THE METHOD OF BOSON EXPANSIONS IN QUANTUM THEORY

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Abstract:

We give a review of Boson expansion methods applied in the quantum theory as e.g. expansions of spin, bifermion and Fermion operators in cases of the finite and infinite number of degrees of freedom. The basic purpose of the paper is to formulate the most general criterions allowing to get so called finite spin approximation of any given Bose field theory, and the class of Fermion theories associated with it. Quite the converse, we need also to be able to reconstruct the primary Bose field theory while any finite spin or Fermi systems are given.

0. Introduction

Within last twenty years, the method of Boson expansions has proved useful in the quantum theory of many-Fermion systems. For example it allowed to build a contemporary theory of spin waves in the low temperature description of the Heisenberg ferromagnet [1–9, 15], where for long time it was known that the ideal magnon gas perfectly simulates the behaviour of the Heisenberg crystal itself. A similar situation appears in the study of the weak excitation limit of atomic nuclei in the microscopic model, where the spectra of low lying excited states are similar to these of the excited system of weakly coupled quadrupole Bosons [7, 18–26].

On the other hand recent developments on the connection between the Thirring and Sine–Gordon systems in the two space-time dimensions, resulted in a couple of papers connected with the Boson expansion methods in quantum field theory, see e.g. [38, 44–46, 105–128], but also [16, 17, 27–35, 40–42, 76].

All the approaches mentioned above were developed independently and as applications to quite different physical phenomena. Therefore, there appears the fascinating problem of considering Boson expansion methods in the quantum theory of Fermion systems from a unified point of view, i.e., to establish whether there exist any global physical conditions under which this method admits concrete applications, governing all Fermion systems, independent of the number of degrees of freedom in the theory. Here there appears also the question of the mutual relations between the different approaches proposed so far.

We include here the new results following from the construction of representations of the canonical anticommutation relations algebra in the Fock representation of the canonical commutation relations algebra. (Throughout the paper we use a shorthand notation CAR and CCR respectively while using these representations.)

We claim that the physical essence of the Boson expansion methods reads:

Any quantum Boson in the weak excitation limit can exhibit Fermion properties, which then prevail the original Boson ones. We call it a Fermion-like behaviour. This is the reason for which Boson systems can be in sufficiently low temperatures used to approximate properties of Fermion systems. Quite conversely, if the higher excitations, as, e.g., the weak coupling limit of the theory, are admitted, then the starting Fermion system can exhibit the Boson properties. We call it a Boson-like behaviour.

The above statement is obviously not true for isolated systems, but if the contact with a suitable environment is taken into account, then in quantum mechanics, many-body theory and quantum field theory one has justified quite serious treatment of the question of metamorphosis of Fermions into Bosons, and conversely, see e.g. [44].

Obviously such a metamorphosis can not always appear for the case when both the starting Boson and the final Fermion are physical objects. It may happen (especially in quantum field theory where the spin-statistics theorem should be taken into account if the number of space-time dimensions is equal to four) that the starting point is an ideal nonphysical, ghost Boson, whose weak excitation limit acquires the properties of a physical Fermion.

From the mathematical point of view, the majority of essential results is based on the Boson expansion theory developed by the present author [16, 17, 27–35, 76], but in the course of the paper we give a review of the related topics which seem essential for better understanding of the method. Our statement on the Fermion–Boson reciprocity is formulated on the basis of the equivalence theorems proved in [17] for the example of Heisenberg ferromagnet, and in what follows for the microscopic model of the atomic nuclei. They result from the projection theorems proved in [76], and collected in the Appendix.

In the case of the Heisenberg lattice it means that its Hamiltonian H can be received as a reduction $P_0 H_B P_0 = H$ to the Hilbert space of spin states of a suitable pure Boson (magnon gas) Hamiltonian, where P_0 projects onto the spin space in the Boson Fock space $\mathcal{F}_B$: $P_0 \mathcal{F}_B = \mathcal{F}_0$. A similar situation appears in the case of the atomic nuclei.

In application to quantum field theory, we were able to prove that with each normal ordered operator series $:\Omega(\psi, \bar{\psi}):$ of free asymptotic Dirac fields $\psi, \bar{\psi}$ one can associate the corresponding functional power series $\Omega^c(\psi^c, \bar{\psi}^c)$ of the classical (commuting ring of functions) spinor fields. The transition from Fermions to classical spinors is realized through the subsidiary (but unphysical as involving the spinor fields which obey Bose statistics) Boson level: $\psi^B, \bar{\psi}^B$. The Boson spinors satisfy the free Dirac equation, and $\mathbf{1}_F{:}\Omega^c(\psi^B, \bar{\psi}^B){:}\mathbf{1}_F \mathcal{F}_F = :\Omega(\psi, \bar{\psi}):\mathcal{F}_F$ what is the equivalence relation between the Bosons and Fermions on the Fermion Fock space. On the Boson level one has the concept of the coherent state expectation value of the operator what realizes a correspondence principle: to the mediating Bosons we have assigned their classical images: $\langle :\Omega^c(\psi^B, \bar{\psi}^B):\rangle = \Omega^c(\psi^c, \bar{\psi}^c)$.

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1. Boson expansions of spin operators

Let H denote a finite dimensional Hilbert space: $\dim H = 2s + 1$ with $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Assume to have given in H an irreducible unitary representation of the SU(2) group whose infinitesimal generators obey the relation: $a, b, c = 1, 2, 3$; $\boldsymbol{S} = \{S_a\}_{a=1,2,3}$

$$[S_a, S_b]_- = i\epsilon_{abc} S_c, \qquad S_\pm = (1/\sqrt{2})(S_1 \pm iS_2) \Rightarrow [S_+, S_-]_- = S_3, \qquad [S_3, S_\pm]_- = \pm S_\pm; \tag{1.1}$$

$$\boldsymbol{S}^2 = \sum_a S_a^2 = S_+ S_- + S_- S_+ + S_3^2 = s(s+1)\mathbf{1}. \tag{1.2}$$

Given further an infinite dimensional Hilbert space $\mathcal{F}$, and a Fock representation of the CCR (canonical commutation relations algebra) in it, realized by the triple $\{a^*, a, \Omega\}$ with:

$$[a, a^*] = \mathbf{1}, \qquad a\Omega = 0. \tag{1.3}$$

We state the following question: find the representation of (1.1) in a certain finite-dimensional subspace of $\mathcal{F}$. By virtue of the irreducibility of the pair $\{a^*, a\}$ we can expect the operator $\boldsymbol{S}$ to be fully expressed in terms of Boson generators.

Let us begin from Holstein–Primakoff [1] solution:

$$\begin{aligned} \boldsymbol{S}^{\mathrm{H}}\colon \sqrt{2}S_+ &= S_x + \mathrm{i}S_y = \sqrt{2s}a^*\sqrt{1-a^*a/2s} \\ \sqrt{2}S_- &= S_x - \mathrm{i}S_y = \sqrt{2s}\sqrt{1-a^*a/2s}\cdot a \\ S_z &= s - a^*a. \end{aligned} \tag{1.4}$$

The square roots are understood formally as infinite series with respect to $a^*a/2s$.

The Hilbert space $\mathcal{F}$ consists here of functions of the occupation number parameter n (the eigenvalue of the operator $\hat{n} = a^*a$).

One can easily check that the operators S_x, S_y, S_z obey the commutation relations (1.1). However the condition of mutual adjointness for S_+, S_- and the selfadjointness for S_z, as well as (1.2) do not hold on the whole of $\mathcal{F}$.

The operators (1.4) leave invariant subspaces of $\mathcal{F}$ consisting of functions which depend either on the occupation number parameter $n = 0, 1, \ldots, 2s$ or $n \geq 2s+1$. The first, $2s+1$ dimensional, subspace we denote H. The orthogonal complement of H in $\mathcal{F}$ is called a nonphysical space, as it involves spin values greater than s.

In practical applications one must restrict considerations to H only, where the functional argument n does not exceed $2s$. However this subsidiary condition is not easy to realize directly. One can use here an indirect method [2]. Namely, let us in the place of Boson operators consider quasi-Boson ones, obeying the relations:

$$\begin{aligned} [a, a^*]_- &= \left(1 - \frac{2s+1}{(2s)!}\cdot a^{*2s}a^{2s}\right) \\ a^{2s+1} &= a^{*2s+1} = 0. \end{aligned} \tag{1.5}$$

Then, by virtue of the relation:

$$a^{*2s+1}a^{2s+1} = \prod_{p=0}^{2s-1}(a^*a - 1) = \prod_{p=0}^{2s-1}(\hat{n} - p) \tag{1.6}$$

and:

$$\hat{n}f(n) = nf(n) \tag{1.7}$$

we get:

$$\prod_{p=0}^{2s-1}(\hat{n} - p)f(n) = \prod_{p=0}^{2s-1}(n - p)f(n) \tag{1.8}$$

which implies that for $n \leq 2s-1$, one of factors appearing on the right-hand side of (1.6), necessarily vanishes.

In consequence, the action of (1.5) onto any function from $\mathcal{F}$ gives a non-zero result only if $f \notin H$. In that case, the quasi-Boson operators differ from Boson operators outside H only. This is the case, when the higher excitations of the quantum system $\{a^*, a, \Omega\}$ can appear with a considerable probability.

The approximation of spin operators by quasi-Bosons becomes better with increasing spin value s.

One can also make use of the power series expansions of (1.4), (compare [3]):

$$\sqrt{2}S_+ = \sqrt{2s}\left(a - \frac{1}{4s}\hat{n}a - \frac{1}{32s^2}\hat{n}^2 a + \cdots\right)$$
$$\sqrt{2}S_- = \sqrt{2s}\left(a^* - \frac{1}{4s}a^*\hat{n} - \frac{1}{32s^2}a^*\hat{n}^2 + \cdots\right) \tag{1.9}$$

and neglecting all terms except a few.

The approximate formulas for the spin operators obtained in this way become good for $s \gg 1$, and are surely not exact for $s = \frac{1}{2}$.

In connection with Dyson's theory of spin waves in a Heisenberg ferromagnet [48, 49], the so called Dyson–Maleev slution [4] was introduced:

$$\boldsymbol{S}^{\mathrm{D}}: \quad \sqrt{2}S_+ = \sqrt{2s}a^*$$
$$\sqrt{2}S_- = \sqrt{2s}\left(1 - \frac{1}{2s}a^*a\right)\cdot a \tag{1.10}$$
$$S_z = s - a^*a.$$

The operators S_+, S_- are here never mutually adjoint inside $\mathcal{F}$ (hence the operator $\boldsymbol{S}$ cannot be Hermitean) though the commutation relations (1.1) obviously hold in $\mathcal{F}$. Moreover, the eigenvalues of S_z are equal to $-s, -s+1, \ldots$ and are not bounded from above, while for a real spin operator we would have the upper bound equal to s.

The difficulties with adjointness and the spectrum of S_z can be removed [5–7], by a suitable choice of the metric in $\mathcal{F}$. Namely, $\mathcal{F}$ is a Hilbert space with respect to a scalar product:

$$\mathcal{F} \ni f, g \Rightarrow (f, g) \in \mathbb{C}, \qquad f_n = \frac{1}{\sqrt{n!}}a^{*n}\Omega \tag{1.11}$$
$$(f_n, f_m) = \delta_{nm}.$$

Let us consider in $\mathcal{F}$ a sesquilinear form:

$$(f, g)_F = (f, Fg) \tag{1.12}$$

where F is a Hermitean operator. We impose on F the additional restriction:

$$(f, \boldsymbol{S}_g^{\mathrm{D}})_F = (\boldsymbol{S}^{\mathrm{D}}f, g)_F \tag{1.13}$$

where $\boldsymbol{S}^{\mathrm{D}}$ is the Dyson–Maleev spin operator.

By virtue of (1.13) we have satisfied in $\mathcal{F}$ the two identities:

$$FS_\pm = S_\mp F, \qquad FS_z = S_z^* F \tag{1.14}$$

which, by taking into account the Boson expansions (1.10), lead to:

$$a^*(1 - a^*a/2s)F = Fa^*$$
$$a^*aF = Fa^*a \tag{1.15}$$

proving that F can be diagonalized in $\mathcal{F}$ together with $\hat{n} = a^*a$. Hence:

$$(1 - \hat{n}/2s)F_n a^* f_n = F_{n+1}a^* f_n \tag{1.16}$$

where:

$$F_n = (f_n, Ff_n) \tag{1.17}$$
$$F_{n+1} = (1 - n/2s)F_n$$
$$F_n = 1 \cdot (1 - 1/2s) \cdots (1 - (n-1)/2s).$$

It is obvious that for non-physical spin values, namely for $n \geq 2s + 1$, the matrix element F_n of F vanishes, because, by virtue of (1.11), (1.12), we have:

$$(f_n, f_m)_F = F_n \delta_{nm}. \tag{1.18}$$

The metric defined by the scalar product $(f, g)_F$ is indefinite: $(f_n, f_n)_F$ vanishes for $n > 2s$. It distinguishes however in $\mathcal{F}$ a $2s + 1$ dimensional, proper subspace H of physical states, on which $(f, g)_F$ defines a Hilbert space topology.

The two discussed solutions $\boldsymbol{S}^{\mathrm{H}}$ and $\boldsymbol{S}^{\mathrm{D}}$ are not fully independent, and a connection between them can be found. Namely [5–9], the square root $F^{1/2}$ of F induces the following identities on H:

$$S_{\pm}^{\mathrm{H}} = F^{1/2} S_{\pm}^{\mathrm{D}} F^{-1/2}$$
$$S_z^{\mathrm{H}} = S_z^{\mathrm{D}} = F^{1/2} S_z^{\mathrm{D}} F^{-1/2}. \tag{1.19}$$

More detailed considerations of this subject can be found in [7–9]. Together with $\boldsymbol{S}^{\mathrm{H}}$ and $\boldsymbol{S}^{\mathrm{D}}$, one can imagine the more general Cooke–Loly solution, given in [10, 11]:

$$\boldsymbol{S}^{\mathrm{C}}: \quad \sqrt{2} S_+ = \sqrt{2s}\, a^* (1 - \hat{n}/2s)^{1-x}$$
$$\sqrt{2} S_- = \sqrt{2s} (1 - \hat{n}/2s)^x \tag{1.20}$$
$$S_z = s - a^* a,$$

where $0 \leq x \leq 1$, and $\boldsymbol{S}^{\mathrm{C}} = \boldsymbol{S}^{\mathrm{D}}$ for $x = 0$ or $x = 1$, while for $x = \frac{1}{2}$ we get $\boldsymbol{S}^{\mathrm{C}} = \boldsymbol{S}^{\mathrm{H}}$.

One can develop a few more approaches, such as, e.g., Schwinger's method of paired Bosons [12, 13], the use of two sets of Bosons, being responsible for the dynamics and the kinematics, respectively of the spinning system [14], as well as the introduction of two sets of Fermions [15]. However all these approaches meet difficulties analogous to these exposed in connection with $\boldsymbol{S}^{\mathrm{H}}$ and $\boldsymbol{S}^{\mathrm{D}}$.

A concrete algebraic realization of the infinitesimal generators of the SU(2) group, does still not guarantee physically correct results, and subsidiary conditions (including a proper definition of the state space and the scalar product in it) are unavoidable.

Recently, in refs. [16, 17], another Boson expansion for spin operators was suggested, giving exact results for spin $\frac{1}{2}$, and which is free of all disadvantages discussed above. We have

$$\boldsymbol{S}^{\mathrm{G}}: \quad \sqrt{2} S_+ = a^* : \exp(-a^* a) :$$
$$\sqrt{2} S_- = : \exp(-a^* a) : a$$
$$S_z = -\tfrac{1}{2} \mathbf{1}_{\mathrm{F}} + a^* : \exp(-a^* a) : a \tag{1.21}$$

where : . : symbolizes the normal ordering of generators a^*, a, and one can equivalently introduce

$$S_z = \tfrac{1}{2} \mathbf{1}_{\mathrm{F}} - a^* : \exp(-a^* a) : a.$$

Here:

$$\mathbf{1}_{\mathrm{F}} = : \exp(-a^* a) : + a^* : \exp(-a^* a) : a \tag{1.22}$$

so that:

$$S_z = -\tfrac{1}{2}[:\exp(-a^*a):-a^*:\exp(-a^*a):a]. \tag{1.23}$$

Because the operator $:\exp(-a^*a):$ projects onto the ground state $\Omega = f_0$ in $\mathscr{F}$, the spin operator components (1.20) make invariant a two-dimensional subspace $H = \mathbf{1}_F\mathscr{F}$ of $\mathscr{F}$, which is spanned by the two basis vectors f_0 and $f_1 = a^*f_0$. In this basis, the matrix realization of the operators (1.21) is exactly given by Pauli matrices.

