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FUNCTIONAL REPRESENTATIONS OF THE CANONICAL ANTICOMMUTATION RELATIONS AND THEIR APPLICATION IN QUANTUM FIELD THEORY

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We construct functional Fock representations of the CAR algebra. The *S*-operator quantum theory of interacting Fermi fields is formulated. It is found that the functional version of this theory does not require the use of functionals taking values from a Grassmann algebra. All functionals used are *C*-valued quantities.

1. Introduction

The LSZ reduction formulas for the *S*-matrix elements suggest that the scattering operator *S*, in the case of spin $-1/2$ Fermi fields can be expressed by the functional series over normal products of field operators (see [15], [3], [4]).

$$:\Omega(\psi_0, \bar{\psi}_0): = \sum_{kl} \frac{1}{k!l!} \sum_{\sigma\tau} (\omega_{kl}^{\sigma\tau}, :\psi_{0\sigma}^k \bar{\psi}_{0\tau}^l:) = \sum_{kl} \frac{1}{k!l!} \sum_{\sigma\tau} \int dx_1 \ldots \int dx_k \int dy_1 \ldots \int dy_l \times$$
$$\times \omega^{\sigma_1 \ldots \sigma_k \tau_1 \ldots \tau_l}(x_1, \ldots, x_k, y_1, \ldots, y_l) : \psi_{0\sigma_1}(x_1) \ldots \psi_{0\sigma_k}(x_k) \bar{\psi}_{0\tau_1}(y_1) \ldots \bar{\psi}_{0\tau_l}(y_l): \quad (1.1)$$

where $\psi_0, \bar{\psi}_0$ are the free operator-valued solutions of the Dirac equations, $: \cdot :$ denotes a normal ordering of the creation and annihilation operators (in the sequence $a^+, \mathring{a}^+, a^-, \mathring{a}^+$). Introducing a formal functional representation of the CAR algebra, where functionals take values from *G* (*G* is a Grassmann algebra), one may obtain (for details see Section 4):

$$:\Omega(\psi_0, \bar{\psi}_0): (\alpha, \mathring{\alpha}, \beta, \mathring{\beta}) = e^{(\alpha \cdot \mathring{\alpha}) + (\mathring{\beta} \cdot \beta)} \Omega(\psi_0, \bar{\psi}_0)$$
$$= e^{(\alpha \cdot \mathring{\alpha}) + (\mathring{\beta} \cdot \beta)} \sum_{kl} \frac{1}{k!l!} \sum_{\sigma\tau} (\omega_{kl}^{\sigma\tau}, \psi_{0\sigma}^k \bar{\psi}_{0\tau}^l), \quad (1.2)$$

where $\psi_0, \bar{\psi}_0$ are the classical *G*-valued solutions of the free Dirac equations expressed by linear forms in α, β and $\mathring{\alpha}, \mathring{\beta}$, respectively, where $\alpha, \beta, \mathring{\alpha}, \mathring{\beta} \in G$. It provides the reduction of an operator problem to a *G*-number problem. The series $\Omega(\psi_0, \bar{\psi}_0)$ in (1.2) is the basic object of the quantum field theory for spin $-1/2$ Fermi fields, given in [15]. One obtains there interacting Fermi field operators. However, the disadvantage of this theory is the fact that quantities used are not ordinary *C*-valued functionals, but elements of *G*. This was the reason for investigations performed in [7], where an example of *C*-valued anticommuting functional operators was constructed. Here we continue these investigations following (a preliminary) formulation of the problem given in [9].

In Section 2 we obtain C-valued functional Fock representations of the CAR algebra $U_C(K)$. The case of G-valued representation $U_G(K)$ is also discussed.

In Section 3 we introduce a functional representation of the LSZ reduction formulas in the case of spin $-1/2$ Fermi fields.

In Section 4 a multiplication formula for operators of the form (1.1) is found. It is expressed with the help of functional derivatives with respect to Fermi operators ordered in normal products.

In Section 5 we formulate the S-operator quantum theory of interacting Fermi fields motivated by the theory given in [15]. Interacting Fermi fields are expressed by infinite, series over normal products of creation and anihilation operators of the free fields. The functional version of the theory is shown to be expressed in terms of C-valued functionals and not elements of the Grassmann algebra G, as in [15].

2. Functional representations of the CAR algebra and the free spin $-1/2$ Fermi field

Let K be a separable complex Hilbert space with an involution *. Let $U(K)$ be an algebra of the canonical anticommutation relations (the CAR algebra) over K. $U(K)$ is a C^*-algebra with the property that there exists a linear map U: $\xi \to b(\xi)$ of K into $U(K)$, whose range generates $U(K)$

$$[b(\xi)^*, b(\eta)]_+ = (\xi, \eta^*)I,$$
$$[b(\xi), b(\eta)]_+ = [b(\xi)^*, b(\eta)^*]_+ = 0, \tag{2.1}$$

where $(\cdot, \cdot)$ is a bilinear form in K, I is a unit in $U(K)$, $\xi, \eta \in K$. It is a well known fact that if $U(K)$ and $U'(K)$ are the CAR algebras over K generated by $b(\xi)$ and $b'(\xi)$ respectively, $\xi \in K$, then these algebras are *-isomorphic (see [5], [11], [12]).

If there exists a vacuum vector f_0, $b(\xi)f_0 = 0$ we will call $U(K) = \{b, b^*, f_0\}$ a Fock representation of the CAR algebra.

Let $K = \overset{N}{\underset{1}{\oplus}} k$, where K is a separable complex Hilbert space.

$$K \ni \xi, \eta, \quad \xi = \{\xi_i\}_{i=1,\dots,N}, \quad \|\xi\|^2 = \sum_{i=1}^{N} (\overset{*}{\xi}_i, \xi_i) = \sum_{i=1}^{N} \|\xi_i\|^2$$

and $(\cdot, \cdot)$ is a bilinear form in K. An induced bilinear form in K is given by $(\overset{*}{\xi}, \eta) = \sum_{i=1}^{N} (\overset{*}{\xi}_i, \eta_i)$.

Now, we will construct a C-valued functional Fock representation $U_C(K)$ of the CAR algebra.

