REPRESENTATIONS OF THE CAR GENERATED BY THE REPRESENTATIONS OF THE CCR III. NON-FOCK EXTENSION

PIOTR GARBACZEWSKI and ZIEMOWIT POPOWICZ
Institute of Theoretical Physics, University of Wrocław, Wrocław, Poland

(*Received March 29, 1976*)

An idea to construct a representation of the CAR algebra in a representation of the CCR algebra, which succeeded in the Fock case, is verified for the simplest non-Fock extensions of the formalism.

1. Introduction

It was shown in [15], referred to as the paper I of the present series, that one is able to construct a Fock representation of the CAR algebra in a given Fock representation of the CCR algebra. The construction was further generalized in [20] referred to as II, to join pairs of (Fock) representations describing different numbers of degrees of freedom. A growing interest in non-Fock representations of the canonical relations, motivates the attempt to establish a possible non-Fock extension of the above-mentioned construction. In this place an intriguing idea emerges of possible application of the whole theory developed for bosons, to the induced Fermi algebras. For this purpose, we extensively use the theory of canonical transformations for representations of the CCR algebra, developed in [1]–[11], [13].

2. Direct product representations of the CCR

Given an enumerable infinite sequence of Hilbert spaces $\{h_k\}_{k=1,2,\dots}$ with an involution "$-$" and bilinear form $(\cdot\,,\,\cdot)$ implementing in each h_k a scalar product $(\bar{\cdot}\,,\,\cdot)$, let us choose in $\{h_k\}_{k=1,2,\dots}$ an (infinite) sequence $\{x_k\}_{k=1,2,\dots}$, $x_k \in h_k$ with the property $\prod_{k=1}^{\infty} \|x_k\|^2 < \infty$. Such sequence is assumed to constitute a product vector $x = \prod_{k=1}^{\infty} x_k \in \prod_{k=1}^{\infty} \otimes h_k$. Two different vectors x, x' are said to be *equivalent* if $\prod_{k=1}^{\infty} (\bar{x}_k, x'_k)$ converges, i.e. $\sum_{k=1}^{\infty} |(\bar{x}_k, x'_k)-1| < \infty$, and weakly equivalent if $\sum_{k=1}^{\infty} ||(\bar{x}_k, x'_k)|-1| < \infty$. If x and x' are weakly equivalent, there always exists such a sequence $\{y_k\}_{k=1,2,\dots}$ of real numbers that x is strongly equivalent to $\prod_{k=1}^{\infty} \otimes e^{iy_k} \cdot x'_k = x''$. For equivalent product vectors we establish the notion of scalar product in $\prod_{k=1}^{\infty} \otimes h_k$ by $(x, x') = \prod_{k=1}^{\infty} (\bar{x}_k, x'_k)$. With respect to this scalar product $\prod_{1}^{\infty} \otimes h_k$ is a closure of the set of finite linear combinations of product vectors. This Hilbert space is nonseparable and can be decomposed into a direct sum of *incomplete direct product spaces* (IDPS) generated by *inequivalent product vectors*, which we write as $\mathrm{IDPS}(x) = \prod_{k=1}^{\infty}{}^{x} \otimes h_k$. Elements of a given IDPS(x) differ from x in a finite number of x_k only. Each IDPS is separable and for x, x' taken from different IDPS, we have $(\bar{x}, x') = 0$ (see [1], [4]). IDPS are the underlying Hilbert spaces, from which we take domains for representations of the CCR algebra. Appropriate representations are called direct product representations. If a representation space is generated by the product vector x, we say about *x-generated representation of the* CCR *algebra*. Assume $\{H_n\}_{n=0,1...}$ to be an orthonormal complete set in $L^2(R^3)$ consisting of Hermite functions H_n. Any direct product representation of the CCR algebra is called *discrete* if and only if it is generated by the vector $x = \prod_{k=1}^{\infty} \otimes H_{n_k}$ where H_{n_k} is the n_kth Hermite function. If we assume $n_k = 0$ independently on k, we obtain a Fock representation of the CCR algebra:

$$H_{n_k}(s) = H_0(s) = \frac{1}{\sqrt[4]{\pi}} \exp(-s^2/2),$$

and a corresponding (Fock) representation space is IDPS (Ω), $\Omega = \prod_{n=1}^{\infty} \otimes (H_0)_n$.