By virtue of $\boldsymbol{S}^G f = 0$ for $f \notin H$, and $(\boldsymbol{S}^G)^2 = \frac{3}{4}\mathbf{1}_F$ the spin value s is equal $\frac{1}{2}$.

The expansions (1.21) can be written in the form:

$$\begin{aligned}
\sqrt{2}S_+ &= a^* - a^{*2}a + \tfrac{1}{2}a^{*3}a^2 - \cdots \\
\sqrt{2}S_- &= a - a^*a^2 + \tfrac{1}{2}a^{*2}a^3 - \cdots \\
S_z &= -\tfrac{1}{2} + a^*a - \tfrac{3}{2}a^{*2}a^2 + \cdots .
\end{aligned} \tag{1.24}$$

If compared with (1.10), we find at once that the solution $\boldsymbol{S}^D$ in case $s = \frac{1}{2}$ is a particular form of (1.21). To check it, it is enough to neglect higher order terms in the expansions (1.24). In addition one must take $-S_z$ in the place of S_z. Hence, it is not surprising that $\boldsymbol{S}^D$ cannot be an exact Boson expansion for spin $\frac{1}{2}$, and the subsidiary conditions are necessary in that case.

Let us further notice that H is two-dimensional and is spanned by the basis $\{f_0, f_1\}$. If to remember that correct spin $\frac{1}{2}$ commutation relations are provided by (1.21), where H is invariant under the action of spin operator components, it is convenient to use the simplified formulas:

$$\sqrt{2}S_+ \equiv a^*, \qquad \sqrt{2}S_- \equiv a, \qquad S_z \equiv -\tfrac{1}{2} + a^*a \tag{1.25a}$$

being the finite version of the correct, infinite expansions:

$$\begin{aligned}
&\sqrt{2}S_+ = \mathbf{1}_F a^* \mathbf{1}_F, \qquad \sqrt{2}S_- = \mathbf{1}_F a \mathbf{1}_F \\
&S_z = \mathbf{1}_F\{-\tfrac{1}{2} + a^*a\}\mathbf{1}_F.
\end{aligned} \tag{1.25b}$$

Obviously one can equally well use $-S_z$ in the place of S_z. (1.25b) is a projected set of operators (1.25a).

The question of higher spins will be considered in below, in connection with Boson expansions of Fermion operators.

All above considerations were pure quantum mechanical in spirit. The transition to the description of the infinite assembly of spins is here immediate. If to consider a Fock representation of the CCR, generated by the triple $\{a_l^*, a_l, \Omega_B\}_{l=1,2,\ldots}$

$$\begin{aligned}
&[a_k, a_l^*]_- = \delta_{kl}\mathbf{1}_B \\
&[a_k, a_l]_- = 0 = [a_k^*, a_l^*]_-.
\end{aligned} \tag{1.26}$$

$a_k\Omega_B = 0$ for all $k = 1, 2, \ldots$, and repeating arguments given previously for each single $l = 1, 2, \ldots$, we get a corresponding sequence of bosonized spin operators $\{S_l\}_{l=1,2,\ldots}$, $a, b, c = 1, 2, 3$

$$[S_{ka}, S_{lb}]_- = S_{kc} \cdot \delta_{kl} \cdot i\epsilon_{abc}. \tag{1.27}$$

This sequence is, in fact, used in practical applications.

2. Boson expansions of bifermion operators

Let the triple $\{b_l^*, b_l, \Omega_F\}_{l=1,2,\ldots}$ generate a Fock representation of the CAR (canonical anticommutation relations algebra):

$$\begin{aligned} &[b_k, b_l^*]_+ = \delta_{kl}\mathbf{1}_F, \qquad b_k\Omega_F = 0 \qquad \text{for all } k = 1, 2, \ldots \\ &[b_k, b_l]_+ = 0 = [b_k^*, b_l^*]_+. \end{aligned} \tag{2.1}$$

$\mathbf{1}_F$ denotes here an operator unit in the algebra. Each element of the representation, being quadratic in b_k^*, b_l we call a bifermion operator. Following papers [18–25], we restrict our considerations to operators $b_k^* b_l^*$ and $b_k^* b_l$, which together with their adjoints satisfy the commutation relations:

$$\begin{aligned} &[b_k^* b_l, b_r^* b_s]_- = \delta_{lr} b_k^* b_s - \delta_{sk} b_r^* b_l, \\ &[b_k^* b_l, b_r^* b_s^*]_- = \delta_{lr} b_k^* b_s^* - \delta_{ls} b_k^* b_r^*, \\ &[b_k^* b_l, b_r b_s]_- = \delta_{lr} b_k^* b_s - \delta_{kr} b_l^* b_s + (\delta_{ls}\delta_{rk} - \delta_{ks}\delta_{lr})\mathbf{1}_F + \delta_{ks} b_l^* b_r - \delta_{ls} b_k^* b_r, \\ &[b_k^* b_l^*, b_r^* b_s^*]_- = 0 = [b_k b_l, b_r b_s]_-. \end{aligned} \tag{2.2}$$

We are interested not only in concrete $b_k^* b_l^*, b_k^* b_l$ but in the whole class of operators $U b_k^* b_l^* U^*, U b_k^* b_l U^*$ determined by (2.2) up to a unitary transformation.

Now, let us assume to have given a Fock representation of the CCR algebra. We wish to prove that there really exists a unitary transformation U such that the Boson expansions:

$$\begin{aligned} U b_k^* b_l^* U^* &= \sum_{n=0}^{\infty} \sum_{r_{1+n}} \sum_{s_n} p_n(k, l, \boldsymbol{r}_{1+n}, s_n) a_{r_1}^* \ldots a_{r_{1+n}}^* a_{s_1} \ldots a_{s_m}, \\ U b_k^* b_l U^* &= \sum_{n=0}^{\infty} \sum_{r_n} \sum_{s_n} q_n(k, l, \boldsymbol{r}_n, s_n) a_{r_1}^* \ldots a_{r_n}^* a_{s_1} \ldots a_{s_n} \end{aligned} \tag{2.3}$$

hold on a suitable domain. Here: $\boldsymbol{r} = (r_1, \ldots r_n)$.

In the above conjecture, we have combined the original idea of Belaev and Zelevinsky [18, 22] with Marumori [19–21, 23–25] approach, giving compact formulas for Boson expansions of bifermion operators. This last approach allows us to avoid a wearisome use of the iteration procedure extensively applied in [18].

Let $\{f_l\}_{l=1,2,\ldots}$ constitute a complete orthonormal system in $\mathscr{L}^2(\mathbb{R}^n)$, $\mathbb{R}^n \ni \boldsymbol{p} = (p_1, \ldots, p_n)$

$$\sum_l f_l(\boldsymbol{p}) f_l(\boldsymbol{q}) = \delta(\boldsymbol{p} - \boldsymbol{q}), \qquad \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{p} f_k(\boldsymbol{p}) f_l(\boldsymbol{p}) = \delta_{kl}. \tag{2.4}$$

It allows us to consider in place of discretely indexed operators a_k^*, a_l, see (1.26), continuously indexed ones:

$$a^*(\boldsymbol{p}) = \sum_l a_l^* f_l(\boldsymbol{p}), \qquad a(\boldsymbol{p}) = \sum_l a_l \bar{f}_l(\boldsymbol{p}). \tag{2.5}$$

Let us write $\mathbb{R}^n = \mathbb{R}^i \otimes \mathbb{R}^j$, $i + j = n$, $\boldsymbol{p} = (p_1, \ldots, p_i, p_{i+1}, \ldots, p_n) = (q_1, \ldots, q_i, r_1, \ldots, r_i) = (\boldsymbol{q}, \boldsymbol{r})$, $\boldsymbol{q} \in \mathbb{R}^i$, $\boldsymbol{p} \in \mathbb{R}^j$.

The basis in $\mathscr{L}^2(\mathbb{R}^i)$ we denote by $\{g_l\}_{l=1,2,\ldots}$ and in $\mathscr{L}^2(\mathbb{R}^j)$ by $\{h_l\}_{l=1,2,\ldots}$ respectively. Then, the basis in $\mathscr{L}^2(\mathbb{R}^i) \otimes \mathscr{L}^2(\mathbb{R}^j)$ is given by: $\{g_k \otimes h_l\}_{k,l=1,2,\ldots}$.

Let us consider the set of antisymmetric basis elements in $\mathscr{L}^2(\mathbb{R}^i) \otimes \mathscr{L}^2(\mathbb{R}^j)$ given by: $\{f_{kl}\}_{k,l=1,2,\ldots}$

where

$$f_{kl} = \frac{1}{\sqrt{2}}(g_k \otimes \bar{h}_l - g_l \otimes \bar{h}_k)$$

$$f_{kl}(\boldsymbol{q}, \boldsymbol{r}) = \frac{1}{\sqrt{2}}[g_k(\boldsymbol{q})\bar{h}_l(\boldsymbol{r}) - g_l(\boldsymbol{q})\bar{h}_k(\boldsymbol{r})]. \tag{2.6}$$

We have:

$$f_{kl}(\boldsymbol{q}, \boldsymbol{r}) = f_{kl}(\boldsymbol{p}). \tag{2.7}$$

Hence, the operators (2.5) can be transformed to the form:

$$\int \mathrm{d}\boldsymbol{p} a^*(\boldsymbol{p}) f_{kl}(\boldsymbol{p}) = a^*_{kl}$$

$$\int \mathrm{d}\boldsymbol{p} a(\boldsymbol{p}) \bar{f}_{kl}(\boldsymbol{p}) = a_{kl} \tag{2.8}$$

and, by virtue of:

$$(f_{kl}, f_{st}) = \int_{\mathbf{R}^i} \mathrm{d}\boldsymbol{q} \int_{\mathbf{R}^j} \mathrm{d}\boldsymbol{r} f_{kl}(\boldsymbol{r}, \boldsymbol{q}) \bar{f}_{st}(\boldsymbol{r}, \boldsymbol{q}) = \delta_{ks}\delta_{lt} - \delta_{kt}\delta_{ls} \tag{2.9}$$

the operators (2.8) satisfy:

$$\begin{aligned} &[a_{kl}, a^*_{st}]_- = (\delta_{ks}\delta_{lt} - \delta_{kt}\delta_{ls})\mathbf{1}_{\mathrm{B}} \\ &[a_{kl}, a_{st}]_- = 0 = [a^*_{kl}, a_{st}]_- \\ &a_{kl}\Omega_{\mathrm{B}} = 0 \quad \text{for all } k, l = 1, 2, \ldots \end{aligned} \tag{2.10}$$

and also: $a_{kk} = 0 = a^*_{kk}$.

We denote the Boson Fock space generated by (2.10) $\mathscr{F}_{\mathrm{B}}$. Its most general element can be written in the form:

$$F = \sum_m (F_m, |m)_{\mathrm{B}}) \tag{2.11}$$

where:

$$|m)_{\mathrm{B}} = (a^*_{k_1 l_1})^{n_1} \ldots (a^*_{k_m l_m})^{n_m}\Omega_{\mathrm{B}} \tag{2.12}$$

and:

$$(F_m, |m)_{\mathrm{B}}) = \sum_{\boldsymbol{n}, \boldsymbol{k}, \boldsymbol{l}} F_m^{\boldsymbol{nkl}} \cdot |m)_{\mathrm{B}}, \qquad \boldsymbol{n} = (n_1, \ldots, n_m).$$

Here $F_m^{\boldsymbol{nkl}}$ is a tensor, totally symmetric in the variables $\boldsymbol{n}, \boldsymbol{k}, \boldsymbol{l}$. Let us restrict our considerations to a subspace H of $\mathscr{F}_{\mathrm{B}}$ distinguished by the requirement $1 = n_1 = \cdots = n_m$ for all m.

Then if tensors $F_m^{\boldsymbol{kl}}$ which are totally antisymmetric with respect to $\boldsymbol{k}, \boldsymbol{l}$ are used to be multiplied by basis vectors and summed over $\boldsymbol{k}, \boldsymbol{l}$, we can introduce into consideration the following antisymmetric basis system in H:

$$|m)_H = (1/\sqrt{(2m-1)!!}) \sum_{\mathrm{perm}} (-1)^p a^*_{k_1 l_{p1}} \ldots a^*_{k_m l_{pm}}\Omega_{\mathrm{B}}. \tag{2.13}$$

This definition makes sense only under the sign of the bilinear form:

$$(F_m, |m)_H) = \sum_{kl} F_m^{kl} |m)_H.$$

Let us further denote by $\mathscr{F}_F$, a Fermion Fock space for the representation (2.1). The basis vectors in $\mathscr{F}_F$ are totally antisymmetric and by analogy to (2.13) can be defined in the form:

$$|m)_F = b_{k_1}^* b_{l_1}^* \ldots b_{k_m}^* b_{l_m}^* \Omega_F. \tag{2.14}$$

Now, we take into consideration the tensor product space $\mathscr{F}_B \otimes \mathscr{F}_F$, the vacuum in which is denoted: $\Omega_B \otimes \Omega_F = |0)_B |0)_F$.
Furthermore, we restrict $\mathscr{F}_B \otimes \mathscr{F}_F$ to $H \otimes \mathscr{F}_F$.
Here $H \otimes \mathscr{F}_F$ is spanned by vectors $|m)_H |0)_F$, and $\Omega_B \otimes \mathscr{F}_F$ by $|0)_B |m)_F$.