In [7] an operator E_n was introduced, bounded in $H_n = \overset{n}{\underset{1}{\otimes}} h$ with the properties:

$$E_n^+ = E_n, \quad E_n^2 = E_n, \quad P_{ik} E_n = -E_n P_{ik}. \tag{2.2}$$

h is a separable Hilbert space, P_{ik} denotes the permutation of ith and kth Hilbert spaces in the tensor product H_n. Let us denote: ω—the space of all real sequences, $V(n)$ —some Euclidean vector space, $\dim V(n) = n$.

DEFINITION 1. $A_n \overset{n}{\underset{1}{\otimes}} (\omega \otimes V) \ni \varepsilon_{i_1\mu_1\ldots i_n\mu_n}$ —*generalized Levi-Civita tensor,*

$$P_{ik}(\mu) P_{ik}(i) \varepsilon_{i_1\mu_1\ldots i_k\mu_k\ldots i_l\mu_l\ldots i_n\mu_n} = -\varepsilon_{i_1\mu_1\ldots i_l\mu_l\ldots i_k\mu_k\ldots i_n\mu_n}, \tag{2.3}$$
$$P_{ik}(\mu) P_{ik}(i) = P_{ik},$$

where A_n is an antisymmetrizing operator in the nth tensor product.

Let $h = L^2(R^m)$ and $h \ni \{e_i\}_{i=1,2,\ldots}$ be an orthonormal set in h; * denotes the complex conjugation in h.

LEMMA 1. *The integral kernel:*

$$E_n^{\mu\nu}(x_n, y_n) = E^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n}(x_1, \ldots, x_n, y_1, \ldots, y_n)$$
$$= \sum_i \sum_j e_{i_1}(x_1) \ldots e_{i_n}(x_n) \varepsilon_{i_1\mu_1\ldots i_n\mu_n} \delta_{i_1j_1} \ldots \delta_{i_nj_n} \delta_{\mu_1\nu_1} \ldots \delta_{\mu_n\nu_n} \overset{*}{e}_{j_1}(y_1) \ldots \overset{*}{e}_{j_n}(y_n) \tag{2.4}$$

defines an operator E_n, *bounded in* $H_n = \overset{n}{\underset{1}{\otimes}} h' = \overset{n}{\underset{1}{\otimes}} (\overset{N}{\underset{1}{\oplus}} h)$ *with the properties* (2.2). *Here* $h' \ni u \Rightarrow \|u\|^2 = \sum_{i=1}^{N} \|u_i\|^2$.

Proof: A straightforward calculation gives:

$$P_{ik}(x) P_{ik}(\mu) E_n^{\mu\nu}(x_n, y_n) = -E_n^{\mu\nu}(x_n, y_n) P_{ik}(y) P_{ik}(\nu).$$

Similarly,

$$\overset{+}{E}{}_n^{\mu\nu}(x_n, y_n) = \overset{*}{E}{}_n^{\nu\mu}(y_n, x_n) = E_n^{\mu\nu}(x_n, y_n).$$

From

$$\sum_{\sigma_1} \ldots \sum_{\sigma_n} \int dz_1 \ldots \int dz_n E_n^{\mu\sigma}(x_n, z_n) E_n^{\sigma\nu}(z_n, y_n)$$
$$= \sum_{ij} e_{i_1}(x_1) \ldots e_{i_n}(x_n) \varepsilon^2_{i_1\mu_1\ldots i_n\mu_n} \delta_{i_1j_1} \ldots \delta_{i_nj_n} \delta_{\mu_1\nu_1} \ldots \delta_{\mu_n\nu_n} \overset{*}{e}_{j_1}(y_1) \ldots \overset{*}{e}_{j_n}(y_n)$$

and

$$\varepsilon^2_{i_1\mu_1\ldots i_n\mu_n} = \varepsilon_{i_1\mu_1\ldots i_n\mu_n}$$

we obtain

$$E_n^{2\mu\nu}(x_n, y_n) = E_n^{\mu\nu}(x_n, y_n).$$

It is seen that the E_n defined by (2.4) satisfies conditions (2.2).

DEFINITION 2. Let $h' = \overset{N}{\underset{1}{\oplus}} h$ be a real separable Hilbert space, $h' \ni u, v$

$$K = \overset{N}{\underset{1}{\oplus}} k, \quad K \ni \xi, \eta, \quad K \subseteq h',$$

$$b(u, v, \xi) := \sum_{\alpha}^{N} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mu\nu\varrho}^{N} (\xi_\alpha u^n_\mu E^{\alpha\mu}_n, E^{\mu\alpha\varrho}_{1+n} v^{1+n}_\varrho), \tag{2.5}$$

$$b(u, v, \xi)^*: = \sum_{\alpha}^{N} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mu\nu\varrho} (u^{1+n}_\varrho E^{\varrho\alpha\mu}_{1+n}, \xi_\alpha E^{\mu\nu}_n v^n_\nu),$$

where we have used the symbolic notation:

$$(\xi_\alpha u^n_\mu E^{\alpha\mu}_n, E^{\mu\alpha\varrho}_{1+n} v^{1+n}_\varrho) = \int dx_1 \ldots \int dx_n \int dy_1 \ldots \int dy_{1+n} \int dz_1 \ldots \int dz_n \int dq \times$$
$$\times \xi_\alpha(q) u_{\mu_1}(x_1) \ldots u_{\mu_n}(x_n) E^{\mu_1 \ldots \mu_n \varrho_1 \ldots \varrho_n}(x_1, \ldots, x_n, z_1, \ldots, z_n) \times$$
$$\times E^{\alpha\varrho_1 \ldots \varrho_n \varrho_1 \ldots \varrho_{1+n}}(q, z_1, \ldots, z_n, y_1, \ldots, y_{1+n}) v_{\varrho_1}(y_1) \ldots v_{\varrho_{1+n}}(y_{1+n}).$$

Now we turn to the proof of

THEOREM 1. *The set $\{b, b^*, f_0\}$, where $f_0 \in C$ and b, b^* are defined above, generates a Fock representation of the* CAR *algebra over K.*