3. Canonical transformations

The following statement holds: *Two irreducible direct product representations of the* CCR *algebra x and x' generated respectively, are unitarily equivalent if and only if x and x' are weakly equivalent* (the proof can be found in [1], [3]).

We see at once that different discrete representations are unitarily inequivalent. Our main task is to find a certain class of unitarily inequivalent representations of the CCR algebra, emerging from a Fock representation through any canonical transformation which is able to break the weak equivalence of generating vectors. Let us restrict considerations to linear inhomogeneous canonical transformations of generators a^*, a; see [1]–[3], [5], [6], [11], [13].

Consider the triple $\{a^*, a, \Omega\}$ generating a Fock representation $U_B(K)$ of the CCR algebra over $K = L^2(R^3)$ and acting irreducibly in $D \subset \mathrm{IDPS}(\Omega)$ being a Fock representation space. Taking orthocomplete set $\{e_j\}_{j=0,1,...}$ in K, let us write $a_j = a(e_j)$. The most general ([2], [5]) one-particle canonical map of $U_B(K)$ into $\tilde{U}_B(K)$ is given by a sequence of unitary operations $\{V_j = V_j^{(1)} V_j^{(2)} V_j^{(3)} V_j^{(4)}\}_{j=0,1,...}$ implemented by

$$V_j^{(1)} = \exp(\lambda_j a_j^* - \bar{\lambda}_j a_j), \qquad V_j^{(2)} = \exp(i\alpha_j a_j^* a_j),$$
$$V_j^{(3)} = \exp\left[\frac{\nu_j}{2}(a_j^* a_j^* - a_j a_j)\right], \quad V_j^{(4)} = \exp(i\beta_j a_j^* a_j) \tag{3.1}$$

with α, β, ν real, $\bar{\lambda}$ complex conjugate to λ.

These maps result in

$$a_j \to \tilde{a}_j = e^{i(\alpha_j+\beta_j)}\mathrm{ch}\nu_j \cdot a_j + e^{i(\alpha_j-\beta_j)}\mathrm{sh}\nu_j \cdot a_j^* + \lambda_j,$$
$$a_j^* \to \tilde{a}_j^* = e^{-i(\alpha_j+\beta_j)}\mathrm{ch}\nu_j \cdot a_j^* + e^{-i(\alpha_j-\beta_j)}\mathrm{sh}\nu_j \cdot a_j + \bar{\lambda}_j. \tag{3.2}$$

In this connection we have the following statement:

Let there be given $U_B(K)$, a Fock representation of the CCR *algebra over K in $D \subset$* IDPS(Ω), *and an infinite sequence $\{V_j\}_{j=0,1,\ldots}$ of unitary maps defined by* (3.1)–(3.2). *A sufficient and necessary condition for $\{V_j\}_{j=0,1,\ldots}$ to realize a unitary transformation of $U_B(K)$ is the simultaneous fulfilment of: $\sum_j |\lambda_j|^2 < \infty$, $\sum_j |\nu_j|^2 < \infty$.*

For the proof see [2], [3], [6].

