ON GENERATING FUNCTIONALS FOR ANTISYMMETRIC FUNCTIONS AND THEIR APPLICATION IN QUANTUM FIELD THEORY

PIOTR GARBACZEWSKI and JAN RZEWUSKI

Institute of Theoretical Physics, University of Wrocław, Wrocław, Poland

(*Received March 13, 1973*)

An isomorphism between certain subspaces of the Hilbert spaces of symmetric and antisymmetric n-point functions (or, more generally, symmetric and antisymmetric tensor products of a Hilbert space) is described. It permits a construction of generating functionals for sets of antisymmetric functions. In this way the theory of Hilbert spaces of functional power series as described in [7] and [8] can be extended to the case of antisymmetric coefficients. As an application, the functional representation for the anticommutation relations is derived. It enables to obtain a functional formulation of quantum field theory also in the antisymmetric case without the use of Grassman algebras.

Introduction

The aim of the present paper is to discuss, once again, the construction of generating ctionals for sets of antisymmetric functions and the corresponding applications to ıntum field theory. The difficulty in the antisymmetric case arises from the fact that ı-linear forms of the type

$$F_n(\varphi) = \int dx_1 \dots \int dx_n f_n(x_1, \dots, x_n)\, \varphi(x_1) \dots \varphi(x_n) \tag{1.1}$$

y the symmetric part of the function $f_n(x_1, \dots, x_n)$ contributes to the integral due to symmetry of the tensor product $\varphi(x_1) \dots \varphi(x_n)$. Thus, there exists no straightforward y to extend the method of generating functionals (functional power series), which are entially sums of terms of the type (1.1), to n-point functions $f_n(x_1, \dots, x_n)$ of any other ımetry type.

Due to the fact that the functional method (the method of generating functionals) ›ves very useful for the understanding of theories containing sets $\{f_n^S\}_{n \in N}$ of symmetric ›oint functions, various attempts have been made to extend this method to sets of antiımetric functions $\{f_n^A\}_{n \in N}$

There are two standard methods to overcome, or rather circumvent, the difficulty. e first one (cf. e.g. [1], [5], [6]) consists in replacing the functions $\varphi(x)$ in the tensor product

$\varphi(x_1) \dots \varphi(x_n)$ in (1.1) by anticommuting functions. Since the simplest realization of such objects can be found in the framework of Grassman algebras, in this case the generating functionals themselves are elements of such an algebra and not c-numbers as in the symmetric case. The second one (cf. e.g. [2], [3]) uses an ordering sign function $\sigma(x_1, \dots, x_n)$ antisymmetric in the variables $x_1, \dots, x_n$, vanishing whenever two of the variables are equal and assuming the absolute value 1 otherwise. With the help of this function one obtains a mapping

$$\begin{aligned} f_n^{A}(x_1, \dots, x_n) &= \sigma(x_1, \dots, x_n) f_n^{S}(x_1, \dots, x_n), \\ f_n^{S}(x_1, \dots, x_n) &= \sigma(x_1, \dots, x_n) f_n^{A}(x_1, \dots, x_n) \end{aligned} \tag{1.2}$$

between the spaces of symmetric and antisymmetric functions, thus relating problems of Fermi and Bose statistics.

In this paper we wish to investigate mappings of the type (1.2) in some greater detail and describe representations of such mappings in terms of operators. For this purpose the operators connecting the various symmetry types are defined in an abstract way by a set of algebraic relations (Section 2).

The domains and ranges of these operators are determined as well as their properties with respect to unitary transformations of the underlying Hilbert space. Examples and realizations follow in Section 3. Section 4 is devoted to the construction of generating functionals for antisymmetric functions and to the generalization of the theory of Hilbert spaces of functional power series to these functionals. In Section 5 we derive the functional representation of the creation and annihilation operators in the anticommuting case and obtain a functional formulation of quantum field theory on mass shell in the antisymmetric case (Fermi statistics).

Off-mass-shell generalizations of the functional method are derived in Section 6 for the case of one real scalar anticommutating field. In particular, the functional form for the reduction formulae which connect the on and off mass shell functionals is derived for the antisymmetric case.

2. Certain representations of square roots of projectors in tensor products

Let $\mathscr{H}_n = \overset{n}{\underset{1}{\otimes}} h$ be the nth tensor product of the Hilbert space h and E_n a bounded operator in $\mathscr{H}_n$ satisfying the relations

$$E_n^+ = E_n, \quad E_n^3 = E_n. \tag{2.1}$$

It follows from (2.1) that E_n^2 is a projector. Conversely, any square root of a projector satisfies (2.1).

The corresponding decomposition of $\mathscr{H}_n$ is $\mathscr{H}_n = \overset{1}{\mathscr{H}}_n \oplus \overset{2}{\mathscr{H}}_n$, where

$$\overset{1}{\mathscr{H}}_n = E_n^2 \mathscr{H}_n, \quad \overset{2}{\mathscr{H}}_n = (1 - E_n^2) \mathscr{H}_n. \tag{2.2}$$

On account of (2.1) we have

$$E_n \overset{1}{\mathscr{H}}_n = \overset{1}{\mathscr{H}}_n, \quad E_n \overset{2}{\mathscr{H}}_n = 0. \tag{2.3}$$

Proof of (2.3):

(i) $E_n \overset{2}{\mathscr{H}}_n = E_n(1-E_n^2)\mathscr{H}_n = 0$;

(ii) $(1-E_n^2)E_n\mathscr{H}_n = 0 \Rightarrow E_n\mathscr{H}_n \subset \overset{1}{\mathscr{H}}_n$;

(iii) $\overset{1}{\mathscr{H}}_n = E_n^2\mathscr{H}_n = E_n(E_n\mathscr{H}_n) \Rightarrow \overset{1}{\mathscr{H}}_n \subset E_n\mathscr{H}_n$.

E_n is, therefore, an automorphism of $\overset{1}{\mathscr{H}}_n$ (a homomorphism $\mathscr{H}_n \to \overset{1}{\mathscr{H}}_n$ with the kernel ker $E_n = \overset{2}{\mathscr{H}}_n$ or a monomorphism $\overset{1}{\mathscr{H}}_n \to \mathscr{H}_n$).