Proof: According to the rules of functional calculus (see [15], [7]), we have

$$b\left(u, \frac{d}{dw}, \xi\right) b(w, v, \eta)^*|_{w=0} = (\xi, \eta) E^2(u, v) - \sum_n \frac{1}{n!} \sum_{\substack{\mu\nu\varrho \\ \alpha\beta}}^{N} (u^{1+n}_\varrho E^{\varrho\alpha\mu}_{1+n}, \eta_\alpha \xi_\beta E^{\beta\mu\nu}_{1+n} v^{1+n}_\nu),$$

where

$$(\xi, \eta) = \sum_{\alpha=1}^{N} (\xi_\alpha, \eta_\alpha),$$

$$E^2(u, v) = \sum_n \frac{1}{n!} \sum_{\mu\nu} (u^n_\mu, E^{2\mu\nu}_n v^n_\nu).$$

One may obtain also:

$$(f_k, b^k)(u, v) = \sum_n \frac{1}{n!} \sum_{\alpha\mu\nu} (f^{\alpha}_k u^n_\mu E^{\alpha\mu}_n, E^{\alpha\mu\nu}_{k+n} v^{k+n}_\nu),$$

$$(f_k, b^{*k})(u, v) = \sum_n \frac{1}{n!} \sum_{\alpha\mu\nu} (u^{k+n}_\mu E^{\mu\alpha}_{k+n}, f^{\alpha}_k E^{\alpha\nu}_n v^n_\nu,$$

$$((f_k, b^{*k}), (g_l, b^l))(u, v) = \sum_n \frac{1}{n!} \sum_{\alpha\beta} \sum_{\mu\nu\varrho} (u^{k+n}_\mu E^{\mu\alpha}_{k+n}, f^{\alpha}_k g^{\tilde{\beta}}_l E^{\beta\nu}_{l+n} v^{l+n}_\nu).$$

Here $\sim$ denotes a reversed order of the indices. From these equalities we get at once

$$[b(\xi), b(\eta)^*]_+(u, v) = (\xi, \eta) E^2(u, v),$$
$$[b(\xi), b(\eta)]_+(u, v) = [b(\xi)^*, b(\eta)^*]_+(u, v) = 0.$$

These relations are valid in the Hilbert space spanned by functional vectors of the form $V(u) = \sum_n \frac{1}{\sqrt{n!}} \sum_p (v_n^p, E_n^p u_n^p)$ and for $f_0 \in C$, $b(\xi) f_0 = b\left(u, \frac{d}{dw}, \xi\right) f_0|_{w=0} = 0$. This completes the proof.

Let us remark that taking into account $K = k$ we reduce the above results to the case discussed in detail in [7]. Now we shall give a formal construction which provides the basis for considerations usually performed in functional theories of fermions (see for example [15], [2]).

Let G be a Grassmann algebra with a bilinear form $(\cdot, \cdot)$; for details see [15], [2], [8], the most detailed study is given in [2]:

$$G \ni \alpha, \beta, \quad \alpha = \{\alpha_s\}_{s=1,2,\ldots,N},$$
$$(\alpha, \beta) = -(\beta, \alpha).$$

DEFINITION 3.

$$G \ni \alpha, \beta, \quad K \ni \xi, \eta,$$
$$b(\alpha, \beta, \xi)^* = e^{(\alpha, \beta)}(\alpha, \xi), \tag{2.6}$$
$$b(\alpha, \beta, \xi) = e^{(\alpha, \beta)}(\beta, \xi).$$

Here, $e^{(\alpha, \beta)} = \sum_n \frac{1}{n!} (\alpha, \beta)^n$ and $(\alpha, \beta) = \sum_{i=1}^{N} (\alpha_i, \beta_i)$ is a bilinear form in G, (α, ξ) denotes the smearing of $\alpha \in G$ with the help of $\xi \in K$.

THEOREM 2. *The set* $\{b, b^*, f_0\}$, *where* $f_0 \in C_G$ (see [2]) *and* b, b^* *are defined in Definition* 3, *generates a Fock representation of the* CAR *algebra over* K.

Proof: An immediate calculation with use of rules of functional calculus ([15], [2]) yields

$$(A \cdot B)(\alpha, \beta) = A\left(\alpha, \frac{d}{d\gamma}\right) B(\gamma, \beta)|_{\gamma=0},$$

$$e^{\left(\alpha, \frac{d}{d\beta}\right)} A(\beta) = A(\alpha + \beta),$$

$$A\left(\alpha, \frac{d}{d\beta}\right) e^{(\gamma, \beta)} = e^{(\gamma, \beta)} A\left(\alpha, \gamma + \frac{d}{d\beta}\right),$$

where

$$A(\alpha, \beta) = \sum_{nm} \frac{1}{\sqrt{n!m!}} \sum_p (a_{nm}^p, \alpha_n^p \beta_m^p).$$

We use here the symbolic notation again. According to these rules, we obtain:

$$[b(\xi), b(\eta)^*]_+ (\alpha, \beta) = (\xi, \eta) e^{(\alpha, \beta)},$$
$$[b(\xi), b(\eta)]_+ (\alpha, \beta) = [b(\xi)^*, b(\eta)^*]_+ (\alpha, \beta) = 0.$$

These relations are valid in the Hilbert space spanned by functional vectors of the form $V(\alpha) = \sum_n \frac{1}{\sqrt{n!}} \sum_\beta (c_n^\alpha, \alpha_n^\beta)$. $\mathscr{I}^{(\alpha-\beta)}$ is a unit operator in $\{b, b^*, f_0\} = U_G(K)$. Now we will prove a useful lemma.

LEMMA 2. *Consider* $U(K)$, $K = \overset{4}{\underset{1}{\oplus}} k$. *The operators*:

$$a^+ = \frac{1}{\sqrt{2}} \begin{bmatrix} b_1^* + ib_3^* \\ b_2^* + ib_4^* \end{bmatrix}, \quad a^- = \frac{1}{\sqrt{2}} \begin{bmatrix} b_1 + ib_3 \\ b_2 + ib_4 \end{bmatrix},$$
$$\overset{*}{a}{}^+ = \frac{1}{\sqrt{2}} \begin{bmatrix} b_1^* - ib_3^* \\ b_2^* - ib_4^* \end{bmatrix}, \quad \overset{*}{a}{}^- = \frac{1}{\sqrt{2}} \begin{bmatrix} b_1 - ib_3 \\ b_2 - ib_4 \end{bmatrix}, \tag{2.7}$$

fulfil the relations:

$$[a^+(\chi), \overset{*}{a}{}^-(\lambda)]_+ = (\chi, \lambda)I = [a^-(\chi), \overset{*}{a}{}^+(\lambda)]_+, \tag{2.8}$$

the other anticommutators vanishing. Here $\chi = \{\chi_\mu\}_{\mu=1,2}$, $\lambda = \{\lambda_\mu\}_{\mu=1,2}$,

$$\lambda, \chi \in K = \overset{2}{\underset{1}{\oplus}} k, \quad (\lambda, \chi) = \sum_{i=1}^{2} (\lambda_i, \chi_i).$$

I is a unit operator in $U(K)$.

