*_x \, \phi_x \, dx + \frac{c}{2}\int dx \, \phi^*(x)^2 \phi(x)^2 \, , \qquad c > 0$$

$$[\phi(x), \phi^*(y)]_- = \delta(x - y)$$

$$[\phi(x), \phi(y)]_- = 0 \qquad \phi(x)\Psi_0 = 0 \qquad \forall x \in R^1 \qquad (8)$$

and if to omit the contributions from sets of Lebesgue measure zero in R^n (i.e. these from

$$\overset{2}{F}_B = \overset{\infty}{\underset{0}{+}} (1 - E_n^2) S_n K^{\times n})$$

we would obtain:

$$\phi(f_1)^* \ldots \phi(f_n)^* \Psi_0 = \int_{x_1 < \ldots < x_n} dx_1 \ldots \int dx_n \, \mathrm{per}(f_i(x_j)) \, ,$$

$$\phi^*(x_1) \ldots \phi^*(x_n)\Psi_0 \qquad (9)$$

However the naive action of H on vectors of the form (9) would reduce the non-trivial model to the free field case which is known to arise in either c=0 or c=∞ strong operator limits. The respective boson or fermion free field models are equivalent [6] in the (boson) Fock space.

However for $0 < c < \infty$ we must address the following problem:

$$|f\rangle = \int dx_1 \ldots \int dx_n\, f(x_1,\ldots,x_n)\phi^*(x_n) \ldots \phi^*(x_n)\Psi_0$$

$$H|f\rangle = \int dx_1 \ldots \int dx_n \left\{(-\frac{1}{2}\sum_{j=1}^{n} \nabla_j^2 + \frac{c}{2}\sum_{i\neq j} \delta(x_i - x_j)) f(x_1,\ldots,x_n)\right\}\phi^*(x_1) \ldots \phi^*(x_n)\Psi_0 \quad (10)$$

and the many-body (hard-core Bose gas) Hamiltonian H_n non-trivially mixes ${}^1F_B$ and ${}^2F_B$ sectors in F_B.

An immediate conclusion is that:

not all boson field theory models allow for a pure boson reconstruction (unless the boson Hamiltonian acts invariantly in ${}^1F_B$) although the reverse is always true.

Let us end up with two remarks [3]:

(1) The situation in continuum is drastically different from this for the lattice systems (even infinite). There is no way at all to give a fermion reconstruction of the Bose system unless a restriction to the appropriate (state) subspace is made or irreducibility of representations is abandoned. The boson representation of the Fermi system does always exists, although in general it may be non-local [2,7]. The boson-fermion Fock space unification argument, nevertheless allows for reasonable local approximations of lattice Fermi systems in terms of Bose ones [8], see also [9].

(2) For each Fermi system and equivalent Bose one can be found (irrespective of what is the space-time adopted). Since the total set of exponential vectors (coherent states) spans the domain for equivalent Bose and Fermi systems, the standard tree approximation methods allow us to attribute an unambigous meaning to the classical relative for any Fermi field, which is a c-number (commuting function ring) field theory, see e.g. [2.10].

Acknowledgement

Since the XXVI Universitätswochen für Kernphysik was organized to celebrate the 60th birthday of Professor Walter Thirring, let me mention that the analysis of the three (Bose, Fermi and classical) versions of the massive Thirring model played a predominant role in our investigations.

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