Let us introduce inversion operators P_{ik} in $\mathscr{H}_n$ interchanging the indices of the ith and the kth element of the tensor product $\overset{n}{\underset{1}{\otimes}} h$. We shall impose now an additional set of conditions on the representation E_n, determining its symmetry character:

$$P_{ik}E_n = -E_n P_{ik} \quad (i, k = 1, 2, \ldots, n). \tag{2.4}$$

To explain the notation in (2.4) we note that E_n, as well as P_{ik}, are bounded operators in $\mathscr{H}_n$ and, therefore, can be considered as elements of the algebraic tensor product $\mathscr{H}_n \otimes \mathscr{H}_n$ completed in the norm $\|\cdot\|$ of bounded operators in $\mathscr{H}_n$. P_{ik} acting on the left (right) interchanges indices in the left (right) factor of this tensor product.

An arbitrary element f_n of $\mathscr{H}_n = \overset{n}{\underset{1}{\otimes}} h$ can be represented as the sum of irreducible representations of the S_n with the help of Young's idempotent operators Y_n

$$f_n = \sum_Y Y_n f_n. \tag{2.5}$$

From among the various Y_n's we shall be particularly interested in the symmetrizing and antisymmetrizing operators

$$S_n = \frac{1}{n!}\sum P_n, \quad A_n = \frac{1}{n!}\sum (-1)^P P_n, \tag{2.6}$$

where the sums are extended over all permutations P_n of n elements. Young's operators have their duals Y_n^{d} among themselves, e.g.: $S_n = A_n^{\mathrm{d}}$, $A_n = S_n^{\mathrm{d}}$.

The symmetry properties of the operator E_n defined by equations (2.4) are reexpressed in terms of Young's operators by the following

THEOREM.

$$P_{ik}E_n = -E_n P_{ik} \Leftrightarrow Y_n E_n = E_n Y_n^{\mathrm{d}}. \tag{2.7}$$

Proof: ($\Rightarrow$) Each permutation P_n can be written as the sum $P_n = P_n^e + P_n^\sigma$ of a term P_n^e containing only products of an even number of inversions P_{ik} and a term P_n^σ containing only products of an odd number of inversions.

Thus, the Young operator $Y_n = \sum_Q (-1)^Q Q_n \sum_P P_n$ can be written in the form

$$Y_n = \sum_Q (-1)^Q (Q_n^e + Q_n^\sigma) \sum_P (P_n^e + P_n^\sigma) = \sum_Q (Q_n^e - Q_n^\sigma) \sum_P (P_n^e + P_n^\sigma).$$

Here $\sum_P, \sum_Q$ denote the sums over all permutations in the lines and in the columns respectively of the corresponding Young scheme. Applying Y_n to E_n we obtain, because of (2.4),

$$Y_n E_n = E_n \sum_Q (Q_n^e + Q_n^\sigma) \sum_P (P_n^e - P_n^\sigma) = E_n Y_n^d,$$

where Y_n^d is the operator corresponding to the dual Young scheme, i.e. the scheme with interchanged lines and columns.

($\Leftarrow$) $Y_n E_n = E_n Y_n^d$ for all Y_n

$$\Rightarrow \sum_Q (Q_n^e - Q_n^\sigma) \sum_P (P_n^e + P_n^\sigma) E_n = E_n \sum_Q (Q_n^e + Q_n^\sigma) \sum_P (P_n^e - P_n^\sigma) \text{ for all } Y_n$$

$$\Rightarrow P_{ik} E_n = -E_n P_{ik}.$$

Relations (2.1), as well as any algebraic relation, remain invariant with respect to the group $G_n(U)$ of all unitary transformations $A_n' = U_n A_n U_n^+$ in $\mathscr{H}_n$, where $U_n \in G_n(U)$ and A_n is an arbitrary bounded operator in $\mathscr{H}_n$. Relations (2.4) which determine the symmetry properties of E_n are, of course, not invariant with respect to the whole group $G_n(U)$ because anticommutation relations with $U_n P_{ik} U_n^+$ may not be equivalent to anticommutation relations with P_{ik}. One can show, however, that relations (2.4), and therefore, the symmetry properties of E_n, remain invariant under the transformations of the subgroup $G_n^S(U)$ of those transformations $U_n \in G_n^S(U)$ which satisfy relations

$$U_n P_{ik} U_n^+ = P_{ik} \quad \text{for all } P_{ik}. \tag{2.8}$$

Indeed, in this case $P_{ik} E_n' + E_n' P_{ik} = U_n \{P_{ik} E_n + E_n P_{ik}\} U_n^+ = 0$ and $E_n' = U_n E_n U_n^+$ shares the property (2.4) with E_n.

The group $G_n^S(U)$ does not exhaust all transformations which leave properties (2.1) and (2.4) invariant. Take, e.g., the group of all transformations consisting of products $P_{i_1 k_1} P_{i_3 k_2} \ldots P_{i_m k_m}$ of inversions. Since P_{ik} is unitary, these products are unitary and, therefore, leave (2.1) invariant. They also leave property (2.4) invariant because $E_n' = P_{ik} E_n P_{ik} = -E_n$.

One easily checks up, however, that they do not satisfy the relations (2.8).

A subgroup of $G_n^S(U)$ is the group $G_n(U^n) \subset G_n^S(U)$ of all those unitary transformations u^n in $\mathscr{H}_n$ which are induced by an arbitrary unitary transformation u in h. (By an induced transformation u^n in $\mathscr{H}_n$ we understand here the tensor product of the transformations u in h.)