Proof: It suffices to remark that

$$a^+ = \frac{1}{\sqrt{2}} (c_1 + ic_2), \quad a^- = \frac{1}{\sqrt{2}} (c_1' + ic_2'),$$
$$\overset{*}{a}{}^+ = \frac{1}{\sqrt{2}} (c_1 - ic_2), \quad \overset{*}{a}{}^- = \frac{1}{\sqrt{2}} (c_1' - ic_2'),$$

where the operators c_1, c_2, c_1', c_2' fulfil

$$[c_1(\chi), c_2(\lambda)]_+ = 0 = [c_1'(\chi), c_2'(\lambda)]_+,$$
$$[c_\mu(\chi), c_\nu(\lambda)]_+ = \delta_{\mu\nu}(\chi, \lambda)I. \tag{2.9}$$

This is a straightforward consequence of the CAR relations. Applying (2.9) in (2.8) we complete the proof of the lemma. Moreover, $(a^+)^* = \overset{*}{a}{}^-$, $(a^-)^* = \overset{*}{a}{}^+$.

The operators (2.7) are creation and annihilation operators of charged spin $-1/2$ Fermi field ([1], [3]). Therefore, we my write

DEFINITION 4. *Let* $f \in S$, *the set of bispinor test functions. The operator valued free spin* $-1/2$ *Fermi field is given by*:

$$\psi_0(\bar{f}) = \sum_{r=1}^{2} \int d\vec{k} \left\{ \sum_\sigma \int dx \{ \bar{f}_\sigma(x) v^{+*}_\sigma(\vec{k}, x) a_r^+(\vec{k}) + \bar{f}_\sigma(x) v_\sigma^{-*}(\vec{k}, x) a_r^-(\vec{k}) \} \right\},$$
$$\bar{\psi}_0(f) = (\psi_0(\bar{f}))^* = (\bar{\psi}_0, f). \tag{2.10}$$

Here $\bar{f}$ denotes the Dirac conjugation of f, $v_{\sigma}^{\pm *}$ are defined in [3], $x \in M_4$—the Minkowski space. These operators fulfil the relations

$$[\psi(\bar{f}), \bar{\psi}(g)]_+ = \sum_{\sigma\tau} \int dx \int dy \bar{f}_{\sigma}(x) \frac{1}{i} S_{\sigma\tau}(x-y) g_{\tau}(y) I, \tag{2.11}$$

which are a straightforward consequence of Lemma 1.

The other anticommutators vanish, $S_{\sigma\tau}(x-y)$ is the fermion anticommutator function, $(\bar{f}, \psi)$ is an invariant in the bispinor space.

3. Functional representation of the LSZ reduction formula for the S-matrix elements in the spin 1/2 Fermi case

Let $:\Omega(\psi_0, \bar{\psi}_0):$ be the S-operator. We have the following

STATEMENT 1. *Given $U_C(K)$. Then*

$$:\Omega(\psi_0, \bar{\psi}_0):(u, v) = \sum_{m,n=0} \frac{1}{\sqrt{n!m!}} \sum_{\substack{\mu\nu \\ \sigma\tau}}^{4} (S_{nm}^{\sigma\tau}, E_n^{\mu\sigma} E_m^{\nu\tau} u_{\sigma}^{\mu} v_{\tau}^{\nu}) = S(u, v),$$

where

$$\begin{aligned} S_{nm}^{\sigma\tau}(\vec{k}_m, \vec{p}_m) = & \sum_{t,r=0}^{m} \sum_{s,p=0}^{n} \sqrt{\binom{s}{p}\binom{t}{r}\binom{n}{s}\binom{m}{t}} \times \sum_{\substack{\mu'\nu' \\ \varrho'\eta'}}^{2} \int d\vec{q}_p \int d\vec{q}_{s-p} \times \\ & \times \int d\vec{u}_r \int d\vec{u}'_{t-r}\, \delta_p(\mu\vec{k}, \mu'\vec{q})\, \delta_{s-p}(\mu\vec{k}, \nu'+2\vec{q}')\, \delta_{n-s,m-t} \times \\ & \times \delta^{n}_{m-t}(\mu\vec{k}, \nu\vec{p})\, \delta_r(\varrho'\vec{u}, \nu\vec{p})\, \delta_{t-r}(\eta'+2\vec{u}, \nu\vec{p}) \times \\ & \times \frac{1}{\sqrt{p!(s-p)!r!(t-r)!}} 2^{p+r-\frac{1}{2}(s+t)}\, i^{s-p+t-r} \sum_{j}^{s-p} \binom{s-p}{j} (-i)^j \times \\ & \times \sum_{j'}^{t-r} \binom{t-r}{j'} (-i)^{j'} \sum_{\sigma\tau} \sum_{\sigma'\tau'} \int dx_s \int dy_t\, \omega^{\sigma\sigma'\tau\tau'}_{p,s-p,r,t-r}(x_s, y_t) \times \\ & \times v_{\sigma_1}^{+\mu_1}(x_1, \vec{q}_1) \ldots v_{\sigma_p}^{+\mu_p}(x_p, \vec{q}_p) \overset{*}{v}{}_{\tau_1}^{+\nu_1}(x_{p+1}, \vec{q}'_1) \ldots \overset{*}{v}{}_{\tau_{s-p}}^{+\nu_{s-p}}(x_s, \vec{q}'_{s-p}) \times \\ & \times v_{\sigma'_1}^{-\varrho_1}(y_1, \vec{u}_1) \ldots v_{\sigma'_r}^{-\varrho_r}(y_r, \vec{u}_r) \overset{*}{v}{}_{\tau'_1}^{-\eta_1}(y_{r+1}, \vec{u}'_1) \ldots \overset{*}{v}{}_{\tau'_{t-r}}^{-\eta_{t-r}}(y_t, \vec{u}'_{t-r}). \end{aligned} \tag{3.1}$$