Let us define the operator U_M:

$$U_M = |0)_{FF}(0| \sum_{n=0}^{\infty} \frac{1}{(2n)!!} \frac{1}{(2n-1)!!} \left(\sum_{kl} a_{kl}^* b_k b_l \right)^n |0)_{BB}(0| \tag{2.15}$$

which, if acting in $H \otimes \mathscr{F}_F$ has the remarkable property to realize the map:

$$U_M: \quad \Omega_B \otimes \mathscr{F}_F \longrightarrow H \otimes \Omega_F. \tag{2.16}$$

In this connection compare [7, 19, 20]:

$$U_M |0)_B |m)_F = |m)_H |0)_F \tag{2.17}$$
$$_F(0|_H(m|U_M^* = {}_F(m|_H(0|$$
$$_F(m|_B(0|U_M^* U_M |0)_B |m'_F) = {}_F(0|_B(m \mid m')_B |0)_F = \delta_{mm'}$$

which proves that $U_M^* U_M$ plays the rôle of the unit operator on $\Omega_B \otimes \mathscr{F}_F$. An arbitrary operator T acting in $\Omega_B \otimes \mathscr{F}_F$ has its image τ in $H \otimes \Omega_F$:

$$_F(m|_B(0|T|0)_B|m')_F = {}_F(0|_B(m|U_M T U_M^*|m')_B|0)_F = {}_F(0|_B(m|\tau|m')_B|0)_F. \tag{2.18}$$

Let us notice that a projection onto $\Omega_B \otimes \mathscr{F}_F$:

$$U_M U_M^* = \sum_m |0)_B |m)_{FF}(m|_B(0| \tag{2.19}$$

has an obvious image:

$$U_M(U_M U_M^*) U_M^* = \sum_m |m)_B |0)_{FF}(0|_B(m| = \hat{P} \tag{2.20}$$

which is a projection onto $H \otimes \Omega_F$.
Because of:

$$U_M^* \hat{P} = U_M^*, \qquad \hat{P} U_M = U_M, \qquad \hat{P}^2 = \hat{P}^* = \hat{P} \tag{2.21}$$

we get:

$$\tau = \hat{P} U_M T U_M^* \hat{P} = \hat{P} \tau \hat{P} \tag{2.22}$$

which proves that τ possesses nonzero matrix elements between physical states in $\mathscr{F}_B \otimes \mathscr{F}_F$ only, i.e.

between elements of $H \otimes \Omega_F$. Following [7], we shall give an explicit formula for the Boson image τ of the operator T:

$$\begin{aligned}
\tau = & \sum_{m,m'} |m)_{BF}(m|T|m')_{FB}(m'| = {}_F(0|T|0)_F|0)_{BB}(0| \\
& + \sum_{m=1}^{N} \frac{1}{(2m)!} \sum_{(k,l)} {}_F(0|b_{l_m} b_{k_m} \dots b_{l_1} b_{k_1} T|0)_F \frac{1}{\sqrt{(m-1)!!}} \sum_{\text{perm}} (-1)^p a^*_{k_1 l_{p_1}} \dots a^*_{k_m l_{p_m}} |0)_{BB}(0| \\
& + \sum_{m=1}^{N} \frac{1}{(2m)!} \sum_{(k,l)} {}_F(0|T b^*_{k_1} b^*_{l_1} \dots b^*_{k_m} b^*_{l_m}|0)_F|0)_{BB}(0| \frac{1}{\sqrt{(2m-1)!!}} \sum_{\text{perm}} (-1)^p a_{k_m l_{p_m}} \dots a_{k_1 l_{p_1}} \\
& + \sum_{n,m} \frac{1}{(2n)!} \frac{1}{(2m)!} \sum_{k,l} \sum_{r,s} {}_F(0|b_{l_n} b_{k_m} \dots b_{l_1} b_{k_1} T b^*_{r_1} b^*_{s_1} \dots b^*_{r_m} b^*_{s_n}|0)_F \frac{1}{(2n-1)!!} \\
& \times \sum_{\text{perm}} (-1)^p a^*_{k_1 l_1} \dots a^*_{k_n l_n} |0)_{BB}(0| \frac{1}{(2m-1)!!} \sum_{\text{perm}} (-1)^p a^*_{r_m s_m} \dots a^*_{r_1 s_1}.
\end{aligned} \tag{2.23}$$

This lengthy formula for a special case of bifermion operators reads:

$$U_M b^*_k b^*_l U^*_M = a^*_{kl} - \tfrac{1}{2}(1 - 1/\sqrt{3}) a^*_{kl} \sum_{nm} a^*_{mn} a_{mn} - \tfrac{1}{3} \sum_{nm} a^*_{km} a^*_{ln} a_{mn} + \dots . \tag{2.24}$$

In [22] there were derived formal, but compact formulas for bifermion operators of interest:

$$U_M b^*_k b^*_l U^*_M = \left(a^*_{kl} - \sum_{nm} a^*_{km} a^*_{ln} a_{mn} \right) \frac{\hat{P}}{\sqrt{1+\hat{N}}} = \hat{P}(a^* \sqrt{1-\hat{\rho}})_{kl}, \tag{2.25}$$

$$U_M b^*_k b_l U^*_M = \sum_n a^*_{kl} a_{ln} \hat{P}$$

where:

$$\hat{\rho}_{kl} = \sum_n a^*_{nl} a_{kn}, \qquad \hat{N} = \sum_k \hat{\rho}_{kk}. \tag{2.26}$$

We have thus found the Hilbert space in which operators $U_M b^*_k b^*_l U_M$, $U_M b^*_k b_l U^*_M$ satisfy the commutation relations (2.2) and moreover admit the Boson expansions (2.23)–(2.26), which by virtue of (2.5)–(2.8) proves our starting conjecture. In this way we have additionally disclosed the conditions under which Belaev–Zelevinsky and Marumori Boson expansions for bifermion operators, can coincide. In this last connection compare also [7, section 5].

The infinite Boson expansions (2.25)–(2.26) are not convenient for practical applications, especially because the convergence of these series is not sufficiently quick. Therefore, one usually either neglects all expansion terms except a few, or one tries to develop a separate theory of finite Boson expansions, see e.g. [26].

Let us consider in the place of U_M the operator U:

$$U = |0)_{FF}(0| \exp\left(\tfrac{1}{2} \sum_{kl} a^*_{kl} b_k b_l \right) |0)_{BB}(0|. \tag{2.27}$$

This operator, though mapping $\Omega_B \otimes \mathcal{F}_F$ onto $H \otimes \Omega_F$, spoils the normalizability of the basis vectors:

$$|m)_B|0)_F = \frac{1}{\sqrt{(2n-1)!!}} U|0)_B|m)_F$$
$$_F(m|_B(0| = \frac{1}{\sqrt{(2m-1)!!}} {}_F(0|_B(m|U. \tag{2.28}$$

To improve this defect, let us define an operator $\tilde{U}$:

$$\tilde{U} = |0)_{BB}(0| \sum_{m=0}^{\infty} \frac{1}{(2m)!} \left(\sum_{kl} b_k^* b_l^* a_{kl} \right)^m |0)_{FF}(0| \tag{2.29}$$

(notice that $\tilde{U}$ is a slight modification of U), satisfying:

$$|0)_B|m)_F = \sqrt{(2m-1)!!}\,\tilde{U}|m)_B|0)_F, \tag{2.30}$$
$$_F(0|_B(m| = \sqrt{(2m-1)!!}\, {}_F(m|_B(0|\tilde{U}.$$

One can easily check that the operator $\tilde{U}U$ is a unit operator in $\Omega_B \otimes \mathcal{F}_F$ while $\hat{P} = U\tilde{U}$ in $H \otimes \Omega_F$ respectively. By analogy to previous considerations, for an arbitrary operator T acting in $\Omega_B \otimes \mathcal{F}_F$, we can easily get its Boson image τ acting in $H \otimes \Omega_F$:

$$\tau = UT\tilde{U}. \tag{2.31}$$

By virtue of the relation $\tilde{U}^* \neq U$, $T^* = T$ does not here imply $\tau^* = \tau$. By making a few, not too difficult calculations, one can derive finite Boson expansions for bifermion operators:

$$Ub_k^* b_l^* \tilde{U} = \left(a_{kl}^* - \sum_{nm} a_{kn}^* a_{lm}^* a_{nm} \right) \hat{P},$$
$$Ub_l b_k \tilde{U} = a_{kl} \hat{P}, \qquad Ub_k^* b_l \tilde{U} = \sum_r a_{kr}^* a_{lr} \hat{P}, \tag{2.32}$$

implying obviously the commutation relations (2.2).

In connection with both infinite and finite Marumori expansions one can express a few objections, being in close analogy with these appearing in the discussion of Boson expansions of spin operators.

A practical use of the infinite case, meets essential difficulties, though it gives a Hermitean Hamiltonian. Usually one takes into account the two first terms of the expansions, as e.g. in [18, 20, 23, 24], not worrying whether this assumption is mathematically and physically correct. On the other hand, even the application of full infinite expansions can appear to be doubtful, because there is no rigorous proof of their convergence.

In case of finite expansions, it is not possible to get a Hermitean Hamiltonian, by virtue of the fact that for example $Ub_k b_l \tilde{U}$ cannot be transformed into $Ub_l^* b_k^* \tilde{U}$ by the use of the * operation; compare also a discussion of the Dyson–Maleev expansion, where an analogous question appeared.

3. Boson expansions of Fermion operators

To explain the leading idea motivating approaches to the question of Boson expansions for Fermions, let us make a citation from [38]: "It has been taken for granted that a fundamental theory

of elementary particles must involve Fermion fields in the basic formalism, but need not involve Bosons. The spinor theory of Heisenberg [36] is the most developed example of this philosophy (compare also considerations on the two-neutrino theory of photons [93]). More recently the opposite point of view has been proposed: that is, a theory in which only observable fields, necessarily uncharged Bosons, occurring in the basic formalism, might be capable of describing Fermions, or uncharged Bosons. An early paper of Skyrme [37] goes so far as to give explicit formulas for the Fermion field in terms of the Boson one in a two-dimensional field theory".

3.1. Representations of the CAR generated by representations of the CCR

Now, we shall collect a few results of the Boson expansion theory developed in the series of papers [27–34, 16, 17, 76]. As its extremely exciting feature there appears the fact that we deal with a kind of universal Boson expansions for Fermion operators, involving applications in quantum theory as a whole, beginning from quantum mechanics through many-body problems and ending in quantum field theory. From a physical point of view this theory governs the behaviour of systems whose excitation level is so lowered that a probability of occupying other than the 0th and 1st energy levels of each single degree of freedom (normal mode) is very small.

Let us begin from the case of the infinitely many degrees of freedom. We denote K a complex Hilbert space. By $\mathcal{U}_{\mathrm{F}}(K)$ we denote a Fock representation of the CAR (canonical anticommutation relations algebra) over K, acting on the representation space $\mathcal{F}_{\mathrm{F}}$, and defined by the triple:

$$\begin{aligned}
&\{b^*, b, \Omega_{\mathrm{F}}\}_K, \qquad f, g \in K\\
&[b(f), b(g)^*]_+ = (f, g)\mathbf{1}_{\mathrm{F}}\\
&[b(f), b(g)]_+ = 0 = [b(f)^*, b(g)^*]_+\\
&b(f)\Omega_{\mathrm{F}} = 0 \qquad \text{for all } f \in K.
\end{aligned} \tag{3.1}$$

By $\mathcal{U}_{\mathrm{B}}(K)$ we denote a Fock representation of the CCR (canonical commutation relations algebra) over K, generated by the triple $\{a^*, a, \Omega_{\mathrm{B}}\}_K$

$$\begin{aligned}
&[a(f), a(g)^*]_- = (f, g)\mathbf{1}_{\mathrm{B}}\\
&[a(f), a(g)]_- = 0 = [a(f)^*, a(g)^*]_-\\
&a(f)\Omega_{\mathrm{B}} = 0 \text{ for all } f \in K.
\end{aligned} \tag{3.2}$$

The representation space we denote by $\mathcal{F}_{\mathrm{B}}$.
Let further E_n be a bounded operator acting on the nth tensor product $K^{\otimes n} = \mathcal{H}_n$, with properties:

$$E_n^3 = E_n, \qquad E_n^* = E_n, \qquad P_{ik}E_n = -E_nP_{ik}, \tag{3.3}$$

where P_{ik} is an operator of permutation of the ith and kth K in $K^{\otimes n}$ compare also [27, 29] where examples for E_n are given.

By virtue of (3.3) E_n^2 is a projector: $\mathcal{H}_n^1 = E_n^2\mathcal{H}_n$, $\mathcal{H}_n^2 = (1 - E_n^2)\mathcal{H}_n$. Let us denote by $A_n\mathcal{H}_n$ and $S_n\mathcal{H}_n$ respectively the totally antisymmetric and symmetric subspaces of $\mathcal{H}_n$. E_n realizes an isomorphism:

$$A_n\mathcal{H}_n^1 \longleftrightarrow S_n\mathcal{H}_n^1 \tag{3.4}$$

which under an additional restriction on E_n:

$$A_n\mathcal{H}_n^2 = 0 \tag{3.5}$$

extends to:

$$A_n\mathcal{H}_n \longleftrightarrow S_n\mathcal{H}_n^1. \tag{3.6}$$

In connection with E_n compare also arguments following (2.12). Let us choose $K = \mathcal{L}^2(\mathbb{R}^n)$, $E_n(\boldsymbol{k}_n, \boldsymbol{p}_n)$ is then an integral kernel of E_n, $\boldsymbol{k}_n = (k_1, \ldots, k_n)$, $k, p \in \mathbb{R}^n$.

The Fock representation $\mathcal{U}_B(K)$ of the CCR algebra acts in the domain $\mathcal{D} \subset \mathcal{F}_B$. We denote $(a^*, a) = \int_{\mathbb{R}^n} \mathrm{d}k a^*(k)a(k)$. Then the operators $b(f), b(g)^*$ with:

$$b(f) = :\exp\{-(a^*, a)\} \sum_{nm} \frac{1}{\sqrt{n!m!}} \int \mathrm{d}\boldsymbol{k}_n \int \mathrm{d}\boldsymbol{p}_m f_{nm}(\boldsymbol{k}_n, \boldsymbol{p}_n) a^*(k_1) \ldots a^*(k_n)a(p_1) \ldots a(p_m):\tag{3.7}$$

where:

$$f_{nm}(\boldsymbol{k}_n, \boldsymbol{p}_m) = \sqrt{n+1}\, \delta_{m,1+n} \int \mathrm{d}\boldsymbol{q}_n \int \mathrm{d}r E_n(\boldsymbol{k}_n, \boldsymbol{q}_n)\bar{f}(r)E_{1+n}(r, \boldsymbol{q}_n, \boldsymbol{p}_{1+n}) \tag{3.8}$$

$$\mathrm{d}\boldsymbol{k}_n = \mathrm{d}k_1 \ldots \mathrm{d}k_n$$

generate a Fock representation $\mathcal{U}_F(K)$ of the CAR algebra acting on the following subspace of $\mathcal{F}_B$: $\mathcal{F}_B^1 = \mathbf{1}_F\mathcal{F}_B$, where: $\mathcal{F}_B^1 = \mathcal{F}_F$

$$\mathbf{1}_F = :\exp\{-(a^*, a)\} \cdot \sum_n \frac{1}{n!} \int \mathrm{d}\boldsymbol{k}_n \int \mathrm{d}\boldsymbol{p}_n \int \mathrm{d}\boldsymbol{r}_n$$
$$\times E_n(\boldsymbol{k}_n, \boldsymbol{r}_n)E_n(\boldsymbol{r}_n, \boldsymbol{p}_n)a^*(k_1) \ldots a^*(k_n)a(p_1) \ldots a(p_n):\tag{3.9}$$

The canonical anticommutation relations for $b(f), b(g)^*$ are proved in [29].

In the above $\mathcal{F}_F$ appears as a physical space H employed in the previous considerations.

The Boson expansions of Fermion operators (3.7)–(3.9) can be applied in quantum field theory. Now let us study a transition to a finite number of degrees of freedom.

We have proved in [32, 16, 17] that the Boson expansions (3.7)–(3.9) can be reduced to a finite number of degrees also. Namely, let $K = \bigoplus_{l=1}^{\infty} K_l$ where $\dim K_l < \infty$ for each l. Then we can construct what we call the truncated representation of the CAR, consisting of mutually commuting segments, inside which the usual CAR hold. The lth segment is generated by the triple $\{b^*, b, \Omega_B\}_l$ given by:

$$b(f_l) = :\exp\Big(-\sum_{\alpha=0}^{N_l} a_\alpha^* a_\alpha\Big) \cdot \sum_{n,m=0}^{N_l-1} (1/\sqrt{n!m!})$$
$$\times \sum_{\alpha_1 \ldots \alpha_n=0}^{N_l} \sum_{\beta_1 \ldots \beta_m=0}^{N_l} f_{nm}(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_m) a_{\alpha_1}^* \ldots a_{\alpha_n}^* a_{\beta_1} \ldots a_{\beta_m}:\tag{3.10}$$

with:

$$f_{nm}(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_m) = \sqrt{n+1}\,\delta_{m,1+n} \sum_{\gamma_1 \ldots \gamma_n=0}^{N_l} \sum_{\sigma=0}^{N_l} E_n(\boldsymbol{\alpha}_n, \boldsymbol{\gamma}_n)\bar{F}_\sigma \cdot E_{1+n}(\sigma, \boldsymbol{\gamma}_n, \boldsymbol{\beta}_{1+n})\tag{3.11}$$

and

$K \ni f_l = \sum_{\alpha=0}^{N_l} F_\alpha f_{l\alpha}$, $\{f_{l\alpha}\}_{\alpha=0,1,\ldots,N_l}$ being the basis system in K_l

$$a_\alpha^* = a(f_{l\alpha})^* = \int \mathrm{d}k f_{l\alpha}(k)a^*(k), \qquad \dim K_l = 1 + N_l.$$

Furthermore:

$$E_n(\alpha_n, \gamma_n) = \epsilon_{\alpha_1 \dots \alpha_n} \delta_{\alpha_1 \gamma_1} \dots \delta_{\alpha_n \gamma_n} \tag{3.12}$$

where $\epsilon_{\alpha_1 \dots \alpha_n}$ is the generalized Levi–Civitta tensor vanishing if any two indices coincide and taking the value ± 1 depending on the odd or even permutation of indices.

