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BOSON-FERMION CORRESPONDENCE IN QUANTUM THEORY AND QUANTIZATION OF SPINOR FIELDS+

by

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1. INTRODUCTION

Recent developments on the connection between Thirring and Sine-Gordon systems in two space-time dimensions resulted in a couple of papers on the question of Fermion-Boson correspondence in quantum field theory (mysterious metamorphosis of Fermions into Bosons, as S. Coleman said), see e.g. [3]. The mentioned correspondence is not a particular feature of q.f.t.only. For example, under the name of the method of Boson expansions, it was employed to build a contemporary theory of spin waves in the low-temperature description of Heisenberg ferromagnet. In this last case, there was known for long time that the ideal magnon gas, in the weak ex-

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citation limit, perfectly simulates the behaviour of the Heisenberg crystal itself, despite of the spin value assigned to the sites of the lattice.

A similar situation appears in the study of the weak excitation limit of the atomic nuclei, in the microscopic model, where the spectra of low lying excited states are similar to these of the weakly excited system of quadrupole Bosons. All that allows to expect that each quantum Boson in the weak excitation limit (not true for isolated systems, one needs any regulation mechanism establishing the needed excitation level), can exhibit Fermion properties, which then prevail the original Boson ones (Fermion-like behaviour). Here, the weak excitation (low temperature) limit of the Boson theory can be also considered as its strong coupling limit provided the strong coupling potential (large distance phenomena in case of Heisenberg ferromagnet) prevents the Boson system from occupying more than a few, low lying, energy levels.

Quite conversely, if the higher excitations (weak coupling limit) are admitted, then starting from the Fermion system, we can expect that Boson properties will prevail the original Fermion ones (Boson-like behaviour of the Fermion).

Above conjectures have an unrestricted validity in the nonrelativistic quantum theory, or if the number of space-time dimensions is less than 4. In either case, the spin-statistics theorem must be taken into account.

2. ϕ_2^4 JUSTIFICATION: WHAT CAN BE DRAWN FROM THE BOSON SYSTEM

To support our thesis that, in a few cases at least, Bosons can be treated as more fundamental than Fermions, let us discuss the ϕ_2^4 example, following [1]. The ϕ_2^4 Hamiltonian is given by:

$$H = \int dx \; \{\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + \lambda(\phi^2 - f^2)^2\} \; . \tag{2.1}$$

This continuous model can be approximated by its <u>lattice version</u> (linear lattice, with the inverse spacing constant Λ and the number of 2N+1 sites). Due to the finite volume, the allowed momenta are

$$k = \frac{2\pi}{L} n, \; n = 0, \pm 1, \ldots, \pm N, \; L = \frac{2N+1}{\Lambda} \; ,$$

and

$$H = \frac{1}{\Lambda} \sum_s \{\tfrac{1}{2}\pi_s^2 + \tfrac{1}{2}(\nabla\phi_s)^2 + \lambda(\phi_s^2 - f^2)^2\} \quad , \tag{2.2}$$

where s enumerates lattice sites, and the gradient term should be still properly defined (in case of Bosons $\nabla\phi_s = \Lambda(\phi_{s+1} - \phi_s)$ can be introduced). In the rescaled form:

$$[\pi_s, \phi_t]_- = -i\Lambda\delta_{st} \rightarrow [p_s, x_t]_- = -i\delta_{st},$$

we have

There is useful to note here, that the gradient term carries an interaction between lattice sites:

$$H = \sum_s \{H^{(s)}_{self} + H^{(s)}_{int}\} \quad ,$$

while the single site term $H^{(s)}_{self}$ describes a Schrödinger problem of a particle in an anharmonic potential. Neglection of the gradient leaves us with the chain of noninteracting anharmonic solutions, for which a Fock construction exists, resulting in the single site basis

$$|\psi\rangle = \prod_s |\psi_s\rangle \; ,$$

$$\langle\psi_s|\psi_t\rangle = \delta_{st}, \; |\psi_s\rangle = \sum_{n_s=0}^{\infty} c_{n_s} \; |n_s\rangle \; , \; |n_s\rangle = \frac{1}{\sqrt{n_s!}}(a_s^*)^{n_s}|0_s\rangle,$$

where $|0_s\rangle$ is the s-th site vacuum.