Proof: Let us start from the series (1.1). From the definition of the normal ordering we obtain:

$$\begin{aligned} :\Omega(\psi_0, \bar{\psi}_0): &= \sum_{nm} \frac{1}{n!m!} \sum_{\sigma\tau} (\omega_{nm}^{\sigma\tau}, :\psi_{0\sigma}^{n} \bar{\psi}_{0\tau}^{m}:) \\ &= \sum_{\substack{nm \\ kl}} \frac{1}{\sqrt{n!m!k!l!}} \sum_{\substack{\mu\nu \\ \varrho\eta}}^{2} (n_{nmkl}^{\mu\nu\varrho\eta}, :a_{\mu}^{+n} \overset{*}{a}{}_{\nu}^{+m} a_{\varrho}^{-k} \overset{*}{a}{}_{\eta}^{-l}:). \end{aligned}$$

Taking into account relations (2.7), (2.9), we obtain:

$$:\Omega(\varphi_0, \bar{\varphi}_0): = \sum_{\substack{nm\\kl}} \frac{1}{\sqrt{n!m!k!l!}} \sum_{\substack{\mu\nu\\\varrho\eta}}^{2} (\mathrm{n}'^{\mu\nu\varrho\eta}_{nmkl}, :c^{n}_{1\mu} c^{m}_{2\nu} c'^{k}_{1\varrho} c'^{l}_{2\eta}:),$$

where

$$c_{1\mu} = b^{*}_{\mu}, \qquad c'_{1\varrho} = b_{\varrho},$$
$$c_{2\nu} = b^{*}_{\nu+2}, \quad c'_{2\eta} = b_{\eta+2}, \quad \mu, \nu, \varrho, \eta = 1, 2,$$

and therefore,

$$:\Omega(\varphi_0, \bar{\varphi}_0):(u, v) = \sum_{\substack{nm\\kl}} \frac{1}{\sqrt{n!m!k!l!}} (\mathrm{n}'^{\mu\nu\varrho\eta}_{nmkl}, :b^{*n}_{\mu} b^{*m}_{\nu+2} b^{k}_{\varrho} b^{l}_{\eta+2}:) (u, v)$$
$$= \sum_{\substack{nm\\kl}} \frac{1}{\sqrt{n!m!k!l!}} \sum_{pr} \frac{1}{\sqrt{p!r!}} \sum_{\substack{\mu\nu\\\varrho\eta}}^{2} \sum_{\alpha\beta\gamma}^{4} (u_{\alpha}^{n+m+p} E^{\alpha, \mu\nu+2\gamma}_{n+m+p},$$
$$\mathrm{n}'^{\mu\nu\varrho\eta}_{nm\widetilde{kl}} \delta_{pr} \delta^{\alpha}_{r}(\gamma, \sigma) E^{\varrho\eta+2\sigma, \beta}_{k+l+r} v_{\beta}^{k+l+r}),$$

where $\sim$ denotes a reversed order of variables in the group $\{k, l\}$, $\delta^{\alpha}_{r}(\gamma, \sigma)$ is a symbolic expression for

$$\frac{1}{r!} \sum_{\text{perm}} (-1)^{P} \delta(\vec{p}_1 - \vec{k}_{p_1}) \delta_{\gamma_1\sigma_{p_1}} \ldots \delta(\vec{p}_r - \vec{k}_{p_r}) \delta_{\gamma_r\sigma_{p_r}}.$$

Let us introduce the new expansion coefficient:

$$\mathrm{n}''^{\mu'\nu'}_{nmpklr} = \sum_{\substack{\mu\nu\\\varrho\eta}}^{2} \delta_{n}(\mu', \mu) \delta_{m}(\mu', \nu+2) \mathrm{n}'^{\mu\nu\varrho\eta}_{nm\widetilde{kl}} \delta_{pr} \delta^{\mu'}_{r}(\mu', \nu') \delta_{k}(\varrho, \nu') \delta_{l}(\eta+2, \nu'), \tag{3.4}$$

where, for example $\delta_{n}(\mu', \mu)$ and $\delta_{m}(\mu', \nu+2)$ are the symbolic expressions for $\delta(\vec{p}_1 - \vec{k}_1) \delta_{\mu'_1\mu_1} \ldots \delta(\vec{p}_n - \vec{k}_n) \delta_{\mu'_n\mu_n}$ and $\delta(\vec{p}_{n+1} - \vec{q}_1) \delta_{\mu'_{n+1}, \nu_1+2} \ldots \delta(\vec{p}_{n+m} - \vec{q}_m) \delta_{\mu'_{n+m}, \nu_m+2}$ respectively.

Summations over vector indices represent also integrations over corresponding variables.

Now we obtain (see also formula (2.55) in [15])

$$:\Omega(\varphi_0, \bar{\varphi}_0):(u, v) = \sum_{\substack{nm\\kl\\pr}} \frac{1}{\sqrt{n!m!k!l!p!r!}} \sum_{\substack{\mu'\nu'\\\alpha\beta}}^{4} (u_{\alpha}^{n+m+p} E^{\alpha\mu'}_{n+m+p}, \mathrm{n}''^{\mu'\nu'}_{nmpklr} E^{\nu'\beta}_{k+l+r} v_{\beta}^{k+l+r})$$
$$= \sum_{nm} \frac{1}{\sqrt{n!m!}} \sum_{t,r=0}^{m,l} \sum_{s,p=0}^{n,k} \sqrt{\binom{s}{p}\binom{t}{r}\binom{n}{s}\binom{m}{t}} \times$$

$$\times \sum_{\substack{\mu'\nu'\\ \alpha\beta}}^{4} \left(u_\alpha^{n} E_n^{\alpha\mu'}, \pi^{\prime\prime\mu\nu}_{[illegible]} E_m^{\nu'\beta} v_\beta^{m}\right)$$

$$= \sum_{nm} \frac{1}{\sqrt{n!m!}} \sum_{\substack{\mu'\nu'\\ \alpha\beta}}^{4} \left(s_{nm}^{\mu\nu}, E_n^{\mu'\alpha} E_m^{\nu'\beta} u_\alpha^{n} v_\beta^{m}\right) = S(u, v). \tag{3.5}$$

This completes the proof.