We then get:

$$\begin{aligned} &[b(f_l), b(g_l)^*]_+ = (f_l, g_l)\mathbf{1}_{\mathrm{F}}^l \\ &[b(f_l), b(g_l)]_+ = 0 = [b(f_l)^*, b(g_l)^*]_+ \\ &b(f_l)\Omega_{\mathrm{B}} = 0 \quad \text{for all } f_l \in K_l \end{aligned} \tag{3.13}$$

where $\mathbf{1}_{\mathrm{F}}^l$ is a unit operator in the lth segment of the truncated representation of the CAR, projecting onto the subspace: $\mathscr{F}_{\mathrm{F}}^l = \mathbf{1}_{\mathrm{F}}^l \mathscr{F}_{\mathrm{B}}$ of $\mathscr{F}_{\mathrm{B}}$ with:

$$\begin{aligned} \mathbf{1}_{\mathrm{F}}^l = {:} \exp\Big(-\sum_{\alpha=0}^{N_l} a_\alpha^* a_\alpha\Big) \cdot \sum_{n=0}^{N_l} (1/n!) \\ \times \sum_{\alpha_1 \dots \alpha_n = 0}^{N_l} \sum_{\beta_1 \dots \beta_n = 0}^{N_l} \sum_{\gamma_1 \dots \gamma_n = 0}^{N_l} E_n(\alpha_n, \gamma_n) E_n(\gamma_n, \beta_n) a_{\alpha_1}^* \dots a_{\alpha_n}^* a_{\beta_1} \dots a_{\beta_n} {:}. \end{aligned} \tag{3.14}$$

In the above we have assumed the vacuum Ω_{B} to be common for all segments from the sequence $l = 1, 2, \dots$.

However preserving formulas (3.10)–(3.13) we can without any difficulty consider for each l a separate vacuum Ω_{B}^l. In that case, in the place of a truncated representation of the CAR we deal with a sequence of finite-dimensional Fock representations of the CAR, generated by the appropriate sequence of triples $\{b^*, b, \Omega_{\mathrm{B}}^l\}_{K_l}$. In this connection compare [17].

Restricting our considerations to a concrete lth segment, we achieve the required finitely dimensional case, namely the quantum mechanics of the many-particle system, generated by dim $K_l = 1 + N_l$ number of different Bosons.

The simplest example is here dim $K = 1$. If $\Omega_{\mathrm{B}}^l = \Omega_{\mathrm{B}}$ for all $l = 1, 2, \dots$ we get:

$$\begin{aligned} &b_l^* = a_l^* {:} \exp(-a_l^* a_l) {:}, \qquad b_l = {:} \exp(-a_l^* a_l) {:} a_l, \\ &\mathbf{1}_{\mathrm{F}}^l = {:} \exp(-a_l^* a_l) \cdot [1 + a_l^* a_l] {:} \end{aligned} \tag{3.15}$$

and in $\mathscr{F}_{\mathrm{B}}$, there obviously holds:

$$\begin{aligned} &[b_k, b_l^*]_- = 0 \qquad \text{for } k \neq l \\ &[b_l, b_l^*]_+ = \mathbf{1}_{\mathrm{F}}^l, \qquad b_l^2 = 0 = b_l^{*2}, \qquad b_l \Omega_{\mathrm{B}} = 0 \qquad \text{for all } l = 1, 2, \dots . \end{aligned} \tag{3.16}$$

If with each lth segment to associate a particular lth vacuum Ω_{B}^l and then to restrict considerations to a concrete single segment, we deal with a one-dimensional quantum mechanical system. Omitting the now superfluous index l, we get the following identities for b, b^*:

$$\begin{aligned} &[b, b^*]_+ = \mathbf{1}_{\mathrm{F}}, \qquad b^{*2} = 0 = b^2, \qquad b\Omega_{\mathrm{B}} = 0, \\ &\mathbf{1}_{\mathrm{F}} = {:} \exp(-a^* a) {:} + a^* {:} \exp(-a^* a) {:} a \end{aligned} \tag{3.17}$$

which thus define an irreducible representation of the CAR in the two-dimensional subspace $\mathscr{F}_{\mathrm{F}}$ of the original Hilbert space $\mathscr{F}_{\mathrm{B}}$ generated by $\{a^*, a, \Omega_{\mathrm{B}}\}$. $\mathscr{F}_{\mathrm{F}}$ is spanned by the ground state Ω_{B} and the one-particle state. For more details see also [16, 17]: $a^*\Omega_{\mathrm{B}} = b^*\Omega_{\mathrm{B}}$ (3.17) is the case when the correct Boson expansions for spin $\frac{1}{2}$ can be proposed.

In case of dim $K = 2$ there is possible to find the correct Boson expansions for spin 0 and 1 operators. We deal here with a two-particle quantum system, where by (3.10)–(3.14) a Boson with two internal degrees of freedom $\{a_\alpha^*, a_\alpha, \Omega_B\}_{\alpha=1,2}$ generates a corresponding Fermion $\{b_\alpha^*, b_\alpha, \Omega_B\}_{\alpha=1,2}$ with:

$$b_\alpha = :\exp\Big(-\sum_{\alpha=1}^{2} a_\alpha^* a_\alpha\Big)\cdot\Big[a_\alpha + \sum_{\gamma,\beta} a_\gamma^* E_2(\alpha,\gamma,\beta_1,\beta_2) a_{\beta_1} a_{\beta_2}\Big]: \tag{3.18}$$

which by virtue of (3.12) gives:

$$b_1 = :\exp\Big(-\sum_\alpha a_\alpha^* a_\alpha\Big)\cdot[a_1 + a_2^* a_1 a_2]: \tag{3.19}$$

$$b_2 = :\exp\Big(-\sum_\alpha a_\alpha^* a_\alpha\Big)\cdot[a_2 - a_1^* a_1 a_2]:$$

and:

$$\mathbf{1}_F = :\exp\Big(-\sum_\alpha a_\alpha^* a_\alpha\Big)\cdot[1 + a_1^* a_1 + a_2^* a_2 + a_1^* a_2^* a_1 a_2]:. \tag{3.20}$$

Here, the anticommutation relations can be easily checked through an immediate calculation, by taking into account that $:\exp(-\Sigma_\alpha a_\alpha^* a_\alpha):$ projects onto Ω_B.

The representation space $\mathcal{F}_F$ is now four-dimensional. The basis vectors in $\mathcal{F}_F = \mathbf{1}_F\mathcal{F}_B$ can be here chosen in the form:

$$\begin{aligned}
&|1,1) = a_1^* a_2^* \Omega_B = f_1 \otimes f_0 \\
&|1,0) = (1/\sqrt{2})[a_1^* + a_2^*]\Omega_B = (1/\sqrt{2})[f_1 \otimes f_0 + f_0 \otimes f_1] \\
&|1,-1) = \Omega_B = f_0 \otimes f_0 \\
&|0,0) = (1/\sqrt{2})[a_1^* - a_2^*]\Omega_B = (1/\sqrt{2})[f_1 \otimes f_0 - f_0 \otimes f_1]
\end{aligned} \tag{3.21}$$

while in the Hilbert space of a single Boson, we would have:

$$\Omega_B = f_0, \qquad f_n = (1/\sqrt{n!})a^{*n} f_0.$$

In (3.21) we have indicated spin properties of the quantum system (3.18)–(3.20). Namely, by [17], we have defined in $\mathcal{F}_F$ the representation of the SU(2) group, whose infinitesimal generator $\boldsymbol{S}$ is given by:

$$\begin{aligned}
&\sqrt{2}S_+ = \sum_\alpha a_\alpha^* :\exp(-(a^*, a)): \\
&\sqrt{2}S_- = \sum_\alpha :\exp(-(a^*, a)): a_\alpha \\
&S_z = -\tfrac{1}{2}\sum_\alpha \mathbf{1}_\alpha + :\exp(-(a^*, a))\cdot(a^*, a):
\end{aligned} \tag{3.22}$$

where $(a^*, a) = \Sigma_\alpha a_\alpha^* a_\alpha$ and:

$$\mathbf{1}_\alpha = :\exp(-(a^*, a))\cdot[1 + a_\alpha^* a_\alpha]:. \tag{3.23}$$

One can check that the first of the numbers on the left-hand side of (3.21) indicates the eigenvalue of $\boldsymbol{S}^2$, while the second number indicates that of S_z respectively.

Hence the representation S naturally splits into the two irreducible components corresponding to spin 1 and spin 0 respectively. Let us notice that the formula (3.22) was presented in the way allowing an apparent extension onto an arbitrary many Boson, dim $K = n$, case. Now let us consider the case of Boson expansions of multi-Fermion operators. We assume that a finite, dim $K = 1 + N$, number of Bosons is involved. Then, we can apply expansions (3.10)–(3.14) to find the corresponding expansions for bifermions $b^*_\alpha b_\beta$, $b_\alpha b_\beta$ with $\alpha, \beta = 0, 1, \ldots, N$.

To make an explicit calculation is not difficult, but rather tedious and therefore let us mention a useful method of functional representations of the CAR and CCR, developed in [27–28, 76], see also the Appendix. In [28], in the proof of Theorem 1, we have listed formulas needing a few minor modifications to solve the problem under consideration. Namely, we have:

$$\boldsymbol{\alpha}_k = (\alpha_1, \ldots, \alpha_k), \qquad \boldsymbol{\mu}_{k+n} = (\mu_1, \ldots, \mu_{k+n}),$$

$$b^*_{\alpha_1} \ldots b^*_{\alpha_k} = :\exp(-(a^*, a)) \cdot \Big[\sum_{n=0}^{N-1} \frac{1}{n!} \sum_{\mu, \rho, \nu = 0}^{N} a^*_{\mu_1} \ldots a^*_{\mu_{k+n}}$$

$$\times E_{k+n}(\boldsymbol{\mu}_{k+n}, \boldsymbol{\alpha}_k, \boldsymbol{\rho}_n) \cdot E_n(\boldsymbol{\rho}_n, \boldsymbol{\nu}_n) a_{\nu_1} \ldots a_{\nu_n} :, \tag{3.24}$$

$$b_{\alpha_1} \ldots b_{\alpha_k} = :\exp(-(a^*, a)) \cdot \Big[\sum_{n=0}^{N-1} \frac{1}{n!} \sum_{\mu, \nu, \rho = 0}^{N} a^*_{\mu_1} \ldots a^*_{\mu_n}$$

$$\times E_n(\boldsymbol{\mu}_n, \boldsymbol{\rho}_n) E_{k+n}(\underset{\sim}{\boldsymbol{\alpha}}_k, \boldsymbol{\rho}_n, \boldsymbol{\nu}_{n+k}) a_{\nu_1} \ldots a_{\nu_{n+k}} :, \tag{3.25}$$

$$b^*_{\alpha_1} \ldots b^*_{\alpha_k} b_{\beta_1} \ldots b_{\beta_l} = :\exp(-(a^*, a)) \cdot \Big[\sum_{n=0}^{N} \frac{1}{n!} \sum_{\mu, \nu, \rho = 0}^{N} a^*_{\mu_1} \ldots a^*_{\mu_{k+n}}$$

$$\times E_{k+n}(\boldsymbol{\mu}_{k+n}, \boldsymbol{\alpha}_k, \boldsymbol{\rho}_n) E_{l+n}(\underset{\sim}{\boldsymbol{\beta}}_l, \boldsymbol{\rho}_n, \boldsymbol{\nu}_{l+n}) \cdot a_{\nu_1} \ldots a_{\nu_{l+n}} :, \tag{3.26}$$

$$\underset{\sim}{\boldsymbol{\alpha}}_k = (\alpha_k, \alpha_{k-1}, \ldots, \alpha_1).$$

Identities make sense only under the sign of the bilinear form, where after multiplying by the antisymmetric tensors, summations over indices are performed.

The restriction of (3.24)–(3.26) to $b^*_\alpha b^*_\beta$, $b_\alpha b_\beta$, $b^*_\alpha b_\beta$ is obvious. By taking into account (3.12) we can easily get expansions in terms of Boson operators for concrete values of $\alpha, \beta = 1, 2, \ldots, N$ where tensors E_{k+n}, E_{l+n} involve suitable sign factors.

Let us notice that the expansions, though infinite, are well defined as operators in $\mathscr{F}_F$. No convergence questions appear here in contrast to the Marumori or Belaev–Zelevinsky approach.

The original Boson expansions of Fermion operators constitute, of course, infinite series. However, for practical purposes, there is useful to know that one can always restrict considerations to a few lowest terms only. If to remember that full operators make invariant the representation space $\mathscr{F}$, and to take care of not getting vectors from beyond $\mathscr{F}_F$, no explicit use of the infinite expansions is needed. Let us for example notice that under the sign of the bilinear form, the following identity holds:

$$b^*_{\alpha_1} \ldots b^*_{\alpha_k} \Omega_B = a^*_{\alpha_1} \ldots a^*_{\alpha_k} \epsilon_{\alpha_1 \ldots \alpha_k} \Omega_B = \epsilon_{\alpha_1 \ldots \alpha_k} |\alpha_1, \ldots, \alpha_k)_B = |\alpha_1, \ldots, a_k)_F. \tag{3.27}$$

It suggests that pure Boson operators can be sometimes used in the place of pure Fermion operators. In fact (3.27) is a special example of the quite general correspondence relation between Fermion and Boson algebras.

Let us consider an arbitrary operator:

$$:F(b^*, b): = \sum_{nm} \frac{1}{\sqrt{n!m!}} (f_{nm}, b^{*n} b^m)$$

$$= \sum_{nm} \frac{1}{\sqrt{n!m!}} \int \mathrm{d}\boldsymbol{k}_n \int \mathrm{d}\boldsymbol{p}_m f_{nm}(\boldsymbol{k}_n, \boldsymbol{p}_m) b^*(k_1) \ldots b^*(k_n) b(p_1) \ldots b(p_m) \tag{3.28}$$

$$\mathrm{d}\boldsymbol{k}_n = \mathrm{d}k_1 \ldots \mathrm{d}k_n, \qquad \boldsymbol{k}_n = (k_1, \ldots, k_n), \qquad k \in \mathbb{R}^3$$

whose generating triple $\{b^*, b, \Omega_B\}$ is associated with the starting Boson triple $\{a^*, a, \Omega_B\}$ used to perform the construction of the Fock representation of the CAR in this of the CCR. Here f_{nm} is a totally antisymmetric $n+m$ point function (distribution in general). We have satisfied the following identity:

$$F(b^*, b): = :\exp(-(a^*, a)) \cdot \sum_{nm} \frac{1}{\sqrt{n!m!}} \sum_k \frac{1}{k!} (\sigma_{n+k} f_{n\underset{\sim}{m}} \sigma_{m+k}, a^{*k+n} a^{k+m}): \tag{3.29}$$

where $\sigma_n(\boldsymbol{k}_n)$ is the n-point Friedrichs–Klauder sign (alternating) function, being a continuous generalization of the Levi–Civitta tensor:

$$E_n(\boldsymbol{k}_n, \boldsymbol{p}_n) = \sigma_n(\boldsymbol{k}_n) \delta(k_1 - p_1) \ldots \delta(k_n - p_n) \tag{3.30}$$

and $\underset{\sim}{m}$ denotes that the order of variables is reversed:

$$f_{n\underset{\sim}{m}}(\boldsymbol{k}_n, p_m) = f_{nm}(k_1, \ldots, k_n, p_m, p_{m-1}, \ldots, p_1).$$

One more identity can be further derived:

$$:F(b^*, b): = :\exp(-(a^*, a)) \cdot F^c(a^*, a): = \sum_n \frac{(-1)^n}{n!} (a^{*n}, :F^c(a^*, a):a^n),$$

$$:F(b^*, b):\Omega_B = \sum_{nm} \frac{1}{\sqrt{n!m!}} (\sigma_n f_{n\underset{\sim}{m}} \sigma_m, a^{*n} a^m)\Omega_B = :F^c(a^*, a):\Omega_B \tag{3.31}$$

where $f^c_{nm} = \sigma_n \cdot \sigma_m \cdot f_{n\underset{\sim}{m}}$ is a function symmetric with respect to permutations of variables inside groups (n) and (m) respectively, but antisymmetric with respect to permutations from (n) into (m) and conversely. (3.31) is the generalization of (3.27) and allows to prove that if $\mathbf{1}_F$ is the unit operator in $\{b^*, b, \Omega_B\}$ so that $\mathscr{F}_F = \mathbf{1}_F \mathscr{F}_B$, then the following identity:

$$\mathbf{1}_F :F^c(a^*, a):\mathbf{1}_F \mathscr{F}_F = :F(b^*, b):\mathscr{F}_F \tag{3.32}$$

holds for all operators $:F(b^*, b):$ and $:F^c(a^*, a):$ related by (3.31). As a specialization of this result, one finds at once that the canonical anticommutations relations hold on $\mathscr{F}_F$ for operators $\mathbf{1}_F a(f)^* \mathbf{1}_F$ and $\mathbf{1}_F a(f) \mathbf{1}_F$. The corresponding representation of the CAR is called a projected representation. Notice here that the formal operator expressions for $\mathbf{1}_F a(f)^* \mathbf{1}_F$, $\mathbf{1}_F a(f) \mathbf{1}_F$ respectively, are quite different from these for $b(f)^*$, $b(f)$ and the action of them on vectors from the domain is essential.