Taking the expectation value $\langle\psi|H|\psi\rangle$ of (2.3) in the single site trial state $|\psi\rangle$, through minimization procedures one can calculate the ground state energy of the interacting system (2.3).

Let us now consider the lattice version of ϕ_2^4 system with the nearest neighbor coupling (periodic boundary conditions),

$$H = \Lambda\{\sum_s \frac{p_s^2}{2} + \frac{\mu^2+2}{2}\, x_s^2 + \lambda x_s^4 - x_s x_{s+1}\} \quad . \qquad (2.4)$$

The single site terms describe anharmonic oscillators at each site, so that the single site basis can be introduced at once: $\otimes_s |n_s\rangle$, $0 \le n_s \le \infty$, and further the matrix form of the Hamiltonian (2.4):

$$H = \sum_s (E^s - X^s \otimes X^{s+1}) \; ; \qquad (2.5)$$

$(H \equiv H/\Lambda)$, $E = \{E_n\}$ is a diagonal matrix with single site eigenvalues on the diagonal, $X = \{\langle n|x|m\rangle\}$ its elements do not vanish between even and odd parity states.

Truncation of the single site base: $0 \le n_j \le s-1$ to a finite number S of lowest energy levels corresponds to the approximation of the lattice system (2.4) by the coupled spin system (2s+1 = S, the finite spin approximation of (2.4) is achieved).

In special case of spin 1/2 approximation, the Hamiltonian matrix (2.5) reads:

$$H = \text{const} + \sum_s \{\frac{\varepsilon}{2}\, \sigma_s^z - \Delta(\sigma_s^+ + \sigma_s^-)(\sigma_{s+1}^+ + \sigma_{s+1}^-)\} \qquad (2.6)$$

with

$$\varepsilon = (E_1 - E_0), \quad \Delta = |\langle 0|x|1\rangle|^2, \quad \sigma_{N+1} \equiv \sigma_{-N},$$

σ's are Pauli matrices. This is the case, when the vacuum and single excitation levels of the starting system (2.4) are mostly important (higher excitations appear with a negligible probability).

When Pauli matrices are involved, by the use of so-called Jordan-Wigner trick one can rewrite (2.6) in the equivalent form, where Fermi operators only appear (Fermion approximation of (2.4)):

$$H = LE_o + \varepsilon \sum_{s=-N}^{N} b_s^* b_s - \Delta \sum_{s=-N}^{N} (b_s^* - b_s)(b_{s+1}^* + b_{s+1}) +$$