After these preparations let us decompose $\overset{n}{\underset{1}{\otimes}} h = \mathscr{H}_n$ into the irreducible parts with respect to the symmetry group S_n : $\mathscr{H}_n = \sum_Y Y_n \mathscr{H}_n$, and represent each of these subspaces in the form of an orthogonal sum

$$Y_n \mathscr{H}_n = Y_n \overset{1}{\mathscr{H}}_n \oplus Y_n \overset{2}{\mathscr{H}}_n,$$

where

$$Y_n \overset{1}{\mathscr{H}}_n = Y_n E_n^2 \mathscr{H}_n \quad \text{and} \quad Y_n \overset{2}{\mathscr{H}}_n = Y_n (1 - E_n^2) \mathscr{H}_n.$$

This decomposition

$$\mathscr{H}_n = \sum_Y Y_n \overset{1}{\mathscr{H}}_n \oplus \sum_Y Y_n \overset{2}{\mathscr{H}}_n \tag{2.9}$$

is independent of the order of projections, due to the fact that $E_n^2 Y_n = Y_n E_n^2$, which is a consequence of (2.7). The decomposition (2.9) clearly exposes the action of the operator E_n on the various subspaces. Indeed, in virtue of (2.3) and (2.7), we have

$$E_n Y_n \overset{1}{\mathscr{H}}_n = Y_n^{\mathrm{d}} \overset{1}{\mathscr{H}}_n \tag{2.10}$$

which shows that E_n is an automorphism of $\overset{1}{\mathscr{H}}_n$ consisting of the isomorphisms $Y_n^{\mathrm{d}} \overset{1}{\mathscr{H}}_n \leftrightarrow \leftrightarrow Y_n \overset{1}{\mathscr{H}}_n$.

Let $f_n^{Y_i} \in Y_n \overset{i}{\mathscr{H}}_n$ $(i = 1, 2)$. Due to the structure of the automorphism E_n (2.10) and to the fact that E_n^2 acts in $\overset{1}{\mathscr{H}}_n$ as the unit operator, we have the relations

$$E_n f_n^{Y1} = f_n^{Y^{\mathrm{d}}1}, \quad E_n^2 f_n^{Y1} = f_n^{Y1}. \tag{2.11}$$

3. Examples

In this section we wish to describe examples of operators satisfying the conditions (2.1) and (2.4). For this purpose we assume that h is separable and introduce the orthonormal set $\{e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_n}\}$ in $\mathscr{H}_n$ corresponding to the orthonormal set $\{e_i\}$ in h.

The operator

$$E_n = \sum_{i_1} \ldots \sum_{i_n} e_{i_1} \otimes \ldots \otimes e_{i_n} \varepsilon_{i_1 \ldots i_n} \bar{e}_{i_1} \otimes \ldots \otimes \bar{e}_{i_n} \tag{3.1}$$

defined by the eigenfunctions $\{e_{i_1} \otimes \ldots \otimes e_{i_n}\}$ and the eigenvalues $\varepsilon_{i_1 \ldots i_n}$ satisfies conditions (2.1) and (2.4) if $\varepsilon_{i_1 \ldots i_n}$ is the totally antisymmetric Levi-Civitta tensor with $\varepsilon_{i_1 \ldots i_n}^2 = 1$ for $i_r \neq i_s$ $(r, s = 1, 2, \ldots, n; r \neq s)$ and $\varepsilon_{i_1 \ldots i_n} = 0$ otherwise. The simple proof of this statement is left to the reader.

Due to invariance of the properties (2.1) and (2.4) with respect to $G_n^{\mathrm{S}}(U)$ we obtain from (3.1) a class of operators satisfying the conditions (2.1) and (2.4) by applying to (3.1) an arbitrary unitary transformation $U_n \in G_n^{\mathrm{S}}(U)$: $E_n(U) = U_n E_n U_n^+$.

For this class it is easy to prove the important property

$$E_n^2 A_n \mathscr{H}_n = A_n \mathscr{H}_n,$$

where A_n is the totally antisymmetric Young operator with the dual $A_n^{\mathrm{d}} = S_n$ (cf. (2.6)).

Proof: The proof is based on the symmetry property of the orthonormal system $e_{i_1} \otimes \otimes e_{i_2} \otimes \ldots \otimes e_{i_n}$ consisting in invariance of the product with respect to a simultaneous permutation of the Hilbert spaces and the indices $\{i_1, \ldots, i_n\}$. Due to this symmetry property, the Fourier coefficient $f^{\mathrm{a}}_{i_1 \ldots i_n}$ in the expansion $f^{\mathrm{a}} = \sum_{i_1} \ldots \sum_{i_n} f^{\mathrm{a}}_{i_1 \ldots i_n} e_{i_1} \otimes \ldots \otimes e_{i_n}$ of an antisymmetric function $f_n^{\mathrm{a}} \in A_n \mathscr{H}_n$ is antisymmetric in the indices $\{i_1, \ldots, i_n\}$ and, therefore, vanishes if any two of the indices coincide. The relation $f_n^{\mathrm{a}\prime} = E_n^2 f_n^{\mathrm{a}}$ reads, in terms of Fourier coefficients, $f^{\mathrm{a}\prime}_{i_1 \ldots i_n} = \varepsilon^2_{i_1 \ldots i_n} f^{\mathrm{a}}_{i_1 \ldots i_n} = f^{\mathrm{a}}_{i_1 \ldots i_n}$, the last equation resulting from the fact that the zeros of $\varepsilon^2_{i_1 \ldots i_n}$ and $f^{\mathrm{a}}_{i_1 \ldots i_n}$ coincide. It follows that for all $f_n^{\mathrm{a}} \in A_n \mathscr{H}_n$, $f_n^{\mathrm{a}\prime} = E_n^2 f_n^{\mathrm{a}} = f_n^{\mathrm{a}}$, and, therefore, $E_n^2 A_n \mathscr{H}_n = A_n \mathscr{H}_n \subset \overset{1}{\mathscr{H}}_n$. We have used in the expansion of f^{a} the same orthonormal system which is used in the definition of E_n in (3.1). Since the symmetry properties of the orthonormal system are not changed by transformations of $G_n^{\mathrm{S}}(U)$, the property $E_n^2 A_n \mathscr{H}_n = A_n \mathscr{H}_n$ occurs for all $E_n(U)$.

For illustration we write down the kernel of the operator E_n in the case where $h = L^2(\boldsymbol{R}^n)$:

$$E_n(x_1, \ldots, x_n;\ y_1, \ldots, y_n) = \sum_{i_1} \ldots \sum_{i_n} e_{i_1}(x_1) \ldots e_{i_n}(x_n) \varepsilon_{i_1 \ldots i_n} \bar{e}_{i_1}(y_1) \ldots \bar{e}_{i_n}(y_n) \tag{3.2}$$

where $x_i, y_i \in \boldsymbol{R}^n$, $i = 1, 2, \ldots, n$.