4. Representations of the CAR generated by representations of the CCR algebra in Fock case

Given $K = L^2(R^3)$, write $\mathscr{F} = \bigoplus_0^\infty K^{\otimes n}$,

$$\mathscr{F} := \left\{F = \{F_n\}_{n=0,1,\ldots}, F_n \in K^{\otimes n}, ||F||^2 = \sum_n ||F_n||^2 = \sum_n (\bar{F}_n, F_n) < \infty\right\}. \tag{4.1}$$

Given the triple $\{a^*, a, \Omega\}$ generating a Fock representation of the CCR algebra over K, the representation space IDPS(Ω) is spanned by a sequence of n-particle vectors:

$$\{|n\rangle^B\}_{n=0,1,\ldots}, \qquad |n\rangle^B(\vec{k}_n) = a^*(k_1)\ldots a^*(k_n)\,\Omega. \tag{4.2}$$

Let us denote by $\mathscr{F}_B$ the subspace of IDPS(Ω) consisting of functional vectors

$$F = \sum_n \frac{1}{\sqrt{n!}}(F_n, |n\rangle^B) = \sum_n \frac{1}{\sqrt{n!}} \int d\vec{k}_n F_n(\vec{k}_n)\,|n\rangle^B(\vec{k}_n)$$
$$= \sum_n \frac{1}{\sqrt{n!}}(F_n^S, |n\rangle^B) \tag{4.3}$$

with the property $\{F_n\}_{n=0,1,\ldots} \in \mathscr{F}$, F_n^S denoting the totally symmetric part of F_n for each n. The nonsymmetric part of F_n is annihilated by the bilinear form $(F_n, |n\rangle^B)$.

If completed with respect to the $\mathscr{F}$-topology: $(\bar{F}, F) = \sum_n ||F_n^S||^2 = ||F||^2$, $\mathscr{F}_B$ becomes the underlying Hilbert space from which we take domains for representations of the CCR algebra.

Any functional (4.3) is of course not a c-number, but a product vector again (one deals with c-valued functionals in the theory of functional representations of the CCR [17], [18] and CAR [14] algebras). The whole CCR algebra acting in $D \subset \mathscr{F}_B$ is spanned by functional polynomials in creation and annihilation operators:

$$\mathscr{F}(a^*, a) = \sum_{n,m} \frac{1}{\sqrt{n!\,m!}} \int d\vec{k}_n \int d\vec{p}_m \, \mathscr{F}_{nm}(\vec{k}_n, \vec{p}_m)\, a^*(k_1) \dots a^*(k_n)\, a(p_1) \dots a(p_m)$$
$$= \sum_{n,m} \frac{1}{\sqrt{n!\,m!}} (\mathscr{F}_{nm}, \overset{*}{a}{}^n a^m), \tag{4.4}$$

Acting on the Fock vacuum Ω, these polynomials generate a linear manifold $U_B(K)\Omega \subseteq D$ consisting of vectors of the form (4.3): $F = \mathscr{F}(a^*, a)\Omega$, so that $\mathscr{F}_B$ is the closure of D in the $\mathscr{F}$-topology. It was shown in [15] that it is possible to induce in $U_B(K)$ a Fermi triple $\{b^*, b, \Omega\}_k$, generating in a proper subspace $\overset{1}{\mathscr{F}}_B$ of $\mathscr{F}_B$ a Fock representation of the CAR algebra over K with the same vacuum Ω. This result is a consequence of the isomorphism [14]: $\mathscr{F}_n = K^{\otimes n}$, S_n-symmetrizing, A_n-antysymmetrizing operators in $\mathscr{F}_n$, $\mathscr{F}_n^A = A_n \mathscr{F}_n$, $\mathscr{F}_n^S = S_n \mathscr{F}_n$, $E_n \colon \overset{1}{\mathscr{F}}{}_n^S \leftrightarrow \mathscr{F}_n^A$ realized by the square root of a certain projector E_n^2: $E_n^2 \mathscr{F}_n^S = \overset{1}{\mathscr{F}}{}_n^S$, $\overset{1}{\mathscr{F}}{}^S = \bigoplus_0^\infty \overset{1}{\mathscr{F}}{}_n^S$. Denote $\vec{k}_n = (k_1, \dots, k_n)$, $\vec{p}_n = (p_1, \dots, p_n)$, $k, p \in \mathbf{R}^3$, $d\vec{k}_n = dk_1 \dots dk_n$ and assume $E_n(\vec{k}_n, \vec{p}_n)$ to be an integral kernel of E_n. The annihilation generator for induced Fermi algebra can be introduced as follows:

$$f \in K, \quad (a^*, a) = \int dk a^*(k)\, a(k),$$

$$b(f) = :\exp\{-(a^*, a)\} \sum_{n,m} \frac{1}{\sqrt{n!\,m!}} \int d\vec{k}_n \int d\vec{p}_m f_{nm}(\vec{k}_n, \vec{p}_m) \times$$
$$\times a^*(k_1) \dots a^*(k_n) a(p_1) \dots a(p_m), \tag{4.7}$$

with

$$f_{nm}(\vec{k}_n, \vec{p}_m) = \sqrt{n+1}\, \delta_{m,1+n} \int d\vec{q}_n \int dr\, E_n(\vec{k}_n, \vec{q}_n) f(r) E_{1+n}(r, \vec{q}_n, \vec{p}_{1+n}) \tag{4.8}$$

and $b(f)^*$ given by the $*$-operation applied to $b(f)$, keeping in mind the property $E_n^* = E_n$ for the square root of E_n^2 (see [14], [15]). Generators of induced CAR algebras, if restricted to a proper domain $D \subset \mathscr{F}_B$, can be considered as polynomials (4.5) and can thus be embedded into the CCR algebra. Any domain for $U_F(K)$ consists of vectors of the form

$$F = \sum_n \frac{1}{\sqrt{n!}} (\overset{1}{F}{}_n^S, |n\rangle^B) = \sum_n \frac{1}{\sqrt{n!}} (E_n F_n^A, |\overset{1}{n}\rangle^B)$$
$$= \sum_n \frac{1}{\sqrt{n!}} (F_n^A, E_n^T |n\rangle^B) = \sum_n \frac{1}{\sqrt{n!}} (F_n^A, |n\rangle^F), \tag{4.9}$$

where E_n^T denotes the transposition of E_n:

$$(E_n^T)^2|n^B\rangle := |\overset{1}{n}\rangle^B, \quad E_n^T|\overset{1}{n}\rangle^B = |n\rangle^F, \tag{4.10}$$

with:

$$|n\rangle^F(\vec{k}_n) = b^*(k_1) \dots b^*(k_n)\Omega = \int d\vec{p}_n E_n^T(\vec{k}_n, \vec{p}_n)\, a^*(p_1) \dots a^*(p_n)\Omega, \tag{4.11}$$

$$||F||^2 = \sum_n ||\overset{1}{F}{}_n^S||^2 = \sum_n ||F_n^A||^2 = ||\overset{1}{F}||^2 \tag{4.12}$$

what shows that in the Fock construction a representation space for $U_F(K)$ is really $\overset{1}{\mathscr{F}}_B$. In $\overset{1}{\mathscr{F}}_B$ the triple $\{b^*, b, \Omega\}_K$ acts irreducibly.

The above considerations can be summarized in the following conclusion:

$$U_B(K) \supset U_F(K)|_{\mathscr{D}\cap\overset{1}{\mathscr{F}}_B\neq\emptyset}, \tag{4.13}$$

what is the mentioned embedding of the CAR algebra in the CCR algebra, universally valid in the Fock case. Let us add that the above exposition of main ideas differs slightly, compared with [15]. We make here a significant distinction between $\mathscr{F}$ and $\mathscr{F}_B$, while in [15] (taking into account its topological properties) $\mathscr{F}_B$ was identified with the notion $\mathscr{F}^S$ of the present paper. No direct reference to the Fock construction was made there (see, however, the proof of Lemma 3 in [15]). A Fock construction allows to avoid completely the use of any set of totally antisymmetric functions, conventionally understood as necessary to describe Fermi statistics. In this connection see also [14], [19].