$$+ \Delta (b_N^* - b_N)(b_{-N}^* + b_{-N}) \, (\exp(i\pi n) + 1) \qquad (2.7)$$

where

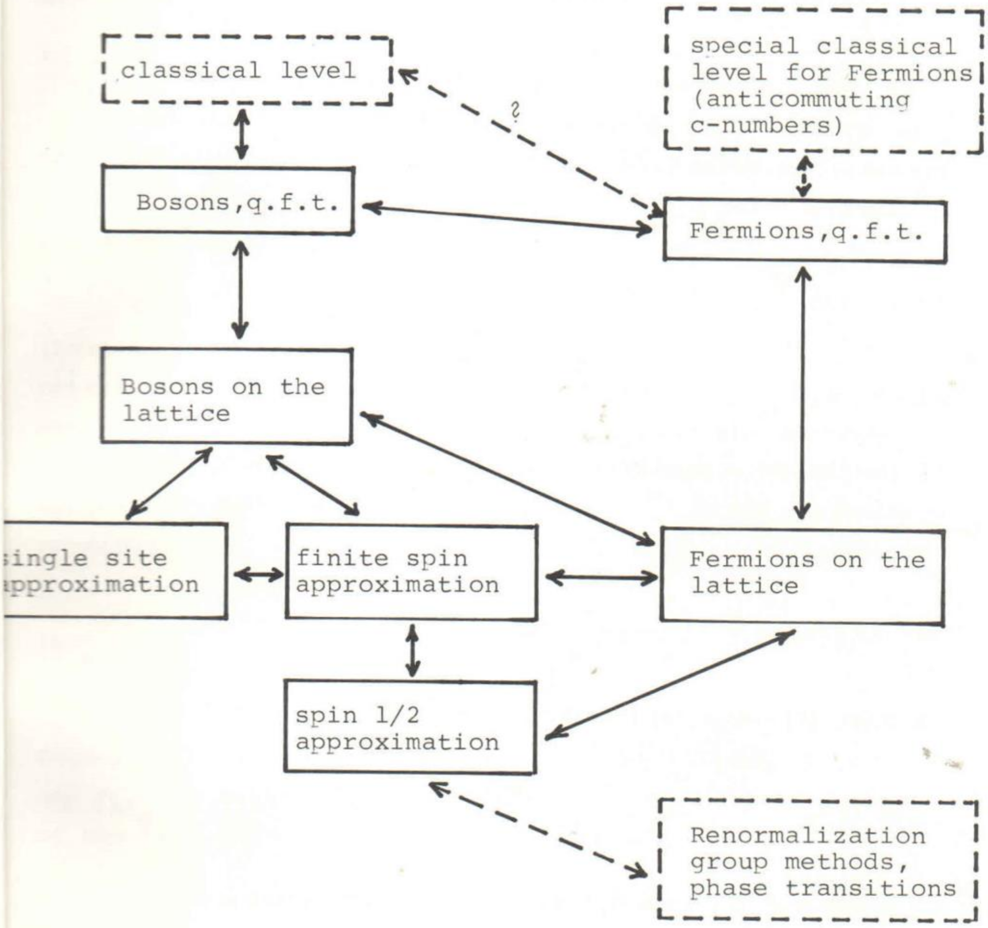
$$n = \sum_s n_s , \qquad n_s = b_s^* b_s \ .$$

In this place one can obviously state the question whether there exists any continuous Fermion theory, whose lattice approximation is (2.7).

Let us emphasize that in the above approximations of the starting Boson system (2.1) we did not bother what were exactly the mechanisms, whose influence could justify the choice of a concrete approximation. The question of interest was rather to identify the physical situations in which the starting Boson system like-transforms (in the approximate sense) into the finite spin or pure Fermion system.

If in addition to introduce into consideration the question of classical basis behind the quantum concepts (as e.g. the kind of correspondence principle realized via coherent state methods), then the diagram of current problems can be completed.

DIAGRAM OF PROBLEMS



In the above, majority of steps can be realized by the use of Boson expansion methods, whose basic aim is to start from the given Boson system (q.f.t. or a corresponding classical level eventually), and generate further as many as possible from the indicated relations.

3. FERMION-BOSON CORRESPONDENCE IN THE FOCK CONSTRUCTION

Let us assume to have given the triple $\{a^*, a, \Omega_B\}_K$, generating a Fock representation of the CCR (canonical commutation relations) algebra over the separable Hilbert space $K \ni f, g$:

$$[a(f), a(g)^*]_- = (f,g)\, 1_B$$

$$a(f)\Omega_B = 0 \quad . \tag{3.1}$$

If to choose a sequence $\{f_k\}_{k=1,2,\ldots}$ of basis vectors in K, then $a_k^* = a(f_k)^*$, allows to define Fock space basis vectors:

$$|k_1, \ldots, k_n\rangle_B = a_{k_1}^* \ldots a_{k_n}^* \Omega_B \ ,$$

so that the Fock space vector is given in the form:

$$F_B \ni |F\rangle_B = \sum_n \sum_{\{k\}} F^S_{k_1 \ldots k_n} |k_1, \ldots, k_n\rangle_B = \sum_n (F^S_n, |n\rangle_B) . \tag{3.2}$$