In general, if the number of N Bosons $\{a^*_\alpha, a_\alpha\}_{\alpha=0,1,\ldots,N-1}$ is involved, Boson expansions of multifermion operators (3.24)–(3.26) can be, by the use of (3.12), rewritten in the form:

$$b^*_{\alpha_1}\dots b^*_{\alpha_k} \stackrel{a}{=} :\exp(-(a^*,a))\cdot \sum_{n=0}^{N-1}\frac{1}{n!}\sum_{\rho}^{N} a^*_{\alpha_1}\dots a^*_{\alpha_k}\epsilon_{\alpha_1\dots\alpha_k\rho_1\dots\rho_n}\epsilon_{\rho_1\dots\rho_n}a^*_{\rho_1}a_{\rho_1}\dots a^*_{\rho_n}a_{\rho_n}: \tag{3.33}$$

$$b_{\alpha_1}\dots b_{\alpha_k} \stackrel{a}{=} :\exp(-(a^*,a))\sum_{n=0}^{N-1}\frac{1}{n!}\sum_{\rho}^{N} a^*_{\rho_1}a_{\rho_1}\dots a^*_{\rho_n}a_{\rho_n}$$
$$\times \epsilon_{\rho_1\dots\rho_n}\epsilon_{\alpha_1,\dots,\alpha_k\rho_1\dots\rho_n}a_{\alpha_1}\dots a_{\alpha_n}:$$

$$b^*_{\alpha_1}\dots b^*_{\alpha_k}b_{\beta_1}\dots b_{\beta_l} \stackrel{a}{=} :\exp(-(a^*,a))\sum_{n=0}^{N}\frac{1}{n!}\sum_{\rho} a^*_{\alpha_1}\dots a^*_{\alpha_l}$$
$$\times a^*_{\rho_1}a_{\rho_1}\dots a^*_{\rho_n}a_{\rho_n}\cdot\epsilon_{\alpha_1\dots\alpha_k\rho_1\dots\rho_n}\epsilon_{\rho_1\dots\rho_n\beta_1,\dots,\beta_l}a_{\beta_1}\dots a_{\beta_l}:$$

where $\epsilon_{\alpha_1,\dots,\alpha_k\rho_1\dots\rho_n} = \epsilon_{\alpha_k\alpha_{k-1}\dots\alpha_1\rho_1\dots\rho_n}$ and: $\sigma_n(\boldsymbol{kn}) = \Sigma_{\boldsymbol{\alpha}}\epsilon_{\alpha_1\dots\alpha_n}f_{\alpha_1}(k_1)\dots f_{\alpha_n}(k_n)$ with $\{f_\alpha\}_{\alpha=1,2,\dots}$ being the basis in $\mathscr{L}^2(\mathbb{R}^3)$. The symbol $\stackrel{a}{=}$ means the validity of identities under the sign of the bilinear form only (after the integrations with test functions).

By the use of (3.31), we get:

$$b^*_{\alpha_1}\dots b^*_{\alpha_k}\mathscr{F}_F \stackrel{a}{=} \epsilon_{\alpha_1\dots\alpha_k}\mathbf{1}_F a^*_{\alpha_1}\dots a^*_{\alpha_k}\mathbf{1}_F\mathscr{F}_F,$$
$$b_{\alpha_1}\dots b_{\alpha_k}\mathscr{F}_F \stackrel{a}{=} \mathbf{1}_F a_{\alpha_1}\dots a_{\alpha_k}\mathbf{1}_F\epsilon_{\alpha_1,\dots,\alpha_k}\mathscr{F}_F,$$
$$b^*_{\alpha_1}\dots b^*_{\alpha_k}b_{\beta_1}\dots b_{\beta_l} \stackrel{a}{=} \mathbf{1}_F a^*_{\alpha_1}\dots a^*_{\alpha_k}a_{\beta_1}\dots a_{\beta_l}\mathbf{1}_F\epsilon_{\alpha_1\dots\alpha_k}\epsilon_{\beta_1,\dots,\beta_l}\mathscr{F}_F. \tag{3.34}$$

In the above, the complete formulas $\mathbf{1}_F{:}F^c(a^*,a){:}\mathbf{1}_F$ are called the infinite Boson expansions equivalent on $\mathscr{F}_F$ to $:F(b^*,b):$, while operators $:F^c(a^*,a):$ are called the finite Boson expansions corresponding to $:F(b^*,b):$.

Notice that in general $:F^c(a^*,a):$ can appear as the infinite operator series, and the word finite informs us that finite parts only of Boson expansions for b^* and b were used to construct $:F^c(a^*,a):$.

3.2. *Jordan–Wigner representation*

In the many-body problems, people frequently employ the Jordan–Wigner construction of the representation of the CAR algebra, which is based on the use of an infinite family:

$$[\sigma^\pm_k,\sigma^\pm_l]_- = 0 \qquad \text{for } k\neq l \tag{3.35}$$
$$[\sigma^+_k,\sigma^-_k]_+ = \mathbf{1}_F, \qquad (\sigma^+_k)^2 = 0 = (\sigma^-_k)^2$$

of spin $\frac{1}{2}$ raising and lowering operators, see e.g. [100–105].

In practical applications, above part-Boson, part-Fermion nature of operator stands for a difficulty in the theory, because no simple linear transformation between σ^+_k's and σ^-_k's such as would be required to diagonalize a quadratic form (the Hamiltonian), leaves these rules invariant.

However, there is not difficult to transform rules (3.35) into a complete set of the CAR. The famous Jordan–Wigner trick is here in order, where the Fermion creation and annihilation operators appear according to:

$$c_k = \exp\left(i\pi\sum_{j=1}^{k-1}\sigma^+_j\sigma_j\right)\cdot\sigma^-_k$$
$$c^*_k = \exp\left(i\pi\sum_{j=1}^{k-1}\sigma^+_j\sigma^-_j\right)\cdot\sigma^+_k \tag{3.36}$$

where:

$$c_k^* c_k = \sigma_k^+ \sigma_k^- \tag{3.37}$$

and the inverse transformation reads:

$$\begin{aligned} \sigma_k^- &= \exp\Big(i\pi \sum_{j=1}^{k-1} c_j^* c_j\Big) \cdot c_k \\ \sigma_k^+ &= \exp\Big(i\pi \sum_{j=1}^{k-1} c_j^* c_j\Big) \cdot c_k^*. \end{aligned} \tag{3.38}$$

The above method was invented many times since the basic paper [100] appeared. It has been used to change spin operators into Fermions in [102], to change electrons into Bosons with a "hard core" [103], and to change the hard core Bosons into Fermions [104]. See also [119, 120, 123–130]. Above considerations are closely connected with the application of Boson expansion methods of the previous section to so called Jordan–Wigner representation of the CAR algebra [100, 43, 74, 83]. Namely by the use of operators (3.35), we can introduce the following Fermion operators:

$$\begin{aligned} &b_j = \prod_{k=1}^{j-1} (1 - 2\sigma_k^+ \sigma_k^-) \cdot \sigma_j^-, \qquad b_j^* = \prod_{k=1}^{j-1} (1 - 2\sigma_k^+ \sigma_k^-) \cdot \sigma_j^+ \\ &[b_k, b_l^*]_+ \subseteq \delta_{kl} \mathbf{1}_F, \qquad [b_k, b_l]_+ = 0 = [b_k^*, b_l^*]_+. \end{aligned} \tag{3.39}$$

This is a Fock representation if $\sigma_k^- \Omega = 0$ for all $k = 1, 2, \ldots$. If to consider spin operators $\sigma^{\pm}$ as constructed in the Fock representation of the CCR algebra, we have:

$$\sigma_k^- = :\exp(-a_k^* a_k): a_k, \qquad \sigma_k^+ = a_k^* :\exp(-a_k^* a_k): \tag{3.40}$$

where $a_k = \int \mathrm{d}p a(p) \bar{f}_k(p)$ and $\{f_k\}_{k=1,2\ldots}$ is the basis system in $\mathscr{L}^2(\mathbb{R}^3)$. If to notice that in case of $\dim K = 2$ for all $k = 1, 2, \ldots$:

$$\begin{aligned} &\mathbf{1}_F^k = :\exp(-a_k^* a_k) \cdot (1 + a_k^* a_k):, \\ &\mathbf{1}_F^k - 2a_k^* :\exp(-a_k^* a_k): a_k = :\exp(-a_k^* a_k): - a_k^* :\exp(-a_k^* a_k): a_k, \end{aligned} \tag{3.41}$$

we get at once:

$$\begin{aligned} b_j &= \prod_{k=1}^{j-1} [\mathbf{1}_F^k - 2a_k^* :\exp(-a_k^* a_k): a_k] \cdot :\exp(-a_j^* a_j): a_j = :\exp\Big(-\sum_{k=1}^{j} a_k^* a_k\Big) \cdot \prod_{k=1}^{j-1} (1 - a_k^* a_k): a_j, \\ b_j^* &= a_j^* :\exp\Big(-\sum_{k=1}^{j} a_k^* a_k\Big) \cdot \prod_{k=1}^{j-1} (1 - a_k^* a_k): \end{aligned} \tag{3.42}$$

which gives one more example of the CAR induced by the CCR, and is a complete "bosonization" formula for the Jordan–Wigner representation [23].

3.3. Kàlnay solution

Let us now present another approach to the question of Boson expansions of Fermi operators in which, in addition to the Bosons of interest, one introduces a finite number of subsidiary Fermion operators. Original considerations can be found in [40–43, 34] as well as in references listed in these papers.

Let there be given a Fock representation of the CAR algebra realized by an infinite sequence $\{F_i\}_{i=1,2,\ldots}$ of matrices

$$([F_i, F_j^*]_+)_{rt} = \sum_s (F_i)_{rs}(F_j^*)_{st} + \sum_s (F_j^*)_{rs}(F_i)_{st} = (F_iF_j^*)_{rt} + (F_j^*F_i)_{rt} = \delta_{ij}\delta_{rt} = \delta_{ij}\mathbf{1}_{rt},$$

$$[F_i, F_j]_+ = 0 = [F_i^*, F_j^*]_+ = 0, \tag{3.43}$$

$$F_i\Theta = 0 \Rightarrow (F_i\Theta)_r = \sum_s (F_i)_{rs}\Theta_s = 0 \qquad \text{for all } r, i = 1, 2, \ldots$$

where Θ is the vacuum vector for the representation. Let us write $K = \bigotimes_1^N \mathscr{L}^2(\mathbb{R}^3)$. Then K is spanned by a complete orthogonal system $\{g_r^m\}_{r=1,2,\ldots}^{m=1,\ldots,N}$

$$\sum_s \bar{g}_s^n(p)g_s^m(q) = \delta_{nm}\delta(p-q)$$

$$\sum_{m=1}^N \int_{\mathbb{R}^3} \bar{g}_r^m(p)g_s^m(p)\,dp = \delta_{rs}, \qquad p, q \in \mathbb{R}^3. \tag{3.44}$$

Then the following trilinear functions can be associated with matrices F

$$F_i^{mn}(q, p) = \sum_{r,s}(F_i)_{rs}\bar{g}_r^m(q)g_s^n(p)$$

$$F_i^{*mn}(q, p) = \sum_{r,s}(F_i^*)_{rs}\bar{g}_r^m(q)g_s^n(p) = F_i^{nm}(p, q)^*. \tag{3.45}$$

They satisfy the identities:

$$\sum_{n=1}^N \int dk[F_i^{mn}(p, k)F_j^{*nl}(k, q) + F_j^{*mn}(p, k)F_i^{nl}(k, q)] = \delta_{ml}\delta(p-q)\delta_{ij}, \tag{3.46}$$

$$\sum_n \int dk[F_i^{mn}(p, k)F_j^{nl}(k, q) + F_j^{mn}(p, k)F_i^{nl}(k, q)] = 0.$$

Let there be given a Fock representation of the CCR algebra (1.26). By the use of the basis system (3.44) we can consider in the representation space $\mathscr{F}_B$ a continuously indexed set of generators, according to:

$$a_m^*(p) = \sum_r a_r^* g_r^m(p), \qquad a_m(p) = \sum_r a_r\bar{g}_r^m(p) \tag{3.47}$$

where:

$$[a_m(p), a_n^*(q)]_- = \delta(p-q)\delta_{nm}\mathbf{1}_B,$$
$$[a_m(p), a_n(q)]_- = 0 = [a_m^*(p), a_n^*(q)]_-, \tag{3.48}$$
$$a_m(p)\Omega_B = 0 \qquad \text{for all } m, p.$$

By the use of above formulas one can easily check that the operators $\{b_i^*, b_i\}_{i=1,2,\ldots}$ given by:

$$b_i = \sum_{m,n=1}^N \int_{\mathbb{R}^3}\int F_i^{mn}(p, q)a_m^*(p)a_n(q)\,dp\,dq$$

$$b_i^* = \sum_{m,n=1}^N \int_{\mathbb{R}^3}\int F_i^{*mn}(p, q)a_m^*(p)a_n(q)\,dp\,dq \tag{3.49}$$

satisfy:

$$[b_i, b_j^*]_+ = \delta_{ij}\mathbf{1}_B, \qquad [b_i, b_j]_+ = 0 = [b_i^*, b_j^*]_+. \tag{3.50}$$

The identities (3.50) are valid in the one-particle sector B^1 of the Boson Fock space $\mathcal{F}_B$.

Moreover, by the use of the vector $\Theta = \{\Theta_r\}_{r=1,2,\ldots}$ we can construct

$$\Theta^m(p) = \sum_r \Theta_r \bar{g}_r^m(p) \tag{3.51}$$

and further:

$$\Omega_F = \int dp \sum_{m=1}^{N} \Theta^m(p) a_m^*(p)\Omega_B \tag{3.52}$$

the vacuum vector for the representation (3.50), which obviously belongs to B^1: $b_i\Omega_F = 0$ for all $i = 1, 2, \ldots$.

Hence, we have defined the triple $\{b_i^*, b_i, \Omega_F\}_{i=1,2,\ldots}$ which in B^1 defines a Fock representation of the CAR algebra.

In the above we have tacitly assumed that the objects (3.45) are c-numbers. In this connection let us add that we have shown elsewhere [34], that there exists a broad class of coefficients (3.45) which are operator-valued and can be derived through a simple construction from an arbitrary Fock representation of the CAR algebra: the matrix representation is then explicitly constructed from the starting one.

Kalnay's theory thus admits Fock representations of the CAR algebra, which are directly Boson constructed, but indirectly through trilinear functions which can appear as operator-valued, depending on certain starting Fock representation of the algebra, quite different from the derived one.

4. Fermion–Boson reciprocity in quantum field theory

4.1. Relations between Fermions and Bosons

Let us start from the two arbitrary families Q_F and Q_B of Fermion and Boson quantum fields respectively. We can consider the following question: do there exist relations assigning to one or more elements of Q_B the one or more elements of Q_F.

Quite popular recently supersymmetry approach [67, 68] seems to offer an example of the a priori requested relation between Fermions and Bosons. The fields are formed into irreducible supermultiplets, allowing to deduce the conservation laws, which reflect the mentioned relation. This is obviously a result of the supersymmetry requirement and not of the intimate structure of Q_B and Q_F.