5. Truncation procedure

(4.13) suggests an extension of the theory of canonical transformations for bosons to the Fermi case. However, except for the unitary map, operators realizing a global canonical transformation of any element (4.5) of $U_B(K)$ (and if specialized to domains, of $U_F(K)$) in general *may* not exist. Therefore we shall perform a certain *truncation* program to avoid this disadvantage. Given the triple $\{a^*, a, \Omega\}_K$ where $K = L^2(R^3)$ is spanned by the orthocomplete, real set $\{e_j\}_{j\in I}$, $a_j = a(e_j)$, let us divide I into the enumerable infinite family of nonintersecting finite sequences of indices: $\{n_1, \dots, n_j\} = (j)$, $(j) \neq (k) \Rightarrow (j)\cap(k) = (\emptyset)$. Any set $\{e_j\}_{j\in(j)}$ spans a linear manifold $K^{(j)}$. In virtue of $(j)\cap(k) = (\emptyset)$ for different sequences, we have in fact $K = \bigoplus_{(j)} K^{(j)}$. Given enumerable infinite set of generators for $U_B(K)$, $\{a_i\}_{i\in I}$ and certain $(j) \in I$, let us define the (j)-truncated annihilation operator by $a_{(j)}(p) = \sum_{(j)} a_j e_j(p)$. In connection with this notion let us introduce $\mathscr{F} \supset \mathscr{F}^{(j)} = \bigoplus_0^\infty (K^{(j)})^{\otimes n}$. The triple $\{a^*_{(j)}, a_{(j)}, \Omega\}_{K^{(j)}}$ we call the *(j)-truncated triple*. It generates in $D \subset \mathscr{F}_B^{(j)}$ a (j)-truncated Fock representation of the CCR algebra over $K^{(j)}$. Here

$$\begin{aligned}
[a_{(j)}(p), a^*_{(k)}(q)]_- &= \sum_{(j),(k)} e_j(p)\, e_k(q)\, [a_j, a_k^*]_- \\
&= \delta_{(j),(k)} \sum_{(j)} e_j(p)\, e_j(q) \cdot \mathbf{1}_B = \delta_{(j)(k)} \cdot P_{(j)}(p, q) \cdot \mathbf{1}_B,
\end{aligned} \tag{5.1}$$

where $P_{(j)}$ is a unit operator in $K^{(j)}$ and $\delta_{(j)(k)} = 0$ if $(j) \neq (k)$ and 1 if $(j) = (k)$. Neglecting (j)-index, with $f \in K^{(j)}$, we have $[a(f), a(g)^*]_- = (\bar{f}, g)\mathbf{1}_B = (\bar{f}, P_{(j)}g)\mathbf{1}_B = (\bar{f}, g)_{(j)}\mathbf{1}_B$. The (j)-truncated representation of the CCR algebra over $K^{(j)}$ let us denote by $U_B^{(j)}(K)$. Its general element is, of course, a polynomial in $\{a_j, a_j^*\}_{j\in(j)}$. Assume $\dim(j) \geq 2$, i.e. the number of indices appearing in (j) to be not less than 2. Now, following considerations of Section 4, we can introduce a *-representation of the CAR algebra, induced in the (j)-truncated Bose one. Formulas for $b(f)_{(j)}$ are of the form (4.7), (4.8) with the only difference that $a_{(j)}, a_{(j)}^*$ appear in the place of $a, a^*, f^{(j)}$ in the place of f and $E_n^{(j)} = P_{(j)}^n E_n P_{(j)}^n$, providing a restriction of E_n to $\mathscr{F}_n^{(j)}$ in the place of E_n. The CAR for $b(f)_{(j)}$, $b(g)_{(j)}^*$ obviously hold in $\overset{1}{\mathscr{F}}{}_B^{(j)}$ due to the *-equivalence relation between $a_{(j)}, a_{(j)}^*$ and a, a^*. Because the original proof [15] of the CAR for $b(f)$, $b(g)^*$ reproduces the Wick theorem, we do not repeat it for truncated generators, referring only to the mentioned *-equivalence. Let us remark that contrary to $U_B^{(j)}(K)$, the new unit operator appears in $U_F^{(j)}(K)$:

$$\begin{aligned} \mathbf{1}_F^{(j)} &= :\exp\{-(a^*, a)_{(j)}\} \sum_n \frac{1}{n!} (a_{(j)}^{*n} E_n^{(j)}, E_n^{(j)} a_{(j)}^n):, \\ \mathbf{1}_F^{(j)} \mathscr{F}_B &= \overset{1}{\mathscr{F}}{}_B^{(j)}. \end{aligned} \tag{5.4}$$