In the same way one can proceed in the Fermi case: $\{b^*, b, \Omega_F\}_K$ is given by

$$[b(f), b(g)^*]_+ = (f,g)\, 1_F$$

$$b(f)\Omega_F = 0 \ , \tag{3.3}$$

so that $b_k^* = b(f_k)^*$ implies

$$|k_1, \ldots, k_n\rangle_F = b_{k_1}^* \ldots b_{k_n}^* \Omega_F \ ,$$

and further

$$F_F \ni |F>_B = \sum_n \sum_{\{k\}} F^a_{k_1 \ldots k_n} |k_1,\ldots,k_n>_F = \sum_n (F^a_n, |n>_F) \ . \qquad (3.4)$$

In the above superscripts s,a denote symmetric and anti-symmetric tensors respectively, while F_B, F_F the Boson and Fermion Fock spaces respectively.

If to introduce now a discrete version $\varepsilon_{k_1 \ldots k_n}$ (Levi-Civitta tensor in n-dimensions) of the continuous Friedrichs-Klauder sign function $\sigma_n(k_1,..,k_n)$, see e.g. [2], then one easily notices that in the Fock construction there is no essential difference between vectors of the form (3.2) and (3.4). For example one can consider $F^a_{k_1 \ldots k_n}$ in the form $F^s_{k_1 \ldots k_n} \varepsilon_{k_1 \ldots k_n}$ so that:

$$|F>_F = \sum_n \sum_{\{k\}} F^s_{k_1 \ldots k_n} \{\varepsilon_{k_1 \ldots k_n} |k_1 \ldots k_n>_F\} \ , \qquad (3.5)$$

suggesting that $|F>_F$ can be as well the element of F_F and F_B, provided suitable restrictions on representations of the CCR and CAR algebra are given.

This is exactly the case, when "schizons" (see Schroer's lectures [3]) are needed. There are the Boson and Fermion representations, whose vacuum and one-particle sectors coincide. In case of $K = L^2(R^n)$, one can even get a very simple example of Boson constructed Fermions:

$$k \in R^n, \ b(k) = \exp(-i\pi \int_k^\infty a^*(p)a(p)dp) \times a(k) \quad .$$

This representation can be always closed on F_F to a more sophisticated example, constructed in [5] (we prefer here a discrete language, but the transition to a continuous one is nearly immediate if in the place of $\varepsilon_{k_1 \ldots k_n}$ to put $\sigma_n(k_1 \ldots k_n)$ and in the place of summations with respect to k's to consider respective integrations):

$$b(f) = :\exp(-\sum_s a_s^* a_s) \cdot \sum_{n,m} \frac{1}{\sqrt{n!m!}} \sum_{\{r\}} \sum_t \cdot$$

$$\cdot \sqrt{n+1}\ \delta_{m,1+n}\ \varepsilon_{r_1 \ldots r_n}\ \bar{f}_t\ \varepsilon_{tr_1 \ldots r_n}\ a_{r_1}^* \ldots a_{r_n}^* \cdot$$

$$\cdot\ a_t a_{r_1} \ldots a_{r_n}\ . \tag{3.6}$$

One can easily check that

$$a(f)\Omega_B = b(f)\Omega_B$$

$$a(f)^*\Omega_B = b(f)^*\Omega_B$$

$$[b(f), b(g)^*]_+ = (f,g)\ 1_F \tag{3.7}$$

where

$$1_F = \sum_n \frac{1}{n!} \sum_{\{r\}} :\ a_{r_1}^* \ldots a_{r_1}^*\ \ \varepsilon_{r_1 \ldots r_n}^2\ a_{r_1} \ldots . a_{r_n} \cdot$$

$$\cdot \exp(-\sum_s a_s^* a_s): \tag{3.8}$$

is a projection in F_B, so that

$$F_F = 1_F \, F_B \quad . \tag{3.9}$$

If one wishes to deal with a finite number of annihilation and creation generators a_s^*, a_s and b_s^*, b_s respectively, there is enough to restrict summations with respect to {r},t in (3.6), (3.8) to a finite number N, say. The most general element of the CAR algebra (3.6)-(3.9) is of the form