Another possibility is concerned with field theories of several quantum fields, obeying the abnormal commutation relations. In that case one introduces so called Klein transformations [69], changing commutation properties of fields under consideration.

The approaches developed in [36, 70, 71, 93] can be summarized in the notion of the fermionization of Bosons program, where all appearing in Nature Boson and Fermion fields are believed to be derivable from a single nonlinear spin $\frac{1}{2}$ "urfeld".

In the framework of the conventional quantum field theory the fermionization program was realized in connection with the Thirring model, see e.g. [47, 116, 117]. Namely, to define a current one

needs a Fock representation of the CCR algebra over $\mathscr{L}^2(\mathbb{R}^1)$ constructed in terms of the Fock representation of the CAR algebra over $\bigotimes_1^2 \mathscr{L}^2(\mathbb{R}^1)$, hence with a doubled number of the internal degrees of freedom in the theory. Quite the converse trend is realized within the Bosonization of Fermions program, where one wishes to get Fermion fields in the field algebras of the given Boson fields. This is the place where the Boson expansion methods are applied.

There was conjectured in [37] that it is more difficult to construct half-integral representations of rotation groups out of the integral than conversely, and it seems to be patiently impossible within the limitations of the polynomial expansion. The author even tried to derive the explicit formulas for the Fermion field in terms of the Boson field, when the number of space-time dimensions was reduced to two. This approach was further analyzed and generalized in [38] with the use of the C*-algebraic techniques what allowed to construct a field algebra of the Boson field, which exhibited the anticommutation rules for certain values of the charge.

Another realization of the program is based on the idea of Boson expansions of Fermion operators; the approaches of previous sections give here the solutions in the indirect way. Namely, the relations between quantum fields appear on the level of the Fock representations of the CCR and CAR algebras, hence free fields. Through the Fourier analysis one goes from the SL(2C) covariant to the SU(2) covariant objects. Because the Haag expansions of interacting fields and scattering operators are power series of normal ordered operator expressions which include the "bare" (free) images of the "dressed" (interacting) field only, one can always express any Fermion field in terms of the appropriate free Boson fields. See e.g. [30, 31, 108–120, 45, 46]. Recently, papers [105–107], threw a new light on the possibility of the Fermion–Boson metamorphosis in gauge theories with the magnetic monopole. Namely, in the SU(2) quantum gauge field theory, with the isospin symmetry broken spontaneously by a triplet of scalar mesons, isospinor degrees of freedom are converted into the spin degrees of freedom under the influence of the magnetic monopole field [105]. Then in the transition formfactor, spin and isospin form an antisymmetric singlet, which implies the nonvanishing of matrix elements of the spinor field between the spinless states.

On the other hand in [106] there was argued that as a consequence of the spin-statistics theorem, in the SU(2) theory of isospin Bosons, in the field of the magnetic monopole, one can get Fermions. In the analogous direction there goes the investigation [107], where an object composed of the spinless electrically charged particle and the spinless magnetically charged particle may bear net half-integer spin, while the two-cluster wave function is symmetric. The study of a relative motion of these clusters proves that this symmetry condition does not violate the spin-statistics theorem. An intuitive Goldhaber's explanation says: perhaps an object whose half-integer spin comes from the charge-pole contribution obeys Fermi–Dirac statistics so that a Fermion can be made out of Bosons.

As a solution to this question there is found that the anomalous relation between the cluster spin and the permutation symmetry of a two-cluster wave function produced by the static fields of charge and pole in a given cluster, is compensated by the anomalous relation between the wave-function symmetry and the quantum numbers which correspond to physical observables. All that follows from the long-range interactions of charges with poles in the different clusters. So, indeed, the two anomalies combine in such a way that Fermions can be made out of Bosons.

4.2. On field theories in the two space-time dimensions

The method of Boson expansions is not a strange concept for people working in the domain of the low temperature description of the (anti)ferromagnetic crystals or the atomic nucleus. There is

however not broadly known its close connection with the so called lattice approximations of Boson (quantum) field theories, see e.g. [119–130] as well as its relation to the investigations of the Fermion–Boson correspondence, especially for the Thirring and Sine–Gordon systems [44–46, 108–120].

The famous Coleman's conjecture [44] on the possible metamorphosis of Fermions into Bosons, results in the statement that for mass zero and the space-time dimension two, the Fock space of a massless Dirac field, contains massless Bose particles. In the study of equivalences between the appearing Fermions and Bosons, the explicit constructions of field operators were performed, see e.g. [45, 46, 109, 114, 115, 116, 118, 126, 130]. An example of the Mandelstam's solution [45] reads:

$$\Psi(x) = \{\Psi_l(x)\}_{l=1,2}$$

$$\Psi_1(x) = (c\mu/2\pi)^{1/2} \exp(\mu/8\epsilon) : \exp\left[-2\pi\, i\beta^{-1} \int_{-\infty}^{x} d\xi \dot{\phi}(\xi) - \tfrac{1}{2}\beta\phi(x)\right] :$$

$$\Psi_2(x) = -i(c\mu/2\pi)^{1/2} \exp(\mu/8\epsilon) : \exp\left[-2\pi\, i\beta^{-1} \int_{-\infty}^{x} d\xi \dot{\phi}(\xi) + \tfrac{1}{2} i\beta\phi(x)\right] : \tag{4.1}$$

where $\phi(x)$ satisfies the so called quantum Sine–Gordon equation:

$$x = (x, t)$$

$$\Box\phi(x) = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi(x, t) = (\mu^2/\beta) : \sin[\beta\phi(x, t)] : \tag{4.2}$$

and the canonical commutation relations, while $\Psi(x)$ is proved to be a Fermion, which under suitable restrictions becomes a Fermion of the massive Thirring model.

There is instructive to know that the introduced so Fermion–Boson correspondence results also in the equivalence of the Thirring and Sine–Gordon models Hamiltonians, see e.g. [46, 115]. In [115] it is shown that any interacting spinor system in the two space-time dimensions can be equivalently described by the scalar system: both theories have a common Hamiltonian. The correspondence of this kind is further extended on the case of the vector-spinor systems and the Yukawa interacting systems, which both can be related to the Sine–Gordon model.

The transition from Fermions to Bosons in the two-dimensional quantum field theory follows from the fact that the Fock space of the free massless Fermion field contains in every charge sector mass zero bound states, so that the two Fermion bound states are connected with the massless Boson field. This result was at the roots of the neutrino theory of light [93]. How to get Bosons from the two-component Fermions see also [47].

Let us add that the structure of the Fermion Fock space, with special respect to the Fermion–Boson reciprocity was studied in detail, in the Uhlenbrock's papers [116, 117]. The transition from Boson's to Fermions is realized by the use of the exponentiated field. Here, we shall only mention an interesting feature of this approach resulting in the so called Kronig identity between the Hamiltonians of the free massless Fermion and Boson fields:

$$H_F = H_B + \tfrac{1}{2}(Q_+^2 + Q_-^2) \tag{4.3}$$

with Q_+, Q_- being the suitably defined charge operators. In case of the Thirring model the authors get the generalization of the Kronig identity in which the renormalized Hamiltonians H_F^∞ and H_B^∞ appear in the place of H_F and H_B.

In the massive case, an analogous identity can be proved if to use the concept of the dressing transformation. It allows to reformulate the physical Hilbert space problem in the Fock space, by the use of pure Boson operators. Denoting d, the scaling dimension, n to be an index of the charge sector, and m denoting the mass, we get:

$$H_F(d, m, n) = H_B(d, m, n) + \tfrac{1}{2}(Q_+^2 + Q_-^2). \tag{4.4}$$

Some efforts are connected with the "choice of quasi-local Fermi fields instead of quasi-local Bose fields so that they are interpolating fields for the same one-particle states, whose corresponding multi-particle states will obey either Fermi or Bose statistics" [114]. The special notion of "schizon" was introduced in this connection. Then, if starting from Bosons:

$$[a(p), a^*(q)]_- = p_0\delta(p-q), \qquad [a(p), a(q)]_- = 0 \tag{4.5}$$

the authors define Fermions by:

$$b^*(p) = a^*(p)\exp\left(-i\pi\int_p^\infty n(q)\,dq\right) \tag{4.6}$$

$$n(q) = (1/q)a^*(q)a(q).$$

Note here a close analogy with the Jordan–Wigner trick. In addition to pure Boson approach to Fermions, it is useful to mention the study of the Fermion–Boson correspondence which is performed on the level of Lagrangeans for simplest models, but for the price of introducing the additional, quite formal (elements of the Grassmann algebra) degrees of freedom being the reminiscent of the supersymmetry approach. They make the Boson constructed fields to anticommute, see e.g. [112, 113].

5. Quantum fields on the spatial lattice: towards the Heisenberg crystal

5.1. Lattice approximation of the Thirring model

The Hamiltonian of the massive Thirring model on a one-dimensional lattice with spacing a and $N = 2r$ sites is given by [119, 120]:

$$H = \sum_{n=-r+1}^{r}\left\{\frac{i}{2a}V(G)\cdot(\phi_n^+\phi_{n+1} - \phi_{n+1}^+\phi_n) + (-1)^n\frac{m_0}{2}(\phi_n^+\phi_{n+1}^+ + \phi_{n+1}\phi_n)\right.$$

$$\left. -\frac{G}{2a}(\phi_n^+\phi_n - \tfrac{1}{2})(\phi_{n+1}^+\phi_{n+1} - \tfrac{1}{2})\right\} - E_0. \tag{5.1}$$

The ϕ_n's are Fermion operators $[\phi_n, \phi_m^+]_+ = \delta_{nm}$ and $\phi_{r+1} = \phi_{-r+1}$. G is a renormalized coupling constant, and the $V(G)$ is the finite renormalization constant needed to make the speed of light equal to unity. Up to the first order in G, we have $V(G) = 1 + (G/\chi)$, m_0 denotes the bare mass and E_0 is included to make the ground state energy vanish: m_0 depends on the lattice constant a.

With the help of the Jordan–Wigner trick we can relate the above model to the Heisenberg spin-chain

problem, the XYZ-model with the Hamiltonian

$$H_{XYZ} = -\tfrac{1}{2} \sum_{k=-r+1}^{r} \{J_X \sigma_k^x \sigma_{k+1}^x + J_Y \sigma_k^y \sigma_{k+1}^y + J_Z \sigma_k^z \sigma_{k+1}^z\} \tag{5.2}$$

periodic boundary conditions are implied.

The appropriate version of the Jordan–Wigner trick reads:

$$\phi_k^+ = \exp\{i(\pi/4)(N+1)\} \cdot \sigma_k^+ \prod_{j=-r+1}^{r} (i\sigma_j^z)$$
$$\sigma_k^+ = \tfrac{1}{2}(\sigma_k^x + i\sigma_k^y), \tag{5.3}$$

so that, under this transformation:

$$H = H_{XYZ} + \tfrac{1}{2}(1 + (-1)^{r+F}) \cdot \{J_x \sigma_r^x \sigma_{-r+1}^x + J_y \sigma_r^y \sigma_{-r+1}^y\} + \text{const.} \tag{5.4}$$

with:

$$J_x = \frac{V}{2a} + \frac{m_0}{2}, \qquad J_y = \frac{V}{2a} - \frac{m_0}{2}, \qquad J_z = \frac{G}{4a}$$
$$(-1)^F = \prod_{k=-r+1}^{r} \sigma_k^z = \exp\left\{i\pi \sum_{k=-r+1}^{r} \phi_k^+ \phi_k\right\}. \tag{5.5}$$

Here $(-1)^F$ commutes with H_{XYZ} and H. It proves that in the sector $(-1)^F = (-1)^{r+1}$ we have $H = H_{XYZ}$, while if $(-1)^F = (-1)^r$, H equals to another Hamiltonian $\tilde{H}_{XYZ}$ given by the Heisenberg ferromagnet formula with the anticyclic boundary conditions: $\sigma_{r+1}^x = -\sigma_{-r+1}^x$, $\sigma_{r+1}^y = -\sigma_{r+1}^y$, $\sigma_{r+1}^z = \sigma_{-r+1}^z$. Unfortunately for $J_z = 0$, which corresponds to the free model, the three Hamiltonians H_{XYZ}, $\tilde{H}_{XYZ}$ and H have different spectra, what slightly spoils the received equivalence. Here the strong dependence of the theory, while formulated in the Heisenberg language, on the choice of the boundary conditions, should be emphasized.

If to introduce the projection operators:

$$P_+ = \tfrac{1}{2}(1 - (-1)^{r+F}), \qquad P_- = \tfrac{1}{2}(1 + (-1)^{r+F})$$

we have

$$[H_{XYZ}, P_\pm]_- = [\tilde{H}_{XYZ}, P_\pm]_- = [H, P_\pm]_-$$

so that

$$H = P_+ H_{XYZ} + P_- \tilde{H}_{XYZ} + \text{const.}$$

and hence:

a) P_+ resp. P_- is the projection on the subspace with an even resp. odd Fermion number above the physical ground state,

b) for large N the spectra of $P_+ H_{XYZ}$ and $P_- H_{XYZ}$ are the same, what partly solves the mentioned spectrum problem.

5.2. From Bosons to Heisenberg antiferromagnet

Let us consider again the spatial lattice with spacing a and the integer lattice label n. If, with each

lattice site we have associated a complex scalar field φ_n satisfying the canonical commutation relations:

$$[\varphi_n^+, \varphi_m]_- = \delta_{nm}, \qquad [\varphi_n, \varphi_m]_- = 0$$
$$\varphi_n \Omega_B = 0 \qquad \text{for all } n = 1, 2, \ldots \tag{5.6}$$

then in the corresponding Fock space we can construct the Fermion unit operator where in the place of creation and annihilation operators a^*, a we put respectively: $\varphi^+, \varphi, \mathbf{1}_F(a^*, a) \Rightarrow \mathbf{1}_F(\varphi^+, \varphi)$. By the transformation: $\mathbf{1}_F \varphi_n \mathbf{1}_F = \phi_n$ we have associated with each lattice site a single component Fermion field ϕ_n:

$$[\phi_n, \phi_m^+]_+ \subseteq \delta_{nm} \mathbf{1}_F, \qquad [\phi_n, \phi_m]_+ = 0. \tag{5.7}$$

Let us consider the Hamiltonian:

$$\mathbf{1}_F H_B \mathbf{1}_F = H_F = \frac{i}{2a} \sum_n \{\phi_n^+ \phi_{n+1} - \phi_{n+1}^+ \phi_n\}, \tag{5.8}$$

compare in this connection (5.1). We have:

$$i[H, \phi_n]_- = \dot{\phi}_n = \frac{\phi_{n+1} - \phi_{n-1}}{2a} = \frac{\Delta \phi_n}{\Delta x} \tag{5.9}$$

so that the time dependence of ϕ_n at even (odd) sites is determined by the spatial difference of ϕ_{n+1} at odd (even) sites. Following the prescription of [127], let us define a two-component field ψ_m:

$$\psi_n = \begin{pmatrix} \phi_e \\ \phi_o \end{pmatrix}_n, \qquad \begin{matrix} \phi_e = \phi_n, & n \text{ even} \\ \phi_o = \phi_n, & n \text{ odd} \end{matrix} \tag{5.10}$$

the components of ψ_n satisfy:

$$\dot{\phi}_0 = \Delta \phi_e / \Delta x, \qquad \dot{\phi}_e = \Delta \phi_0 / \Delta x \tag{5.11}$$

which, at the continuum limit becomes the massless Dirac equation: $(\partial/\partial t)\psi = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \partial\psi/\partial x$ in the standard representation: $\gamma_o = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. By the use of the Jordan–Wigner trick:

$$\psi_n = \prod_{l<n} \{i\sigma_l^3\} \cdot \sigma_n^-$$
$$\psi_n^+ = \prod_{l<n} \{-i\sigma_l^3\} \cdot \sigma_n^+ \tag{5.12}$$

the one-dimensional Fermion problem can be rewritten as the one-dimensional spin problem: σ_n^l is the spin matrix at the nth site, $l = 1, 2, 3 \ldots$ so that:

$$H_F = \frac{1}{2a} \sum_n \{\sigma_n^+ \cdot \sigma_{n+1}^- + \sigma_{n+1}^+ \cdot \sigma_n\} \tag{5.13}$$

what describes the XY antiferromagnetic chain.