Analogously to (4.13) we have $U_B^{(j)}(K) \supset U_F^{(j)}(K)|_{D \cap \overset{1}{\mathscr{F}}{}_B^{(j)} \neq \emptyset}$.

Let us add that the requirement $\dim(j) \geq 2$ insures the appearance of many particle terms in the expansion (4.7), (4.8). Due to commutativity of $a_{(j)}, a_{(k)}^*$ in the case $(j) \neq (k)$, we have a remarkable property of truncated Fermi generators induced in the (j)-truncated representation of the CCR algebra:

$$\begin{aligned} [b(f)_{(j)}, b(g)_{(j)}^*]_- &= 0, \\ [b(f)_{(j)}, b(g)_{(j)}^*]_+ &= (\bar{f}, g)_{(j)} \mathbf{1}_F^{(j)}, \end{aligned} \tag{5.4}$$

allowing to employ these representations in the theory of *spin waves* (*Heisenberg chains*) where Fermi statistics is needed inside any chain centre, while Bose statistics must appear if distinct centres are to be considered ([21]).

6. Non-Fock extension for truncated representations

Given the sequence of (j)-truncated representations of the CCR algebra, each truncated representation is generated by a finite (equal to $\dim(j)$) number of generators $a_j = a(e_j)$, a_j^*. Therefore transformations (3.1)–(3.2) can be realized in any $U_B^{(j)}(K)$ by a unitary operator $V_{(j)} = V_k \ldots V_{k_j}$ with V_k defined by (3.1)

$$\begin{aligned} \tilde{a}_{(j)} = V_{(j)} a_{(j)} V_{(j)}^{-1} &= \sum_{(j)} V_j a_j V_j^{-1} \cdot e_j, \\ \tilde{U}_B^{(j)}(K) &= V_{(j)} U_B^{(j)}(K) V_{(j)}^{-1}. \end{aligned} \tag{6.1}$$

A sequence of unitary maps $V_{(j)}$ in $\mathscr{F}_B$ allows us to consider a corresponding sequence $\{\tilde{U}_B^{(j)}(K)\}_{(j)\in I}$ of transformed truncated representations. Each $V_{(j)}$ provides here an example of the canonical transformation. With reference to the statement at the end of Section 3, we assume λ and ν *not* to obey the asumptions $\sum_i |\lambda_i|^2 < \infty$, $\sum_i |\nu_i|^2 < \infty$. Hence, a sequence $\{V_{(j)}\}_{(j)\in I}$ cannot represent any unitary canonical transformation of $U_B(K)$ proving thus that in $\mathscr{F}_B$, and (if specialized) in any of $\mathscr{F}_B^{(j)}$ we cannot find any vector annihilated by $\tilde{a}_{(j)}(f)$ independently of the choice of (j) and $f \in K$. This result applies at once in the Fermi case. We have here a relation (5.4) and therefore if restricted to domains $D \cap \overset{1}{\mathscr{F}}{}_B^{(j)} \neq \emptyset$ we have $\tilde{b}_{(j)}(f) = V_{(j)} b_{(j)}(f) V_{(j)}^{-1}$ which, together with its Hermitean conjugate, generates $\tilde{U}_F^{(j)}(K)$. Looking at the expansion (4.7) for $b(f)$, we find that a necessary condition for $\tilde{b}_j(f)$ to annihilate any vector F from the domain is the property $\tilde{a}_{(j)}(f)F = 0$ because $\tilde{a}_{(j)}(f)$ is the only first order term in this expansion and cannot be cancelled in any other way. For certain (j) this effect can surely appear, however it cannot be valid independently of $(j) \in I$. Hence $\tilde{b}_{(j)}(f)F = 0$ cannot appear independently of $(j) \in I$ and $f \in K$, despite of the choice of $F \in \mathscr{F}_B$, unless simultaneously $\sum_i |\lambda_i|^2 < \infty$, $\sum_i |\nu_i|^2 < \infty$.