Another example is provided by the function $\sigma(x_1, \ldots, x_n)$ discussed, for example, by Friedrichs [4] and Klauder [5] and mentioned briefly in the introduction to this paper. This function is defined as an antisymmetric real valued function of the points $x_1, \ldots, x_n$ possessing the value 0 whenever any two of the points coincide and the absolute value 1 otherwise. It is easily seen that *the operator E_n with the kernel*

$$E_n(x_1, \ldots, x_n;\ y_1, \ldots, y_n) = \sigma(x_1, \ldots, x_n) \delta(x_1 - y_1) \ldots \delta(x_n - y_n) \tag{3.3}$$

satisfies conditions (2.1) *and* (2.4) *and, in addition, that* $A_n \mathscr{H}_n = A_n \overset{1}{\mathscr{H}}_n$.

Proof: The proof of (2.1) and (2.4) is trivial. The last property ($A_n \mathscr{H}_n = A_n \overset{1}{\mathscr{H}}_n$) follows from $\sigma^2(x_1, \ldots, x_n) f_n^{\mathrm{a}}(x_1, \ldots, x_n) = f_n^{\mathrm{a}}(x_1, \ldots, x_n)$ for all $f_n^{\mathrm{a}} \in A_n \mathscr{H}_n$ and is a consequence of the fact that an antisymmetric element $f_n^{\mathrm{a}}(x_1, \ldots, x_n)$ of $\mathscr{H}_n = \bigotimes_1^n L^2(\boldsymbol{R}^n)$ is zero whenever two of the points $x_1, \ldots, x_n$ coincide and, therefore, the zeros of $f_n^{\mathrm{a}}(x_1, \ldots, x_n)$ coincide with the zeros of $\sigma^2(x_1, \ldots, x_n)$.

The Fourier transform of (3.3) is a tensor

$$\varepsilon_{i_1 \ldots i_n;\, j_1 \ldots j_n} = \int dx_1 \ldots \int dx_n \bar{e}_{i_1}(x_1) \ldots \bar{e}_{i_n}(x_n) \sigma(x_1, \ldots, x_n) e_{j_1}(x_1) \ldots e_{j_n}(x_n) \tag{3.4}$$

with the properties

$$P_{rs} \varepsilon_{i_1 \ldots i_n;\, j_1 \ldots j_n} = -\varepsilon_{i_1 \ldots i_n;\, j_1 \ldots j_n} P_{rs}. \tag{3.5}$$

Equations (3.4) and (3.5) show that the relation between the two representations (3.3) and (3.1) is of dual nature and can be considered as an interchange of the discrete and the continuous indices.

The examples discussed here show that there are many realizations of the relations (2.1), (2.4). One source of this abundance is the indetermination of the definition of the antisymmetric tensor $\varepsilon_{i_1 \dots i_n}$. We can reduce this indetermination to some degree by defining $\varepsilon_{i_1 \dots i_n}$ in terms of a single second order antisymmetric tensor ε_{ik} ($\varepsilon_{ik}^2 = 1$ for $i \neq k$, $\varepsilon_{ik} = -\varepsilon_{ki}$).

Indeed, one easily shows that

$$\varepsilon_{i_1 \dots i_n} = \prod_{s=2}^{n} \prod_{r=1}^{s-1} \varepsilon_{i_r i_s} \tag{3.6}$$

possesses all the required properties.

One can rewrite this expression in terms of E_2 if one considers E_n as an operator in $\mathscr{H}_n$ with $m > n$, according to formulas of the type

$$E_n(x_1, \dots, x_m;\ y_1, \dots, y_m) = E_n(x_{i_1}, \dots, x_{i_n};\ y_{i_1}, \dots, y_{i_n})\, \delta(x_{i_{n+1}} - y_{i_{n+1}}) \dots \delta(x_{i_m} - y_{i_m}). \tag{3.7}$$

Defining the various n-dimensional extensions of the second order operator $E_2(x_1, x_2, y_1, y_2)$ as

$$\begin{aligned} E_{rs}(x_1, \dots, x_n;\ y_1, \dots, y_n) &= \prod_{i=1}^{n}{}''(r, s)\, \delta(x_i - y_i)\, E_2(x_r, x_s;\ y_r, y_s) \\ &= \sum_{i_1} \dots \sum_{i_n} e_{i_1}(x_1) \dots e_{i_n}(x_n)\, \varepsilon_{i_r i_s}\, \bar{e}_{i_1}(y_1) \dots \bar{e}_{i_n}(y_n), \end{aligned} \tag{3.8}$$

where $\prod_{i=1}^{n}{}''(r, s)$ denotes the omission of the factors r and s in the product, we obtain for (3.6)

$$E_n = \prod_{s=2}^{n} \prod_{r=1}^{s-1} E_{rs}. \tag{3.9}$$

One easily shows this representation of E_n is invariant with respect to $G_n(u^n)$ in the sense that $E_n(u^n) = \prod_{s=2}^{n} \prod_{r=1}^{s-1} E_{rs}(u^n)$ and $E_{rs}(u^n)$ is again of type (3.8) with E_2 replaced by $E_2(u^2)$.

3. Generating functionals for antisymmetric functions

In this section we shall construct generating functionals for antisymmetric functions which permit to apply the theory of Hilbert spaces of functional power series to series containing antisymmetric coefficients. We shall also derive some estimates for these series.

Let $\langle x, y \rangle$ be the scalar product defining h and let us assume that there exists an involution $x \to \bar{x}$ in h satisfying $\bar{\bar{x}} = x$ and $\overline{\langle x, y \rangle} = \langle \bar{x}, \bar{y} \rangle$.