As we know from previous considerations each of Pauli operators can be bosonized so that one can once more state the question of the Boson translation of the same (see (5.8)) physical problem.

5.3. *Variations on ϕ_2^4*

The ϕ_2^4 quantum field theory is given by the Hamiltonian:

$$H = \int dx \{\tfrac{1}{2}(\partial\phi/\partial t)^2 + \tfrac{1}{2}(\partial\phi/\partial x)^2 + \lambda(\phi^2 - f^2)^2\}, \qquad f^2 > 0. \tag{5.14}$$

Its lattice approximation on the linear lattice with the dimension L, spacing $a = 1/\Lambda$, $L = (2N+1)/\Lambda$, $V = L$ is described by the Hamiltonian:

$$H = \frac{1}{\Lambda}\sum_j \{\tfrac{1}{2}\pi_j^2 + \tfrac{1}{2}(\nabla\phi_j)^2 + \lambda(\phi_j^2 - f^2)^2\} \tag{5.15}$$

where the gradient term is chosen in the form [123, 124, 128]:

$$\frac{1}{\Lambda}\sum_j \tfrac{1}{2}(\nabla\phi_j)^2 = \frac{V}{2}\sum_k k^2\phi(k)\phi(-k) = \sum_{j,j'}\frac{\Lambda}{2}\phi_{j,j'}\cdot D(j-j') \tag{5.16}$$

with:

$$D(j-j') = \frac{1}{2N+1}\sum_{k=-k_{\max}}^{k_{\max}}\left\{\frac{k^2}{\Lambda^2}\exp[\mathrm{i}(j-j')k/\Lambda]\right\}. \tag{5.17}$$

After rescaling:

$$\begin{aligned} &x_j = \Lambda^{-1/2}\phi_j, \qquad p_j = \Lambda^{-1/2}\pi_j \Rightarrow [p_j, x_k]_- = -\mathrm{i}\delta_{jk}, \\ &\lambda_0 = \lambda\Lambda^{-2}, \qquad f_0^2 = f^2 \end{aligned} \tag{5.18}$$

we get:

$$H = \Lambda\sum_j\left\{\tfrac{1}{2}p_j^2 + \tfrac{1}{2}(\nabla x_j)^2 + \lambda_0(x_j^2 - f_0^2)^2\right\}. \tag{5.19}$$

The lattice version of the starting quantum model is obviously an approximation of the continuous one. However, further approximations can be made within the lattice formulation.

Let us notice that in (5.19) the different single-site terms:

$$H_{ss} = \Lambda\sum_j\{\tfrac{1}{2}p_j^2 + \lambda_0(x_j^2 - f_0^2)^2\} \tag{5.20}$$

are coupled by the gradient terms only, which in fact carry interactions in the model. If to neglect the gradients, we receive the so called single-site approximation, where at each site we have the identical Schroedinger problem of a particle in an anharmonic potential. In this approximation, the eigenstates of the Hamiltonian, are formed by the product of single site eigenstates. The lowest energy eigenstate is $|\psi_o) = \Pi_j|\psi_o)_j$ where $|\psi_o)_j$ is the jth site ground state. The next energy level is achieved if at most one of oscillators is in its first excited state; huge degeneracy appears here.

Introducing the annihilation and creation operators at each site j:

$$\begin{aligned} &x_j = \frac{1}{\sqrt{2\alpha_j}}(a_j + a_j^*), \qquad \mathrm{i}p_j = \sqrt{\frac{\alpha_j}{2}}(a_j - a_j^*) \\ &[a_j, a_k^*]_- = \delta_{jk} \end{aligned} \tag{5.20}$$

we can get the most general, in the single-site approximation, state function of the system:

$$\begin{aligned} &|\psi) = \prod_j |\psi_j), \qquad (\psi_j \mid \psi_k) = \delta_{jk} \\ &|\psi_j) = \prod_{n_j=0}^{\infty} c_{n_j}|n_j). \end{aligned} \tag{5.21}$$

In the so received single site basis, one can approximately calculate the energy of the original, interacting, system:

$$H = \Lambda\Big\{\sum_j H_{ss}(j) + \tfrac{1}{2}\sum_{j_1\neq j_2} D(j_1 - j_2)x_{j_1}x_{j_2}\Big\} \tag{5.22}$$

so that its expectation value in the trial state $|\psi)$:

$$(\psi|H|\psi) = H_\psi = \Lambda\Big\{\sum_j (\psi_j|H_{ss}|\psi_j) + \tfrac{1}{2}\sum_{j_1\neq j_2} D(j_1 - j_2)(\psi_{j_1}|x_{j_1}|\psi_{j_1})(\psi_{j_2}|x_{j_2}|\psi_{j_2})\Big\}\ . \tag{5.23}$$

In case of the translation invariant ground state, needing $\Sigma_{j_1} D(j_1 - j_2) = 0$ so that $\Sigma_{j_1\neq j_2} D(j_1 - j_2) = -\Sigma_j D(0) = -LD(0)$, implies:

$$E_0(\psi) = H_\psi = \Lambda \cdot L\{(\psi|H_{ss}|\psi) - \tfrac{1}{2}D(0)(\psi|x|\psi)^2\}. \tag{5.24}$$

Another kind of approximation of the starting Boson theory is the finite spin approximation. Let us discuss it for the case of ϕ_2^4 theory with the addition of the nearest neighbor coupling term:

$$H = \Lambda\sum_j \{\tfrac{1}{2}P_j^2 + \tfrac{1}{2}(\mu^2 + 2)x_j^2 + \lambda x_j^4 - x_j x_{j+1}\}. \tag{5.25}$$

The periodic boundary conditions are assumed.

In each single site, the corresponding single-site term describes the anharmonic quantum oscillator, whose solution as the Schroedinger problem:

$$(\tfrac{1}{2}p^2 + \tfrac{1}{2}(\mu^2 + 2)x^2 + \lambda x^4)|n) = E_n|n) \tag{5.26}$$

by the use of the basis $\otimes_j|n_j)$, $0 < n_j < \infty$ allows to write the Hamiltonian (5.25) in the matrix form, H being dimensionless $H \equiv H/\Lambda$:

$$H = \sum_j (E^j - X^j \otimes X^{j+1}). \tag{5.27}$$

Here E is the diagonal matrix $\{E_j\}$ consisting of the single site energy values. X has the nonvanishing matrix elements between the even and odd parity states.

The finite spin approximation is received by truncating the base to a finite number S of levels at each site: $\otimes_j|n_j)$, $0 \leq n_j \leq S - 1$ so that the truncated Hamiltonian represents the coupled spin s system with $2s + 1 = S$. Notice that the approximation is base dependent. As a particular example one can consider the spin $\frac{1}{2}$ approximation: $S = 2$ what means that the two low lying states only of the single site Schroedinger problem are retained now. This approximation is reasonable under the existence of the external regulation mechanism (low temperatures, the strong coupling, eq. the weak excitation limits) forbidding the occupation of higher energy levels: the probability that the system is excited to higher than the lowest two, energy levels, is negligibly small.

The Hamiltonian matrix acquires the form:

$$H = \text{const.} + \sum_j \Big\{\frac{\epsilon}{2}\sigma_j^z - \Delta\cdot(\sigma_j^+ + \sigma_j^-)(\sigma_{j+1}^+ + \sigma_{j+1}^-)\Big\} \tag{5.28}$$

where, see e.g. [123, 129] $\epsilon = (E_1 - E_0)$, $\Delta = |(0|x|1)|^2$, $\sigma_{n+1} \equiv \sigma_{-n}$ and σ's are the ordinary Pauli matrices.

Since only the two states are considered for each oscillator, they can be represented by the presence or absence of a Fermion, what motivates the translation of H to the pure Fermion language:

the Jordan–Wigner trick:

$$\sigma_j^+ = \prod_{l=-N}^{j-1} (-1)^{n_l} b_j^*, \qquad n_l = b_l^* b_l, \qquad n = \sum_l^N n_l$$

$$\sigma_j^- = \prod_l^{j-1} (-1)^{n_l} b_j, \qquad \sigma_j^z = 2n_j - 1$$

allows to derive:

$$H = LE_0 + \epsilon \sum_{j=-N}^{N} b_j^* b_j - \Delta \sum_{j=-N}^{N} (b_j^* - b_j)(b_{j+1}^* + b_{j+1}) + \Delta(b_N^* - b_N)(b_{-N}^* + b_{-N})(\exp(i\pi n) + 1) \tag{5.30}$$

what finally enabled us to associate with the starting Boson system the corresponding lattice Fermion. One can here obviously state the question whether (5.30) can be further connected, in the sense of the lattice approximation, with some continuous (quantum field theory) Fermion system.

6. Heisenberg ferromagnet in the low temperature limit

6.1. Spin $\frac{1}{2}$ approximation in quantum mechanics

Let us consider an elementary quantum system in one dimension. Such a system is completely determined by an irreducible pair $\{P, Q\}'' = \lambda \mathbf{1}_B$, $\lambda \in \mathbf{0}^1$ of the momentum and position operators, which are selfadjoint in the suitable Hilbert space $\mathcal{F}_B$.

We assume the quantum motion of the system to be governed by the Hamiltonian H, whose complete eigenfunction system $\{f_n\}_{n=0,1,\ldots}$ $Hf_n = E_n f_n$ spans $\mathcal{F}_B$. We have then given the operators $a = (1/\sqrt{2})(Q + iP)$, $a^* = (1/\sqrt{2})(Q - iP)$ so that $af_0 = 0$, and $f_n = (1/\sqrt{n!})a^{*n} f_0$. Hence, the triple $\{a^*, a, f_0\}$ generates a Fock representation of the CCR in $\mathcal{F}_B$. In accordance with considerations of sections 1, 3.1, in a two-dimensional subspace $\mathcal{F}_F$ of $\mathcal{F}_B$, which is spanned by vectors f_0, f_1, we have given a Fock representation of the CAR algebra $\{b^*, b, f_0\}$ with:

$$b = \mathbf{1}_F a \mathbf{1}_F, \qquad b^* = \mathbf{1}_F a^* \mathbf{1}_F \tag{6.1}$$

and furthermore, an infinitesimal generator $\boldsymbol{S}$ of the irreducible in $\mathcal{F}_F$ representation of the SU(2) group: $\sqrt{2}S^+ = b^*$, $\sqrt{2}S^- = b$, $S_3 = (-1/2)\mathbf{1}_B + b^* b = \mathbf{1}_F[-1/2 + a^* a]\mathbf{1}_F$ where: $\boldsymbol{S}^2 = \frac{3}{4}\mathbf{1}_F$, $S_3 f_0' = -\frac{1}{2} f_0$, $S_3 f_1 = \frac{1}{2} f_1$. Notice that $S_3 = \frac{1}{2}\mathbf{1}_F - b^* b$ implies: $S_3 f_0 = \frac{1}{2} f_0$, $S_3 f_1 = -\frac{1}{2} f_1$.

In consequence, the vectors f_0, f_1 are the common eigenvectors of the three operators: $H, \boldsymbol{S}^2, S_3$. Therefore, if the description of our elementary quantum system can be restricted to $\mathcal{F}_F$ only, we can in principle characterize it fully by a complete family $\{H_F, \boldsymbol{S}^2, S_3\}$ of the commuting in $\mathcal{F}_F$ observables: $H_F = \mathbf{1}_F H \mathbf{1}_F$. The only thing is to disclose the physical conditions, under which such a restriction is possible. It is obviously nonrealizable for the isolated system. Let us therefore assume it to be in contact with a suitable, low-temperature environment (reservoir). In such case, any pumping of the system to energies exceeding E_1 can be made negligibly probable if compared with this to produce either E_0 or E_1. Then, the description of any elementary quantum system, with the good accuracy can be reduced to $\mathcal{F}_F$ only, where a complete family of observables $\{H_F, \boldsymbol{S}^2, S_3\}$ is given.

Another mechanism of this kind can be the strong coupling potential forbidding the system to

occupy higher energy levels. This phenomenon seems to be of special importance in the interacting many-particle systems.

The above considerations can be summarized in the following conjecture: the quantum motion of a one-dimensional spinless system, which is governed by the Hamiltonian H, provided the system is in contact with the low-temperature environment, in the weak excitation limit perfectly simulates the internal spin $\frac{1}{2}$ quantum motion of a spinning object, in its own reference rest frame.

Furthermore, we can treat each spinless elementary quantum system as a superposition of the Bosonic and Fermionic "phases". If higher excitations are allowed, the Bosonic one prevails, however in the weak excitation limit the Fermionic one becomes prevailing.

One can even try to establish certain critical temperature of the reservoir, beginning from which our Boson can be with a good accuracy considered as the spin $\frac{1}{2}$ Fermion. Things would become still more exciting if there would exist a large energy gap between E_1 and higher energy levels of the Hamiltonian. In that case there would be even possible to get the highly stable Fermionic "phase" of our elementary quantum system, in may be large range of energies.

6.2. Bosonization of Fermions on the isotropic lattice

We are interested in the special class of the quantum spin systems called the isotropic Heisenberg models, whose specialized case is the famous Ising model. Let the isotropic spin lattice consist of the number N of equivalent sites, each one occupied by identical atoms: each one with spin s and the magnetic moment μ. With the lth site of the lattice we associate the lth copy $\mathcal{F}_l$ of the finite dimensional Hilbert space $\mathcal{F}$, $\dim \mathcal{F} = 2s+1$, $s = 0, \frac{1}{2}, \ldots$ together with an irreducible unitary representation of the SU(2) group, with the infinitesimal generators $\{\boldsymbol{S}_k\}_{k=1,2,\ldots}$ satisfying:

$$[S_{ka}, S_{lb}]_- = \mathrm{i}\epsilon_{abc} S_{kc} \cdot \delta_{kl} \tag{6.2}$$

where $a, b, c = 1, 2, 3$, (or x, y, z), $\boldsymbol{S}_k^2 = s(s+1)\mathbf{1}_k$ and $\mathbf{1}_k$ is a unit operator in $\mathcal{F}_k$.

We denote $I_{kl} = I(|k-l|)$, $I_{kk} = 0$, $k, l = 1, 2, \ldots, N$, the exchange integral of the lattice, and $\mathcal{H} = (0, 0, \mathcal{H})$ is the magnetic field oriented along the z-axis of the reference rest frame. Then the general Hamiltonian of the lattice [9], reads:

$$H = G_0 - \mu \sum_{k=1}^{N} \mathcal{H} \cdot \boldsymbol{S}_k - \tfrac{1}{2} \sum_{k,l=1}^{N} I_{kl} \boldsymbol{S}_k \boldsymbol{S}_l \tag{6.3}$$

and can be further written in the form:

$$\begin{aligned} &S_k^{\pm} = S_k^x \pm \mathrm{i} S_k^y, \\ &H = E_0 + H_2 + H_4 \\ &E_0 = -N\mu\mathcal{H} \cdot s - \tfrac{1}{2} N s^2 J(0) \\ &H_2 = (\mu\mathcal{H} + sJ(0)) \sum_k (s - S_k^z) - \tfrac{1}{2} \sum_{k,l} I_{kl} S_k^+ S_l^- \\ &H_4 = -\tfrac{1}{2} \sum_{k,l} I_{kl} (s - S_k^z)(s - S_l^z) \end{aligned} \tag{6.4}$$

where $j(\nu) = \Sigma_k I(k) \exp(\mathrm{i}k\nu)$. Moreover, to be in agreement with commonly accepted in solid state physics notation $S^{\pm}$ must be identified with the notion $\sqrt{2}S^{\pm}$, of previous sections.

The description of the Heisenberg ferromagnet in the low temperature limit (Curie point) involves

Bosons in the basic formalism [48, 49]. Namely, in that case, a collection of the ideal spin waves constituting the free magnon gas, is believed to simulate perfectly a behaviour of the crystal: transition probabilities for the ideal spin wave processes with a good accuracy approximate these for a real system. A conventional computational tool in this place is the use of formal substitutions as e.g. these of Holstein and Primakoff or of Dyson and Maleev. They allow to consider in the place of (6.2) a pure Boson Hamiltonian of the ideal spin waves defined in the appropriate space of spin states.