The formula (3.1) provides a functional representation of the LSZ reduction formula for the S-matrix elements in the case of spin $-1/2$ Fermi fields.

The Statement 1 permits us to apply the apparatus developed in [16], [17] to the functionals of the form: $:\Omega(\psi_0, \bar{\psi}_0):(u, v)$. Let us remark that

$$S(u, v) = \sum_{nm} \frac{1}{\sqrt{n!m!}} \sum_{\substack{\mu'\nu'\\ \alpha\beta}} \left(E_n^{\alpha\mu'} E_m^{\beta\nu'} s_{nm}^{\mu\nu}, u_\alpha^{n} v_\beta^{m}\right). \tag{3.6}$$

This implies for example (if $||S||$ does exist):

$$|:\Omega(\psi_0, \bar{\psi}_0):(u, v)| \leqslant ||S|| \exp\{\tfrac{1}{2}||u||^2 + \tfrac{1}{2}||v||^2\} \tag{3.7}$$

and for a unitary S-functional:

$$|S(u, v)| = |:\Omega(\psi_0, \bar{\psi}_0):(u, v)| \leqslant \exp\{\tfrac{1}{2}||u||^2 + \tfrac{1}{2}||v||^2\}, \tag{3.8}$$

where

$$||u||^2 = (\overset{*}{u}, u) = \sum_{\mu=1}^{4} (\overset{*}{u}_\mu, u_\mu).$$

4. Multiplication formula for operators of the form (1.1)

Let us take into account $U_G(K)$ with $K = \overset{4}{\underset{1}{\oplus}} k$. We have

LEMMA 3. *Functional operators:*

$$\begin{aligned} a^+(\xi) &= e^{(\alpha,\overset{*}{\alpha})+(\overset{*}{\beta},\beta)}(\alpha, \xi), & \overset{*}{a}{}^-(\xi) &= e^{(\alpha,\overset{*}{\alpha})+(\overset{*}{\beta},\beta)}(\overset{*}{\beta}, \xi), \\ a^-(\xi) &= e^{(\alpha,\overset{*}{\alpha})+(\overset{*}{\beta},\beta)}(\beta, \xi), & \overset{*}{a}{}^+(\xi) &= e^{(\alpha,\overset{*}{\alpha})+(\overset{*}{\beta},\beta)}(\overset{*}{\alpha}, \xi), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \alpha &= \frac{1}{\sqrt{2}}(\alpha^1 + i\alpha^2), & \overset{*}{\alpha} &= \frac{1}{\sqrt{2}}(\beta^1 - i\beta^2), \\ \overset{*}{\beta} &= \frac{1}{\sqrt{2}}(\alpha^1 - i\alpha^2), & \beta &= \frac{1}{\sqrt{2}}(\beta^1 + i\beta^2) \end{aligned} \tag{4.2}$$

fulfil the relations (2.8).

Proof: We have

$$\alpha^i = \begin{bmatrix} \alpha_1^i \\ \alpha_2^i \end{bmatrix}, \quad \beta^i = \begin{bmatrix} \beta_1^i \\ \beta_2^i \end{bmatrix}.$$

Thus

$$(\alpha, \overset{*}{\alpha}) + (\overset{*}{\beta}, \beta) = (\alpha^1, \beta^1) + (\alpha^2, \beta^2) = \sum_{\nu=1}^{2} \{(\alpha_\nu^1, \beta_\nu^1) + (\alpha_\nu^2, \beta_\nu^2)\}.$$

Putting:

$$\alpha' = \begin{bmatrix} \alpha_1^1 \\ \alpha_2^1 \\ \alpha_1^2 \\ \alpha_2^2 \end{bmatrix} = [\alpha_\mu']_{\mu=1,2,3,4}, \quad \beta' = \begin{bmatrix} \beta_1^1 \\ \beta_2^1 \\ \beta_1^2 \\ \beta_2^2 \end{bmatrix} = [\beta_\mu']_{\mu=1,2,3,4},$$

and for $K = \overset{4}{\underset{1}{\oplus}} k$ defining

$$b(\xi)^* = e^{(\alpha', \beta')}(\alpha', \xi),$$
$$b(\xi) = e^{(\alpha', \beta')}(\beta', \xi),$$

we may apply Lemma 2 and this proves Lemma 3.

Now, we have:

$$\varphi_0(\bar{f})(\alpha, \beta, \overset{*}{\alpha}, \overset{*}{\beta}) = e^{(\alpha, \overset{*}{\alpha}) + (\overset{*}{\beta}, \beta)} \varphi_0(\bar{f})$$
$$= e^{(\alpha, \overset{*}{\alpha}) + (\overset{*}{\beta}, \beta)} \sum_\nu \int d\vec{k} \int dx \bar{f}(x) \{v^\nu + (\vec{k}, x) \alpha_\nu(\vec{k}) + v^{-\nu}(\vec{k}, x) \beta_\nu(\vec{k})\}, \quad (4.3)$$

and similarly,

$$\bar{\varphi}_0(f)(\alpha, \beta, \overset{*}{\alpha}, \overset{*}{\beta}) | e^{(\alpha, \overset{*}{\alpha}) + (\overset{*}{\beta}, \beta)} \bar{\varphi}_0(f) |$$
$$= e^{(\alpha, \overset{*}{\alpha}) + (\overset{*}{\beta}, \beta)} \sum_\nu \int d\vec{k} \int dx f(x) \{\overset{*}{v}{}^\nu + (\vec{k}, x) \overset{*}{\alpha}_\nu(\vec{k}) + \overset{*}{v}{}^{-\nu}(\vec{k}, x) \overset{*}{\beta}_\nu(\vec{k})\}. \quad (4.4)$$

This implies (see also [15]):

$$:\Omega(\varphi_0, \bar{\varphi}_0):(\alpha, \beta, \overset{*}{\alpha}, \overset{*}{\beta}) = e^{(\alpha, \overset{*}{\alpha}) + (\overset{*}{\beta}, \beta)} \Omega(\varphi_0, \bar{\varphi}_0), \quad (4.5)$$

where

$$\Omega(\varphi_0, \bar{\varphi}_0) = \sum_{nm} \frac{1}{n! m!} (\omega_{nm}, \varphi_0^n \bar{\varphi}_0^m).$$

From the rules of the functional calculus for elements of Grassmann algebras (see Theorem 2, [15], [2]), we obtain:

LEMMA 4. *For given $U_G(K)$, $K = \bigoplus_1^4 k$ the following relations hold:*

$$:\Omega_1(\psi_0, \overline{\psi}_0):\left(\alpha, \frac{d}{d\gamma}, \overset{*}{\alpha}, \frac{d}{d\overset{*}{\gamma}}\right):\Omega_2(\psi_0, \overline{\psi}_0):(\gamma, \beta, \overset{*}{\gamma}, \overset{*}{\beta})_{\gamma=\overset{*}{\gamma}=0}$$

$$= :\Omega_{12}(\psi_0, \overline{\psi}_0):(\alpha, \beta, \overset{*}{\alpha}, \overset{*}{\beta})$$

$$= e^{(\alpha, \overset{*}{\gamma})+(\overset{*}{\beta}, \beta)}\Omega_1(\psi, \overline{\psi})\exp\left\{i\frac{\overleftarrow{d}}{d\psi}S^+\frac{\overrightarrow{d}}{d\overline{\psi}} - i\frac{\overleftarrow{d}}{d\psi}S^-\frac{\overrightarrow{d}}{d\overline{\psi}}\right\}\Omega_2(\psi, \overline{\psi})\Big|_{\substack{\psi=\psi_0\\ \overline{\psi}=\overline{\psi}_0}}. \tag{4.6}$$

The proof may be found in [15], Chapter 5. There is, however, a difference between the case discussed here and in [1], which must be kept in mind: in [1] α, β, $\overset{*}{\alpha}$, $\overset{*}{\beta}$ are G-valued bispinor functions and not G-valued vector functions as in (4.18).

Looking at [1], [3], [4], we conclude that the Grassmann algebra techniques were primarily motivated by operations over normal products of Fermi operators.

Let us take into account arbitrary Fermi field operators expressed by linear forms in creation and annihilation operators, and therefore satisfying

$$\begin{aligned} [\psi(\overline{f}), \overline{\psi}(g)]_+ &= (\overline{f}, Fg)I, \\ [\psi(\overline{f}), \psi(\overline{g})]_+ &= [\overline{\psi}(f), \overline{\psi}(g)]_+, \end{aligned} \tag{4.7}$$

$f, g \in S$—the space of bispinor test functions, I is the unit operator in $U(K)$. For free fields we have $F = S$, see (2.11).

Let us formulate the following

DEFINITION 5. *Left-hand-side functional derivative*

$$\begin{aligned} :\left(f, \frac{d}{d\psi}\right)\Omega(\psi, \overline{\psi}): &= \sum_{nm} \frac{1}{(n-1)!m!}(\omega_{1,n-1,m}, f : \psi^{n-1}\overline{\psi}^m:), \\ :\left(\overline{f}, \frac{d}{d\overline{\psi}}\right)\Omega(\psi, \overline{\psi}): &= \sum_{nm} \frac{1}{n!(m-1)!}(\omega_{1,n,m-1}, \overline{f} : \psi^n\overline{\psi}^{m-1}:). \end{aligned} \tag{4.8}$$

Right-hand-side functional derivative

$$\begin{aligned} :\Omega(\psi, \overline{\psi})\left(\frac{\overleftarrow{d}}{d\psi}, f\right): &= \sum_{nm} \frac{1}{(n-1)!m!}(\omega_{n-1,m,1}, : \psi^{n-1}\overline{\psi}^m : f), \\ :\Omega(\psi, \overline{\psi})\left(\frac{\overleftarrow{d}}{d\overline{\psi}}, \overline{f}\right): &= \sum_{nm} \frac{1}{n!(m-1)!}(\omega_{n,m-1,1}, : \psi^n\overline{\psi}^{m-1} : \overline{f}). \end{aligned} \tag{4.9}$$

Looking into [1], Chapter III and into [13], [14] for properties of these derivatives, we can formulate:

THEOREM 3. *Given* $U(K)$, $K = \bigoplus_{1}^{4} k$. *Then*

$$:\Omega_1(\psi_0, \overline{\psi}_0)::\Omega_2(\psi_0, \overline{\psi}_0): = :\Omega_{12}(\psi_0, \overline{\psi}_0): = :\Omega_1(*)\Omega_2:$$

$$= :\Omega_1(\psi, \overline{\psi}) \exp\left\{ i \frac{\overleftarrow{d}}{d\psi} S^+ \frac{\overrightarrow{d}}{d\overline{\psi}} - i \frac{\overleftarrow{d}}{d\psi} S^- \frac{\overrightarrow{d}}{d\overline{\psi}} \right\} \Omega_2(\psi, \overline{\psi}): \Big|_{\substack{\psi=\psi_0 \\ \overline{\psi}=\overline{\psi}_0}}. \tag{4.10}$$

Proof: The computation can be performed in the same way as in the proof of Lemma 4. The result can be written as:

$$:\Omega_{12}(\psi, \overline{\psi}): \Big|_{\substack{\psi=\psi_0 \\ \overline{\psi}=\overline{\psi}_0}} = :\Omega_{12}(\psi_0, \overline{\psi}_0):.$$

Taking into account the $U_G(K)$, we obtain evidently

$$:\Omega_{12}(\psi, \overline{\psi}): = e^{(\alpha\cdot\dot{\alpha}+\dot{\beta}\cdot\beta)}\Omega_{12}(\psi, \overline{\psi}),$$

with $\Omega_{12}(\psi, \overline{\psi})$ given by (4.6).

Thus the theorem is fulfilled for $U_G(K)$, $K = \bigoplus_{1}^{4} k$.

But (4.10) is independent on the choice of the (*-isomorphic) representations of the CAR algebra. Therefore the validity of (4.10) for $U_G(K)$ implies the same for each $U(K)$, $K = \bigoplus_{1}^{4} k$. This proves the theorem.

An essentially functional version of equation (4.10) is provided by

STATEMENT 2. *Given* $U_C(K)$, $K = \bigoplus_{1}^{4} k$. *Then*

$$:\Omega_{12}(\psi_0, \overline{\psi}_0):(u, v) = :\Omega_1(*)\Omega_2:(u, v) = S_{12}(u, v) = (S_1 \cdot S_2)(u, v). \tag{4.11}$$

Proof: The statement is a trivial consequence of (3.1) and (4.10).