We can define in this case a bilinear form in h as $(x, y) := \langle x, \bar{y} \rangle$. The scalar product, bilinear form and involution in h induce the corresponding notions in $\mathscr{H}_n = \overset{n}{\underset{1}{\otimes}} h$:

$$\langle f_n, g_n\rangle, \quad (f_n, g_n) := \langle f_n, \bar{g}_n\rangle, \quad f_n \to \bar{f}_n$$
$$(\bar{\bar{f}}_n = f_n, \ \overline{\langle f_n, g_n\rangle} = \langle \bar{f}_n, \bar{g}_n\rangle; \ f_n, g_n \in \mathscr{H}_n).$$

Consider now the space $\mathscr{F}$ of sequences $\{f_n\}_{n\in N}$ $(f_n \in \mathscr{H}_n, n=0, 1, \ldots; \mathscr{H}_0 = \boldsymbol{C})$ satisfying the condition

$$\|f\| := \sqrt{\sum_n \|f_n\|^2} < \infty,$$

where $\|f_n\| = \sqrt{\langle f_n, f_n\rangle}$;

$$\mathscr{F} = \{f = \{f_n\}_{n\in N}: f_n \in \mathscr{H}_n, \ \|f\| < \infty\} = \bigoplus_{n=0}^{\infty} \mathscr{H}_n. \tag{4.1}$$

Decomposing each $\mathscr{H}_n$ according to (2.9), we obtain the corresponding decomposition of $\mathscr{F}$. We shall be interested, however, in this paper, only in those parts of $\mathscr{H}_n$ which correspond to the symmetric and antisymmetric Young's idempotents:

$$\begin{aligned} \mathscr{F}^{\mathrm{S}} &= \bigoplus_{n=0}^{\infty} \mathrm{S}\mathscr{H}_n = \bigoplus_{n=0}^{\infty} \{\mathrm{S}\overset{1}{\mathscr{H}}_n \oplus \mathrm{S}\overset{2}{\mathscr{H}}_n\} = \overset{1}{\mathscr{F}}{}^{\mathrm{S}} \oplus \overset{2}{\mathscr{F}}{}^{\mathrm{S}}. \\ \mathscr{F}^{\mathrm{A}} &= \bigoplus_{n=0}^{\infty} \mathrm{A}\mathscr{H}_n = \bigoplus_{n=0}^{\infty} \{\mathrm{A}\overset{1}{\mathscr{H}}_n \oplus \mathrm{A}\overset{2}{\mathscr{H}}_n\} = \overset{1}{\mathscr{F}}{}^{\mathrm{A}} \oplus \overset{2}{\mathscr{F}}{}^{\mathrm{A}}, \end{aligned} \tag{4.2}$$

where

$$\overset{i}{\mathscr{F}}{}^{\mathrm{S}} = \bigoplus_{n=0}^{\infty} \mathrm{S}\overset{i}{\mathscr{H}}_n, \quad \overset{i}{\mathscr{F}}{}^{\mathrm{A}} = \bigoplus_{n=0}^{\infty} \mathrm{A}\overset{i}{\mathscr{H}}_n \quad (i=1, 2).$$

Let us consider now the corresponding Hilbert spaces of functional power series (generating functionals) which are defined as mappings $f\colon h \ni x \to f(x) \in \boldsymbol{C}$ in the following way:

$$\mathscr{H}^{\mathrm{S}} = \left\{ f^{\mathrm{S}}: f^{\mathrm{S}}(x) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} (f_n^{\mathrm{S}}, x^n), \ \|f^{\mathrm{S}}\| < \infty \right\},$$

$$\mathscr{H}^{\mathrm{A}} = \left\{ f^{\mathrm{A}}: f^{\mathrm{A}}(x) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} (E_n f_n^{\mathrm{A}}, x^n), \ \|f^{\mathrm{A}}\| < \infty \right\},$$

where

$$\|f^{\mathrm{S}}\|^2 := \sum_{n=0}^{\infty} \|f_n^{\mathrm{S}}\|^2 \quad \text{and} \quad \|f^{\mathrm{A}}\|^2 := \sum_{n=0}^{\infty} \|f_n^{\mathrm{A}1}\|^2. \tag{4.3}$$

The first definition is the conventional one in the symmetric case, the second definition is the generalization to the antisymmetric case. Due to the occurrence of E_n and to the property $E_n f_n^2 = 0$ (cf. (2.3)), only antisymmetric functions from $\overset{1}{\mathscr{H}}_n : f_n^{\mathrm{A}1} \in A_n \overset{1}{\mathscr{H}}_n$ occur in (4.3.)

This is not a serious restriction since we can always choose E_n in such a way (Section 3) that $A_n\overset{1}{\mathscr{H}}_n = A_n\mathscr{H}_n$. In this case the index "1" can be omitted in the second formula (4.3).

Due to the fact that $f^{\mathrm{A}}(x)$ has the same form as $f^{\mathrm{S}}(x)$ with f_n^{S} replaced by $E_n f_n^{\mathrm{A}}$, we can generalize immediately the results of [7] and [8], concerning symmetric functionals, to the antisymmetric case.

We shall discuss here, briefly, only the questions of analyticity and estimates of the power series (4.3).

Given a series $f(x) = \sum_n \frac{1}{\sqrt{n!}}(f_n, x^n)$, we construct the series

$$(g_m, f^{(m)})(x) := \sum_n \frac{\sqrt{(m+n)!}}{n!}(f_{m+n}, g_m x^n) \tag{4.4}$$

obtained from $f(x)$ by differentiating each term in this series m times according to the formula

$$\left(g, \frac{d}{dx}\right) f(x) = \lim_{\varepsilon\to 0} \varepsilon^{-1}\{f(x+\varepsilon g) - f(x)\}, \tag{4.5}$$

where $x, g \in h, \varepsilon \in \boldsymbol{R}$.