$$:F(b^*,b): = \sum_{nm} (f_{nm}, b^{*n} b^m) = \sum_{nm} \sum_{\{r\}} \sum_{\{s\}} \cdot$$

$$\cdot \, f_{r_1 \ldots r_n \, s_1 \ldots s_m} \cdot b^*_{r_1} \ldots b^*_{r_n} \; b_{s_1} \ldots b_{s_m} \; , \tag{3.10}$$

where f_{nm} is the n+m-antisymmetric tensor. One can easily check that

$$:F(b^*,b):\Omega_B = \sum_{nm} (\overset{c}{f}_{nm}, a^{*n} a^m)\, \Omega_B = :\overset{c}{F}(a^*,a):\Omega_B \quad , \tag{3.11}$$

where

$$\overset{c}{f}_{r_1 \ldots r_n \, s_1 \ldots s_m} = f_{r_1 \ldots r_n \, s_1 \ldots s_m} \cdot \varepsilon_{r_1 \ldots r_n} \, \varepsilon_{s_m \ldots s_1} \; . \tag{3.12}$$

Furthermore, if to take into account a few symmetry arguments, concerning especially the decomposition of n-point tensors into irreducible parts with respect to the symmetry group, one can prove [5] the following projection theorem:

$$:F(b^*,b):F_F = 1_F:\overset{c}{F}(a^*,a):F_F \tag{3.13}$$

being the identity on F_F. If specialized, we find that on F_F, the <u>projected</u> Boson generators are in fact Fermion generators:

$$1_F\; a(f)\; 1_F = b(f)$$
$$1_F\; a(f)^* 1_F = b(f)^* \quad . \tag{3.14}$$

Formulas (3.13), (3.14) provide an elegant way of changing the symmetry properties of any theory under consideration, where expansions into series of normal-ordered products of Fock generators are admitted.

We see thus at once that, if physics in any way makes reasonable the reduction of interests concerning the Boson system to $F_F = 1_F\, F_B$, then the approximation of it by the corresponding (associated) Fermion system is justified.

4. SELECTED APPLICATION: ISOTROPIC HEISENBERG LATTICE

As a special example of the projection theorem (3.13) one can study a Boson theory, whose weakly excited (low temperature) limit well approximates properties of the Heisenberg ferromagnet in low temperatures.

Namely, if to reenumerate the set of generators: $(k) \to (k\alpha)$, $k = 1,\ldots,N$, $\alpha = 1,\ldots,n$, we can start from the Hamiltonian

$$H_B = G_o - \mu \sum_k^N \vec{\kappa}\cdot\vec{s}_k - (1/2) \sum_{k,h=1}^N J_{kh}\, \vec{s}_k\, \vec{s}_h\ , \qquad (4.1)$$

where

$$\vec{s}_k = (s_k^x, s_k^y, s_k^z)\ , \qquad \text{and} \qquad s_k^+ = \sum_\alpha^n a^*_{k\alpha}\ ,\ s_k^- = \sum_\alpha^n a_{k\alpha}\ ,$$

$$s_k^2 = \{-(n/2) + \sum_\alpha^n a^*_{k\alpha}\, a_{k\alpha}\}\ .$$

By applying the projector

$$P_o = :\exp(-\sum_{k\alpha} a^*_{k\alpha}\, a_{k\alpha}): + \sum_{k=1}^N P_o^k \qquad (4.2)$$

with

$$P_o^k = 1_F^k - :\exp(-\sum_\alpha^n a^*_{k\alpha}\, a_{k\alpha}):,$$

where 1_F^k is given by (3.8) if specialized to the total number n of Boson generators belonging to the k-th from N different collections of them ($a^*_r \to a^*_{k\alpha}$, summation with respect to α), we get

$$P_o\, F_B = F_o \qquad (4.3)$$

i.e. the Hilbert space of the spin states (finite spin approximation), and further

$$P_o\, H_B\, P_o = H\ ,$$

where $H = H(\vec{s}_k \to \vec{S}_k)$ is the Heisenberg ferromagnet

Hamiltonian, and $\vec{S}_k$ the spin operator at the s-th site of the lattice: $\vec{S}_k = P_o \vec{s}_k P_o$. For n = 1 we get spin-1/2 lattice, while in other cases F_o can be decomposed into subspaces corresponding to the irreducible representations of the SU(2): for n = 2, we get spin 1 and spin 0 examples.