If it is not so, no vacuum vector for the whole sequence $\{\tilde{U}_B^{(j)}(K)\}_{(j)\in I}$ and in consequence of (5.4) for $\{\tilde{U}_F^{(j)}(K)\}_{(j)\in I}$ can exist in $\mathscr{F}_B$. *This fact proves that a sequence of truncated representations of the* CAR *algebras* $\{\tilde{U}_F^{(j)}(K)\}_{(j)\in I}$ *is unitarily inequivalent to the sequence* $\{U_F^{(j)}(K)\}_{(j)\in I}$ *emerging in the Fock representation of the* CCR *algebra* $U_B(K)$.

Summarizing let us add that we cannot rather expect the existence of Fermi algebras in non-Fock representation of the CCR, in a *global* sense analogous to the Fock case. Lack of convergence criteria for series of the form (4.7) seems to exclude such a possibility. *In the case of unitary canonical transformations the whole theory for bosons* ([1]–[11], [13]) *can be immediately adopted for induced Fermi algebras, removing separate Fermi counterpart* ([10], [11], [13]) *as superfluous in the theory of canonical transformations of basic physical systems.*

Acknowledgement

One of us (P.G.) wishes to announce helpful discussions with Mr L. Juszczak.

REFERENCES

[1] J. R. Klauder, J. McKenna, E. J. Woods: *J. Math. Phys.* **7** (1966) 822.
[2] O. I. Zavyalow, V. N. Susko: *Teor. Mat. Fiz.* **1** (1969) 153.
[3] G. G. Emch: *Algebraic methods in statistical mechanics and quantum field theory*, Wiley, Interscience, 1970.
[4] J. von Neumann: *Composito Math.* **6** (1938) 1.
[5] R. Haag: *Lectures in theoretical physics*, vol. 2, Boulder, Colorado 1959.
[6 — : *Mat. Fys. Medd. Dan. Vid. Selsk.* **29**, No. **12** (1955).
[7] L. Gårding, A. S. Wightman: *Proc. Nat. Acad. Sci.* **40** (1954) 617.

[8] D. Shale, W. Stinespring: *Ann. Math.* **80** (1964) 356.
[9] L. Streit: *Comm. Math. Phys.* **4** (1967) 22.
[10] A. Guichardet: *Ann. Sci. Ecole Norm. Sup.* **83** (1966) 1.
[11] M. C. Reed: in *Lectures on elementary particles and q.f.t.*, Brandeis Summer Institute, vol. 2, MIT press, Cambridge 1970.
[12] F. Rohrlich: in *Analytic methods in mathematical physics*, R. P. Gilbert, R. G. Newton (ed.), Gordon and Breach, New York 1970.
[13] F. A. Berezin: *Methods of second quantization* (in Russian), Moscow 1965.
[14] P. Garbaczewski: J. Rzewuski: *Rep. Math. Phys.* **6** (1974) 431.
[15] —: *Comm. Math. Phys.* **43** (1975) 131.
[16] —: *Rep. Math. Phys.* **7** (1975) 321.
[17] J. Rzewuski: *Field theory*, part II, P.W.N. Warsaw, Illife Books Ltd., London 1969.
[18] —: *Rep. Math. Phys.* **1** (1971) 195.
[19] J. R. Klauder: *Ann. Phys.* **11** (1960) 123.
[20] P. Garbaczewski: *Bull. Acad. Pol. Sci. Ser. Math. Phys.* **24** (1976) 201.
[21] —: *Bosonization of Fermions in Heisenberg ferromagnet*, submitted for publ.