Applying Schwarz's inequality several times to (4.4), one obtains the estimate

$$|(g_m, f^{(m)})(x)| \leqslant \|g_m\| \{\|f\|^2 - \sum_{i=0}^{m-1} \|f_i\|^2\}^{\frac{1}{2}} \sqrt{L_m(-\|x\|^2)}\, e^{\frac{1}{2}\|x\|^2}. \tag{4.6}$$

It is seen from (4.6) that the series (4.4) represents entire functionals (in the sense of [6]) of growth $\rho \leqslant 2$ and type $\sigma(\rho=2) \leqslant \frac{1}{2}$ if only $g_m \in \mathscr{H}_m$ and $f \in \mathscr{H}^{\mathrm{S}}$. Due to this property, the sum in (4.4) is equal to the mth derivative of $f(x)$ so that we can interchange summation and differentiation

$$\left(g_m, \frac{d^m}{dx^m}\right) f(x) = (g_m, f^{(m)})(x). \tag{4.7}$$

Conversely, it can be shown that if $|f(x)| \leqslant A e^{\frac{1}{2}\|\lambda x\|^2}$, where λ is a Hilbert–Schmidt operator with norm $\|\lambda\| < 1$, then $f \in \mathscr{H}^{\mathrm{S}}$.

Due to the isomorphisms (2.11), we have the isomorphism of $\mathscr{H}^{\mathrm{A}}$ and $\overset{1}{\mathscr{H}}{}^{\mathrm{S}}$, where

$$\overset{1}{\mathscr{H}}{}^{\mathrm{S}} = \left\{ f^{\mathrm{S}1} : f^{\mathrm{S}1}(x) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} (f_n^{\mathrm{S}1}, x^n), \|f^{\mathrm{S}1}\| < \infty \right\}. \tag{4.8}$$

Thus $\mathscr{H}^{\mathrm{A}} \subset \mathscr{H}^{\mathrm{S}}$ and we can specialize the above results to the antisymmetric case by putting everywhere $f_n^{\mathrm{S}1} = E_n f_n^{\mathrm{A}1}$.

The operator E_n can be omitted in all scalar products due to the properties (2.11) which make $\langle f_n, g_n\rangle$ invariant with respect to E_n if at least one of the elements in $\langle f_n, g_n\rangle$ belongs to $\overset{1}{\mathscr{H}}_n$. Indeed, $f_n' = E_n f_n$, $g_n' = E_n g_n$, $f_n \in \overset{1}{\mathscr{H}}_n \Rightarrow \langle f_n', g_n'\rangle = (\bar{g}_n, E_n^2 f_n) = \langle f_n, g_n\rangle$.

Applying this result to symmetric and antisymmetric functions, we obtain

$$\langle E_n f_n^{A1}, E_n g_n^{A1}\rangle = \langle f_n^{\overset{1}{A}1}, g_n^{A1}\rangle, \qquad \langle E_n f_n^{S1}, E_n g_n^{S1}\rangle = \langle f_n^{\overset{1}{S}1}, g_n^{S1}\rangle \tag{4.9}$$

for all f_n^{A1}, $q_n^{A1} \in A_n \mathscr{H}_n$ and for all f_1^{S1}, $q_n^{S1} \in S_n \mathscr{H}_n$. In particular, if $A_n \mathscr{H}_n = E_n^2 A_n \mathscr{H}_n$ (cf. Section 3), one can omit the upper index 1 in the first equation (4.9).

This shows that the theory of Hilbert spaces of functional power series, as developed in [7] and [8], can be applied to the antisymmetric case, all scalar products containing symmetric functions being replaced by the corresponding scalar products of antisymmetric functions.

It must be noted here that in applications to quantum field theory the coefficients f_n are, in general, not elements of $\mathscr{H}_n$ but belong to a larger space of linear continuous functionals on some linear subspace of $\mathscr{H}_n$, dense in $\mathscr{H}_n$.

The extension of the operators E_n to such dual systems is straightforward. One can use here dual systems of the type introduced in [7] and [8]. One has to keep in mind, however, that, in the case where $f_n \notin \mathscr{H}_n$, the proof of Section 3 concerning the equalities $E_n^2 A_n f_n = A_n f_n$, $(1-E_n^2)A_n f_n = 0$ does not hold any more. These and related problems will be discussed separately.

5. Functional representation of the anticommutation relations

Having thus introduced the Hilbert spaces of functional power series in the symmetric as well as in the antisymmetric cases, we can consider double series as operators acting on these spaces according to the rule

$$f'(x) = (Af)(x) = A\left(x, \frac{d}{dy}\right) f(y)\Big|_{y=0} = \int A(x, \bar{y}) f(y) e^{-\|y\|^2} d(y/\sqrt{\pi}) \tag{5.1}$$

or, in terms of coefficients,

$$f_n' = \sum_m (q_{nm}, f_m), \tag{5.2}$$

where

$$A(x, y) = \sum_n \sum_m \frac{1}{\sqrt{n!\,m!}} (a_{nm}, x^n y^m), \tag{5.3}$$

$x, y \in h$, $a_{nm} \in \mathscr{H}_n \otimes \mathscr{H}_m$, and the differential and integral representations of multiplication in (5.1) are discussed in more detail in [7] and [8] (cf. also [6], Chapter 2).

It is well known that the functional representation of creation and annihilation operators in terms of functional double power series is given in the symmetric case (commutators) by the formulae (cf. [7], [8])

$$a^+(x, y; \xi) = e^{(x, y)}(x, \xi), \qquad a(x, y; \xi) = e^{(x, y)}(\xi, y), \qquad (\xi \in h). \tag{5.4}$$

Taking into account the multiplication prescription

$$(AB)(x, y) = A\left(x, \frac{d}{dz}\right) B(z, y)\Big|_{z=0} \tag{5.5}$$

which is a consequence of (5.1), we easily verify the relations

$$[a(\xi), a^+(\eta)]_- = (\xi, \eta)\mathbf{1} \quad \text{and} \quad a(\xi) f_0 = 0, \tag{5.6}$$

where $\mathbf{1}(x, y) = e^{(x,y)}$ is the unit operator in $\mathscr{H}^S$ and $f_0 \in C$ ($|f_0| = 1$) is the vacuum vector in $\mathscr{H}^S$.

We shall prove now that *in the antisymmetric case* (*anticommutators*) *the functional representation of creation and annihilation operators has the form of the following double series*:

$$b^+(x, y; \xi) = \sum_n \frac{1}{n!}(x^{1+n} E_{1+n}, \xi E_n y^n), \qquad b(x, y; \xi) = \sum_n \frac{1}{n!}(\xi x^n E_n, E_{1+n} y^{1+n}), \tag{5.7}$$

where E_n is assumed to satisfy the relation $E_n^2 A_n \mathscr{H}_n = A_n \mathscr{H}_n$ for all n.