From the physical point of view the above procedure is based on the assumption that the ground state and the first excited level of each single degree of freedom of the Boson system are of importance (spin 1/2 approximation behind the received finally finite spin approximation of the Boson theory). In case when $\alpha = 1,\dots,n$ one can interpret (4.4) as a kind of a condensation of Bosonic degrees of freedom around the lattice sites, so that in the original Heisenberg lattice, one more lattice (of the condensed magnon gas) appears.

5. THE CORRESPONDENCE PRINCIPLE IN Q.F.T.: QUANTIZATION OF SPINOR FIELDS WITH NO USE OF ANTICOMMUTING C-NUMBERS

Under the Haag-LSZ assumptions, the most general element of the scalar Boson field algebra can be written in the form (compare Klauder's lecture)

$$:F(\phi): = \sum_n (f_n, :\phi^n:) \quad , \tag{5.1}$$

where brackets denote integrations with respect to Minkowski space-time variables, $:\phi^n:$ is a shorthand notation for a normal-ordered product of free (asymptotic) fields taken at different space-time points. Let $\bar{\alpha}(k)$, $\alpha(k)$, $k \in R^3$, denote Fourier amplitudes of the classical scalar field $\overset{c}{\phi}(x)$. On the basis of coherent state techniques, one can employ so-called functional representation of the CCR algebra [4],what we symbolize by

$$:F(\phi):(\bar{\alpha},\alpha) = \sum_n (f_n, \overset{c}{\phi}{}^n) \exp(\bar{\alpha},\alpha) \ , \qquad (5.2)$$

and on the r.h.s. of (5.2) the classical free fields $\overset{c}{\phi}(x)$ appear. In the functional representation, $\exp(\bar{\alpha},\alpha) = 1_B(\bar{\alpha},\alpha)$, and is the operator unit (the Fock space transforms in that case into the Bargman space). Obviously $F(\overset{c}{\phi}) = \sum_n (f_n, \overset{c}{\phi}{}^n)$ can appear here as a coherent state expectation value $<:F(\phi):>$ of the operator expression, however the use of functional representation has a great advantage of providing the 1-1 map between the classical and quantum level of a given Boson theory, with no polynomial limitations.

Using the functional representations [4,5] of the canonical relations (CCR and CAR) algebras one can prove the following correspondence rule: Let us extend the Haag-LSZ expansion theorem onto the case of Dirac fields:

$$:\Omega(\psi,\bar{\psi}): = \sum_{nm} (\omega_{nm}, :\psi^n \bar{\psi}^m:) \ . \qquad (5.3)$$

Then:

(i) the subsidiary Boson level of the starting Fermion theory is given, where

$$1_F:\overset{c}{\Omega}(\overset{B}{\psi},\overset{B}{\bar{\psi}}):1_F = :\Omega(\psi,\bar{\psi}): \qquad (5.4)$$

is an identity in F_F (the spinor $\overset{B}{\psi},\overset{\underline{B}}{\psi}$ as obeying the commutation rules should violate assumptions of spin-statistics theorem);

(ii) the unrestricted (by projections 1_F) Boson level $:\overset{c}{\Omega}(\overset{B}{\psi},\overset{\underline{B}}{\psi}):$ admits a straightforward classical map

$$<:\overset{c}{\Omega}(\overset{B}{\psi},\overset{\underline{B}}{\psi}):> = \overset{c}{\Omega}(\overset{c}{\psi},\overset{\underline{c}}{\psi}) , \qquad (5.5)$$

where $\overset{c}{\psi},\overset{\underline{c}}{\psi}$ are classical spinor fields (commuting ring).

The converse procedure can stand for a quantization rule of a given classical spinor system.

More details, as well as considerations concerning the map of the algebraic structure, can be found in [5].

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