5. The S-operator formulation of the quantum field theory for spin $-1/2$ Fermi fields

Our starting point is the S-functional theory formulated in [15], Chapter V, with the help of Grassmann algebra tools. Relations between formal (taking values from G) functional power series provide some relations between their expansion coefficients. Our aim is to express these relations in the operator language. It will give us a theory independent of the choice of (*-isomorphic) representations of the CAR algebra.

The essentially S-functional theory (in terms of functionals taking values from C) is thus a trivial corollary obtained by a proper choice of $U_C(K)$, $K = \bigoplus_{1}^{4} k$.

Let $S = :\Omega(\psi_0, \overline{\psi}_0):$ be the on-mass-shell scattering operator. One may apparently conclude that such an operator belongs to the set of operators fulfilling the following conditions:

(i) *algebraic structure*:

$$S_1 \cdot S_2 = S_{12} = :\Omega_1(*)\Omega_2:, \tag{5.1}$$

(ii) *unitarity*:

$$S^+ \cdot S = I, \tag{5.2}$$

(iii) *relativistic invariance*:

$$:\Omega(\psi_{0L,a}, \bar{\psi}_{0L,a}): = :\Omega(\psi_0, \bar{\psi}_0):, \tag{5.3}$$

where

$$\psi_{L,a}(x) = (S_L \psi)(L^{-1}x - L^{-1}a),$$

$\{L, a\}$ are the 10 parameters of the inhomogeneous Lorentz group.

Let us introduce subsidiary Fermi operators:

DEFINITION 6.

$$\begin{aligned} q(x) &= -\int dy S^F(x-y)\gamma_4 \eta(y), \\ \bar{q}(x) &= -\int dy \bar{\eta}(y)\gamma_4 S^F(y-x), \end{aligned} \tag{5.4}$$

where

$$[\eta(\bar{f}), \bar{\eta}(g)]_+ = (\bar{f}, Fg), \quad f, g \in S. \tag{5.5}$$

The other anticommutators vanish.

This implies of course

$$[q(\bar{f}), \bar{q}(g)]_+ = (\bar{f}, F'g). \tag{5.6}$$

Let us formulate further

DEFINITION 7.

$$S_q = :\Omega(q+\psi_0, \bar{q}+\bar{\psi}_0):. \tag{5.7}$$

The operators S_q are the basic objects of our theory. Its contents lies in the five postulates which limit the arbitrariness in the choice of S_q, preserving at the same time relations between expansion coefficients of the operator series (5.7) resulting from the theory given in [15].

(i) *Algebraic structure*

$$S_q^1 * S_q^2 = S_q^{12}: |:\Omega_q^1(*)\Omega_q^2:|$$

$$= :\Omega_1(q+\psi, \bar{q}+\bar{\psi}) \exp\left\{i \frac{\vec{d}}{d\psi} S^+ \frac{\vec{d}}{d\bar{\psi}} - i\frac{\vec{d}}{d\psi} S^- \frac{\vec{d}}{d\bar{\psi}}\right\} \Omega_2(q+\psi, \bar{q}+\bar{\psi}): \Big|_{\substack{\psi=\psi_0 \\ \bar{\psi}=\bar{\psi}_0}}. \tag{5.8}$$

(ii) *Unitarity*

$$S_q^+ * S_q = I. \tag{5.9}$$

(iii) *Causality*. Let us introduce

$$T = S^+ * S_q$$

and define

$$T_i^{\pm} = S^{+} * S_{\eta_i^{\pm}},$$

where (see also [15])

$$\eta_i^{\pm}(x) = \Theta_i^{\pm}(x)\eta(x).$$

Now, the causality condition reads

$$T_1^3 = T_2^3 * T_1^2. \tag{5.10}$$

(iv) *Relativistic invariance*

$$S_{\eta_{L,a}} = S_{\eta}. \tag{5.11}$$

(v) *Internal structure.* On may write $T = :\tau(q, \bar{q}, \psi_0, \bar{\psi}_0):$. The condition reads

$$:\psi(\bar{f})\tau(q, \bar{q}, \psi_0, \bar{\psi}_0): = :\tau(q, , \psi_0, \bar{\psi}_0)\psi(\bar{f}):. \tag{5.12}$$

From the theory result definitions of interacting Fermi field operators expressed as the infinite series over normal products of creation and annihilation operators

$$\begin{aligned} \chi^{ret}(\bar{f}) &= -i\left(T^{+} * \left(\frac{dT}{d\bar{\eta}}, \bar{f}\right)\right), \\ \bar{\chi}^{ret}(f) &= i\left(T^{+} * \left(\frac{dT}{d\eta}, f\right)\right), \\ \chi^{adv}(\bar{f}) &= -i\left(\left(\frac{dT}{d\bar{\eta}}, \bar{f}\right) * T^{+}\right), \\ \bar{\chi}^{adv}(f) &= i\left(\left(\frac{dT}{d\eta}, f\right) * T\right). \end{aligned} \tag{5.13}$$

They fulfil the condition of locality (see [15])

$$[\chi(\bar{f}), \chi(\bar{g})]_{+} = [\chi(\bar{f}), \bar{\chi}(g)]_{+} = [\bar{\chi}(f), \bar{\chi}(g)]_{+} = 0, \tag{5.14}$$

with f, g taking supports from space-like separated regions supp$f \sim$ supp$g$.

Taking into account the $U_C(K)$ representation of the CAR algebra, we obtain the essentially functional version of the theory under consideration. Thus, for example, we have

$$\chi(\bar{f})(u, v) = \sum_{nm} \frac{1}{\sqrt{n!m!}} (\chi_{nm}(\bar{f}), E_n u^n E_m v^m), \tag{5.15}$$

i.e. interacting Fermi field operators may be scrutinized with the help of methods developed in [16], [17].

We have constructed above a quantum theory of interacting Fermi fields based entirely on representations of the CAR (for problems connected with that matter see [11], [12],

[5]). In this construction we assume that there exists a nontrivial S-operator fulfilling (5.8)–(5.12). Taking into account the $U_c(K)$, we reduce the formal theory [15] to a theory in which only ordinary functionals appear. It is proved that the apparatus developed in [16], [17] for such functionals can be employed in the case of functional theories of fermions. This seemed to be impossible up to now (see however, in this connection [9]).

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