Proof: For the proof it is convenient to write the double series (5.7) as bilinear forms

$$b^+(x, y; \xi) = (b^+(x, y), \xi) \quad \text{and} \quad b(x, y; \xi) = (\xi, b(x, y))$$

where

$$b^+(x, y) = \sum_n \frac{1}{n!}(x^{1+n} E_{1+n}, E_n y^n), \qquad b(x, y) = \sum_n \frac{1}{n!}(x^n E_n, E_{1+n} y^{1+n}), \tag{5.7'}$$

are elements of h in contradistinction to (5.7) where $b^+(x, y; \xi)$ and $b(x, y; \xi)$ are elements of C. To prove the anticommutation relation we need only to calculate the products bb, b^+b^+, bb^+ and b^+b. However, for further applications, we shall proceed more generally. Applying k times (5.5), one obtains

$$\begin{aligned} (f_k, b^k)(x, y) &= \sum_n \frac{1}{n!}(f_{\tilde{k}} x^n E_n, E_{k+n} y^{k+n}), \\ (f_k, b^{+k})(x, y) &= \sum_n \frac{1}{n!}(x^{k+n} E_{k+n}, f_k E_n y^n), \\ ((f_k, b^{+k})(g_i, b^i))(x, y) &= \sum_n \frac{1}{n!}(x^{k+n} E_{k+n}\ f_k g_{\tilde{i}}\ E_{i+n} y^{i+n}), \end{aligned} \tag{5.8}$$

where $g_{\tilde{i}}$ denotes the element g_i with inversed order of indices. It remains to calculate bb^+. We have

$$\begin{aligned} b\left(x, \frac{d}{dz}; \xi\right) b^+(z, y; \eta)\Big|_{z=0} &= \sum_n \frac{1}{n!}\left(\xi x^n E_n, E_{1+n} \frac{d^{1+n}}{dz^{1+n}}\right) \sum_m \frac{1}{m!}(z^{1+m} E_{1+m}, \eta E_m y^m)\Big|_{z=0} \\ &= \sum_n \frac{n+1}{n!}(\xi x^n E_n A_{1+n}, E_{1+n}^2 \eta E_n y^n) = \sum_n \frac{n+1}{n!}(\xi x^n E_n A_{1+n}, \eta E_n y^n) \\ &= (\xi, \eta) \sum_n \frac{1}{n!}(x^n, E_n^2 y^n) - \sum_n \frac{1}{n!}(x^{1+n} E_{1+n}, \eta \xi E_{1+n} y^{1+n}). \end{aligned} \tag{5.9}$$

Specializing the third formula (5.8) for $i=k=1$, we obtain

$$b^+\left(x, \frac{d}{dz}; \eta\right) b(z, y; \xi)|_{z=0} = \sum_n \frac{1}{n!}(x^{1+n}E_{1+n}, \eta\xi E_{1+n} y^{1+n}). \tag{5.10}$$

From (5.9), (5.10) and the first two equations (5.8) specialized for $k=2$ we obtain immediately

$$[b^+(\xi), b^+(\eta)]_+ = [b(\xi), b(\eta)]_+ = 0, \qquad [b(\xi), b^+(\eta)]_+ = (\xi, \eta)E^2 \tag{5.10'}$$

where

$$E^2(x, y) = \sum_n \frac{1}{n!}(x^n, E_n^2 y^n) \tag{5.11}$$

is the projector on $\overset{1}{\mathscr{H}}{}^S$. For $f^S(x) = \sum_n \frac{1}{\sqrt{n!}}(f_n^S, x^n)$ we have namely

$$f^{S1}(x) = (E^2 f^S)(x) = \sum_n \frac{1}{\sqrt{n!}}(E_n^2 f_n^S, x^n) = \sum_n \frac{1}{\sqrt{n!}}(f_n^{S1}, x^n) = \sum_n \frac{1}{\sqrt{n!}}(E_n f_n^{A1}, x^n). \tag{5.12}$$

According to the property $E_n^3 = E_n$, E^2 acts like a unit operator in $\mathscr{H}^A$.

Vacuum is represented as in the symmetric case by $f_0 \in C$ ($|f_0| = 1$)

$$b(\xi)f_0 = 0. \tag{5.13}$$

A formulation of quantum field theory on mass shell in terms of functionals can be obtained in the symmetric case (cf. e.g. [6]) by first expressing all operators as series of normal products of creation and annihilation operators and then using the functional representation (5.4). We proceed similarly in the antisymmetric case. We obtain in this way for an arbitrary operator (use (5.7) and third formula (5.8))

$$:\mathscr{A}(b^+, b):(x, y) := \sum_n \sum_m \frac{1}{\sqrt{n!\,m!}}(\mathfrak{a}_{nm}, b^{+n}b^m)(x, y)$$

$$= \sum_n \sum_m \frac{1}{\sqrt{n!\,m!}} \sum_k \frac{1}{k!}(x^{n+k}E_{n+k}, \mathfrak{a}_{n\tilde{m}} E_{m+k} y^{m+k}), \tag{5.14}$$

where $\tilde{m}$ indicates that the order of the corresponding indices has been inversed.

6. Generating functionals off mass shell in the antisymmetric case

To describe the functional formulation of quantum field theory off mass shell in the antisymmetric case we shall have to go over from the general considerations of Sections 2, 4, 5 to a particular representation. Here we shall carry out the considerations for the (unrealistic) case of scalar real fields obeying Fermi statistics. (An application to $\frac{1}{2}$-spin representations will be described elsewhere.) We have thus creation and annihilation operators being functions of momentum $\vec{k} \in E_3$ only.

Let us note that every operator can be expressed as a single power series of normal products of the free fields

$$:\Omega(q_0(b^+, b)): = \sum_n \frac{1}{n!} (f_n^A, :q_0^n(b^+, b):), \tag{6.1}$$

where

$$q_0(b^+, b; x) = \int d\vec{k} \{\bar{f}(\vec{k}, x) b^+(\vec{k}) + f(\vec{k}, x) b(\vec{k})\}, \tag{6.2}$$

and

$$f(\vec{k}, x) = \frac{1}{(2\pi)^{3/2}} \frac{(\exp(i\vec{k}\vec{x} - i\omega(\vec{k})x_0)}{\sqrt{2\omega(\vec{k})}}, \quad \omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}, \tag{6.3}$$

($x \in M_4$) is the orthogonal set of solutions of the Klein-Gordon equation. Introducing in (6.2) the functional representation (5.7) (5.7′) and replacing the abstract vectors $x, y \in h$ of (5.7) by the functions $\alpha(\vec{k})$, $\beta(\vec{k})$ ($\alpha, \beta \in L^2(E_3)$), we obtain the functional representation of the free field

$$q_0(b^+, b; x)(\alpha, \beta) = \int d\vec{k} \{\bar{f}(\vec{k}, x) b^+(\alpha, \beta; \vec{k}) + f(\vec{k}, x) b(\alpha, \beta; \vec{k})\}$$
$$= \int d\vec{k} \sum_n \frac{1}{n!} \{(\alpha^{1+n} E_{1+n}, \bar{f}(x) E_n \beta^n) + (f(x)\alpha^n E_n, E_{1+n}\beta^{1+n}). \tag{6.4}$$

To obtain the functional representation of (6.1), we first rearrange the sum

$$:\Omega(q_0(b^+, b)): = \sum_n \sum_m \frac{1}{n!\,m!} ((f_{n+m}^A, \bar{f}^n f^m), b^{+n} b^m) \tag{6.5}$$

and then use formula (5.14) with

$$a_{nm} = \frac{1}{\sqrt{n!\,m!}} (f_{n+m}^a, \bar{f}^n f^m).$$

The final result is

$$:\Omega(q_0(b^+, b)): (\alpha, \beta) = \sum_n \frac{1}{n!} (f_n^A, \varphi_n(\alpha, \beta)), \tag{6.6}$$

where

$$\varphi_n(\alpha, \beta) = A_n \sum_l \frac{1}{l!} \sum_{m=0}^{n} \binom{n}{m} ((\alpha^{n-m+l} E_{n-m+l}, \bar{f}^{n-m}), (f^{\tilde{m}}, E_{m+l}\beta^{m+l})). \tag{6.7}$$

For illustration and to explain the notation used throughout this paper we write down formula (6.7) also in the explicit form in the representation considered in this section:

$$\varphi_n(\alpha, \beta; x_1, \ldots, x_n)$$

$$=A_n \sum_l \frac{1}{l!} \sum_{m=0}^{n} \binom{n}{m} \int d\vec{k}_1 \ldots \int d\vec{k}_{n-m+l} \int d\vec{p}_1 \ldots \int d\vec{p}_{n-m} \int d\vec{r}_{n-m+l} \ldots \int d\vec{r}_n \times$$

$$\times \int d\vec{p}_{n-m+1} \ldots \int d\vec{p}_{n-m+l} \int d\vec{s}_1 \ldots \int d\vec{s}_{m+l} \times$$

$$\times \alpha(\vec{k}_1) \ldots \alpha(\vec{k}_{n-m+l}) E_{n-m+l}(\vec{k}_1, \ldots, \vec{k}_{n-m+l}; \vec{p}_1, \ldots, \vec{p}_{n-m+l}) \prod_{i=1}^{n-m} \bar{f}(\vec{p}_i, x_i) \times$$

$$\times \prod_{i=n-m+1}^{n} f(\tilde{r}_i; x_i) E_{m+l}(\vec{r}_{n-m+1}, \ldots, \vec{r}_n, \vec{p}_{n-m+1}, \ldots, \vec{p}_{n-m+l}; \vec{s}_1, \ldots, \vec{s}_{m+l}) \times$$

$$\times \beta(\vec{s}_1) \ldots \beta(\vec{s}_{m+l}), \tag{6.8}$$

where $\tilde{r}_i$ indicates that the sequence $\vec{r}_{n-m+1}, \ldots, \vec{r}_n$ is taken in the reversed order $\vec{r}_n, \ldots, \vec{r}_{n-m+1}$ and corresponds to the reversed order in $f^{\widetilde{m}}$ in formula (6.7). The Young antisymmetrizer A acts on the variables $x_1, \ldots, x_n$.

It is seen from (6.6) that transition to the mass shell corresponds in the antisymmetric case to replacing in the generating functional

$$\Omega^a(q) = \sum_n \frac{1}{n!} (E_n f_n^{\mathrm{A}}, q^n) = \sum_n \frac{1}{n!} (f_n^{\mathrm{A}}, \tilde{E}_n^{\dagger} q^n) \tag{6.9}$$

the function $\tilde{E}_n q^n$ by $\varphi_n(\alpha, \beta)$. This is the analogue of the transition $q^n \to e^{(\alpha,\beta)} q_0(\alpha, \beta)$ in the symmetric case $\Omega^{\mathrm{S}}(q) = \sum_n \frac{1}{n!} (f_n^{\mathrm{S}}, q^n)$ and provides a functional formulation of the so-called *reduction formula* [4].

The formulae derived in this and the preceding section enable to express, step by step, the content of quantum field theory in the case of Fermi statistics in terms of generating functionals. We shall carry out the corresponding programm for the more realistic case of $\frac{1}{2}$-spin representation in a subsequent paper.

REFERENCES

[1] Berezin, F. A., *The method of second quantization* (in Russian), Moscow, 1965.

[2] Friedrichs, K. O., *Mathematical aspects of the quantum theory of fields*, Interscience, New York, 1953.

[3] Klauder, J. R., Annals of Phys. **11** (1960), 123.

[4] Lehmann, H., K. Symanzik, K. Zimmermann, Nuovo Cim. **1** (1955), 205.

[5] Matthews, P. T., and A. Salam, *ibid.* [10] **2** (1955), 120.

[6] Rzewuski, J., *Field theory*, Part II, *Functional formulation of S-matrix theory*, Iliffe Books Ltd. London, PWN, Warsaw, 1969.

[7] –, Reports Math. Phys. **1** (1970), 1.

[8] –, *ibid.* **1** (1971), 195.