The use of the Holstein–Primakoff prescription for spin values $s \gg 1$ allows to consider H in the form:

$$H \to H_{\mathrm{B}} = E_0 + H_2 + H' + H'', \tag{6.5}$$

$$H_2 = (\mu\mathcal{H} + sJ(0)) \sum_k n_k - \sum_{k,l} sI_{kl} a_k^* a_l,$$

$$H' = -\tfrac{1}{2} \sum_{k,l} I_{kl} n_k n_l,$$

$$H'' = \tfrac{1}{4} \sum_{k,l} I_{kl}(1 + (1/8s))(a_k^{*2} a_k a_l + a_k^* a_l^{*} a_l^2)$$

$$+ \sum_{k,l} I_{kl}(1/32s)(a_k^{*3} a_k^2 a_l + a_k^* a_l^{*2} a_l^3) - \sum_{kl} I_{kl}(1/16s) a_k^{*2} a_l^* a_l^2 a_k + \mathrm{O}(s^{-2})$$

while by the use of the Dyson–Maleev prescription we get:

$$\begin{aligned} S_k^+ &\longrightarrow \sqrt{2s}\, a_k^* \\ S_k^- &\longrightarrow \sqrt{2s}(1 - a_k^* a_k/2s) a_k \\ S_k^z &\longrightarrow s - a_k^* a_k = s - n_k. \end{aligned} \tag{6.6}$$

In both cases H_2 is interpreted to describe the noninteracting spin waves, H' their dynamical interaction, H'' being responsible for the kinematical corrections. H' and H'' appear here as small perturbations of $H_2 + E_0$.

In the above the condition $s \gg 1$ automatically excludes from considerations lowest spin lattices, therefore it seems reasonable to apply here a rigorous approach of section 3.1.

Let the triple $(a^*, a, \Omega_{\mathrm{B}})_K$ generate a Fock representation of the CCR algebra over the complex separable Hilbert space $K = \bigoplus_{l=1}^{\infty} K_l$ with $\dim K_l = n$ for all $l, n = 1, 2, \ldots$.

The basis system in K we denote $\{f_{l\alpha}\}_{l=1,2\ldots}^{1 \leqslant \alpha \leqslant n}$ so that the indexation $a(f_{l\alpha}) = a_{l\alpha}$ of modes, induces:

$$\begin{aligned} &[a_{l\alpha}, a_{k\beta}^*]_- = \delta_{kl}\delta_{\alpha\beta} \mathbf{1}_{\mathrm{B}} \\ &[a_{l\alpha}, a_{k\beta}]_- = 0, \qquad a_{l\alpha}\Omega_{\mathrm{B}} = 0 \qquad \text{for all } l, \alpha. \end{aligned} \tag{6.7}$$

The underlying Fock space we denote $\mathcal{F}_{\mathrm{B}}$.

Let us now assume to have defined the subsidiary Boson lattice consisting of a finite (large) number of identical cells: $l = 1, 2, \ldots N$, each one occupied by the n-mode cluster: $\alpha = 1, 2, \ldots, n$ enumerating the components of the cluster. We define the following operators:

$$\begin{aligned} \mathcal{S}_k^+ &= \mathcal{S}_l^x + \mathrm{i}\mathcal{S}_l^y = \sum_{\alpha=1}^{n} a_{l\alpha}^* \\ \mathcal{S}_l^- &= \mathcal{S}_l^x - \mathrm{i}\mathcal{S}_l^y = \sum_{\alpha=1}^{n} a_{l\alpha} \\ \mathcal{S}_l^z &= -(n/2) + (a^*, a)_l \end{aligned} \tag{6.8}$$

with $(a^*, a)_l = \Sigma_{\alpha=1}^{n} a_{l\alpha}^* a_{l\alpha}$ and introduce the Hamiltonian of our lattice Boson: notations are taken from the above, and $I_{kk} = 0$ for all k,

$$H_B = G_0 - \mu \sum_{k=1}^{N} \mathscr{H}\mathscr{S}_k - \tfrac{1}{2} \sum_{k,l=1}^{N} I_{kl}\mathscr{S}_k\mathscr{S}_l. \tag{6.9}$$

This operator is well defined in the suitably chosen domains in $\mathscr{F}_B$. If to admit a contact of the subsidiary Boson lattice with an appropriate thermostat (reservoir), one can imagine the regulation mechanism for the excitation level of the lattice, through the raising or lowering of the temperature.

Temperature changes influence the structure of the set of transition probabilities between lattice states: certain transitions become more probable than the others.

Let us choose a discrete set $\{T_s\}_{s=0,1,\ldots}$ of points along the temperature scale, each one with a corresponding neighbourhood ΔT_s. The separation intervals between the neighbouring points are assumed to be sufficiently large if compared with the corresponding ΔT's. With each discrete T, let us associate a projector $\hat{P}_T$ with the property: .

$$\hat{P}_T\mathscr{F}_B = \mathscr{F}_T \tag{6.10}$$
$$\text{prob}[\mathscr{F}_B] = 1, \qquad \text{prob}[\Omega_B] = 0$$

$1 > \text{prob}[\mathscr{F}_T] \gg \text{prob}[\mathscr{F}_B \backslash \mathscr{F}_T]$ within the interval ΔT where the notion $\text{prob}[\mathscr{F}_T]$ denotes a probability with which the transitions between lattice states from $\mathscr{F}_T$ are realized inside $\mathscr{F}_B$ on the chosen temperature range $T \pm \Delta T/2$. To remove the arbitrariness connected with the relation $\gg$, one can try to associate temperature values with concrete probability values, as e.g. $\text{prob}[\mathscr{F}_B \backslash \mathscr{F}_T] \leqslant 0.001$, say. In that case, the Boson lattice, with a good accuracy can be described in terms of states from $\mathscr{F}_T$ and no necessity to consider the whole of $\mathscr{F}_B$ appears.

By virtue of considerations of section 3.1, for certain critical temperature T_0 ($T_0 \ll T_C$) and a corresponding temperature range ΔT, there exists a projector $\hat{P}_0$ such that within the interval $T_0 \pm \Delta T/2$ the following operator identity:

$$\hat{P}_0 H_B \hat{P}_0 = H = G_0 - \mu \sum_{k=1}^{N} \mathscr{H}\boldsymbol{S}_k - \tfrac{1}{2} \sum_{k,l=1}^{2} I_{kl}\boldsymbol{S}_k\boldsymbol{S}_l \tag{6.11}$$

holds on the Hibert space of spin states $\mathscr{F}_0 = \hat{P}_0\mathscr{F}_B$ which is a finite dimensional subspace of $\mathscr{F}_B$.

Furthermore, each of operators: $\boldsymbol{S}_k = \hat{P}_0\mathscr{S}_k\hat{P}_0$ is an infinitesimal generator of the reducible on $\mathscr{F}_0$ representation of the SU(2) group, whose irreducible components induce a corresponding splitting of $\mathscr{F}_0$ into a direct sum of suitable spin spaces. They are parametrized by the spin values associated with each of the N sites of the Heisenberg lattice.

A few comments are now in order:

The identity (6.11) establishes a connection of the subsidiary Boson lattice with the isotropic lattice in the weak excitation limit. It clearly exhibits the limitations (projectors $\hat{P}_0$) under which the use of Boson expansions in the theory is justified. Formally one can consider a straightforward equivalence relation between H_B and H. However in that case the essential domain questions arise. H_B appears here as the finite Boson expansions corresponding to the infinite expansion for H.

The cluster structure of the subsidiary Boson lattice, introduced by us as the limitation of the theory, plays an essential rôle. Namely, in case of $\dim K_l = 1$ for all l, we get the spin $\frac{1}{2}$ lattice. If $\dim K_l = 2$ for all l, then $\mathscr{F}_0$ can be reduced and with each lattice site one can associate either spin 1 or spin 0 quantum excitation.

Our main task now is to construct explicitly the operator P_0. Let us notice that Ω_B is a common ground state for all clusters of the subsidiary Boson lattice. We denote $\mathscr{F}_B^k$ a Fock space associated with the kth cluster, and hence determined by the triple:

$$\{a_{k\alpha}^*, a_{k\alpha}, \Omega_B\}_{1\leqslant\alpha\leqslant n}.$$

From now on we assume to have allowed the spin $\frac{1}{2}$ approximation of our quantum system, so that the two lowest energy states corresponding to each single normal mode of the system (including the single components of the clusters) are essential. Then with each single degree of freedom we have associated the Fermion:

$$b^* = a^* : \exp(-a^*a):, \qquad b = :\exp(-a^*a): a, \qquad \mathbf{1}_F = :\exp(-a^*a)\cdot[1 + a^*a]:,$$

and furthermore the induced spin $\frac{1}{2}$ operator $\boldsymbol{S}$.

If further to consider the n-mode cluster, and repeat above consideration for each single component, then the operator

$$\boldsymbol{S} = \sum_{\alpha=1}^{n} \boldsymbol{S}_\alpha \tag{6.12}$$

defined by:

$$S^+ = \sum_\alpha b_\alpha^*, \qquad S^- = \sum_\alpha b_\alpha$$

$$S^z = -\tfrac{1}{2}\sum_\alpha \mathbf{1}_F^\alpha + (b^*, b) \tag{6.13}$$

where $(b^*, b) = \Sigma_\alpha b_\alpha^* b_\alpha$, is an infinitesimal generator of the reducible on $\mathscr{F}_F$ representation of the SU(2) group. It follows from the fact that for $\alpha \neq \beta$, the operators b_α^*, b_β do commute:

$$[b_\alpha, b_\beta^*]_- = 0 = [b_\alpha, b_\beta]_- \tag{6.14}$$

while:

$$[b_\alpha, b_\alpha^*]_+ = \mathbf{1}_F^\alpha, \qquad b_\alpha^2 = 0 = b_\alpha^{*2} \tag{6.15}$$

so that operators $\boldsymbol{S}_\alpha$ obey:

$$[S_{\alpha a}, S_{\beta b}]_- = \delta_{\alpha\beta}\, i\epsilon_{abc} S_{\alpha c}$$

$$a, b, c = 1, 2, 3, \qquad \alpha, \beta = 1, 2, \ldots, n. \tag{6.16}$$

By the use of the well known addition theorems for the angular momenta in mind, we can add these generators, getting the new infinitesimal generator $\boldsymbol{S}_k = \Sigma_\alpha \boldsymbol{S}_{k\alpha}$ of the SU(2) group being assigned to the kth n-mode cluster under consideration. Obviously, this generator is defined on $\mathscr{F}_F^k$, and decomposes this Hilbert space into a direct sum of spin subspaces corresponding to the irreducible components of the representation.

In the spin $\frac{1}{2}$ approximation of the Boson system we can apply the Boson expansion formulas to the above operators according to:

$$S_k^+ = \sum_\alpha a_{k\alpha}^* : \exp(-a_{k\alpha}^* a_{k\alpha}):,$$

$$S_k^- = \sum_\alpha : \exp(-a_{k\alpha}^* a_{k\alpha}): a_{k\alpha},$$

$$S_k^z = -\tfrac{1}{2}\sum_\alpha \{:\exp(-a_{k\alpha}^* a_{k\alpha}): - a_{k\alpha}^*:\exp(-a_{k\alpha}^* a_{k\alpha}):a_{k\alpha}\},$$

$$[S_{ka}, S_{lb}]_- = \delta_{kl}\, \mathrm{i}\epsilon_{abc} S_{kc}. \tag{6.17}$$

The Fermion unit operator $\mathbf{1}_{\mathrm{F}}^l$ selects in $\mathscr{F}_{\mathrm{B}}$ a subspace of spin states corresponding to the lth cluster with dim $K_l = n$.

In the above the operator $:\exp(-(a^*, a)_k):$, $(a^*, a)_k = \Sigma_\alpha a_{k\alpha}^* a_{k\alpha}$ is responsible for projecting onto the ground state of the cluster. A global projection onto the ground state for the whole of the lattice is then received by the multiplication of all particular cluster projections (by virtue of the direct product structure of $\mathscr{F}_{\mathrm{B}}$), and reads:

$$:\exp(-(a^*, a)):, \qquad (a^*, a) = \sum_k (a^*, a)_k.$$

Let us now define the following operators:

$$\hat{P}_0^k = \mathbf{1}_{\mathrm{F}}^k - :\exp(-(a^*, a)_k):. \tag{6.18}$$

Each kth one projects onto the non-zero mode subspace of $\mathscr{F}_{\mathrm{F}}^k$. Then, the operator:

$$\hat{P}_0 = :\exp(-(a^*, a)): + \sum_{k=1}^{N} \left\{\hat{P}_0^k :\exp(-\sum_{j\neq k}(a^*, a)_j):\right\} \tag{6.19}$$

is a projector and selects in $\mathscr{F}_{\mathrm{B}}$ a subspace $\mathscr{F}_0 = \hat{P}_0\mathscr{F}_{\mathrm{B}}$ being a closed set-theoretic union of all particular spin spaces $\mathscr{F}_0 = \cup_{k=1}^N \mathscr{F}_{\mathrm{F}}^k$ and thus being the Hilbert space of spin states for the subsidiary Boson lattice.

Now there is quite trivial excercise to check that on $\mathscr{F}_0$ the following operator identities hold:

$$S_k^+ = \hat{P}_0 \sum_\alpha a_{k\alpha}^* \hat{P}_0 = \hat{P}_0 \mathscr{S}_k^+ \hat{P}_0$$

$$S_k^- = \hat{P}_0 \sum_\alpha a_{k\alpha} \hat{P}_0 = \hat{P}_0 \mathscr{S}_k^- \hat{P}_0$$

$$S_k^z = \hat{P}_0[-\tfrac{1}{2} + (a^*, a)_k]\hat{P}_0 = \hat{P}_0 \mathscr{S}_k^z \hat{P}_0 \tag{6.20}$$

where $\hat{P}_0$ prevents us from leaving $\mathscr{F}_0$ while using Bosons only. Moreover, for $k \neq l$ there obviously holds on $\mathscr{F}_0$:

$$\hat{P}_0 \mathscr{S}_k \mathscr{S}_l \hat{P}_0 = \hat{P}_0 \mathscr{S}_k \hat{P}_0 \mathscr{S}_l \hat{P}_0 = \boldsymbol{S}_k \boldsymbol{S}_l \tag{6.21}$$

what follows from the commutativity of the components of the operators $\mathscr{S}_k, \mathscr{S}_l$ and $\boldsymbol{S}_k, \boldsymbol{S}_l$ respectively.

Because $\hat{P}_0$ is the unit operator in $\mathscr{F}_0$, we get thus the desired property $H_{\mathrm{B}} \to \hat{P}_0 H_{\mathrm{B}} \hat{P}_0 = H$ to hold in $\mathscr{F}_0$.

Let us summarize the received results: our starting point was the subsidiary Boson lattice, whose description is in the spin $\frac{1}{2}$ approximation reduced to a particular subspace $\mathscr{F}_0$ of $\mathscr{F}_{\mathrm{B}}$, so that the restricted lattice Hamiltonian $H_{\mathrm{B}} = \hat{P}_0 H_{\mathrm{B}} \hat{P}_0$ is in fact the Hamiltonian of the isotropic Heisenberg lattice.

The physical conditions under which a restriction to $\mathscr{F}_0$ is possible, are clarified by the concept of the spin $\frac{1}{2}$ approximation. We need so low temperature T of the thermostat, that each single mode of the

subsidiary Boson lattice is either singly excited or not excited at all, so that:

$$1 > \text{prob}[\mathscr{F}_0] \gg \text{prob}[\mathscr{F}_B \backslash \mathscr{F}_0]. \tag{6.22}$$

Probabilities with which transitions to, and between the higher energy levels of the system participate in (6.22) are then negligible.

In the pure Boson language we deal here with a kind of the condensation of the magnon gas (its single degrees of freedom in fact) around the lattice sites, so that the magnon condensate perfectly imitates the structure of spin interactions inside the crystal.

Moreover, by the use of spin $\frac{1}{2}$ approximation of the Boson system we have received, what in section 5 did correspond to the finite spin approximation.

The last step in these considerations should be now a comparison of results (Hamiltonians I mean) received by the use of the either of the presented methods. Calculations are here straightforward, and to have a comparison with the Holstein–Primakoff and Dyson–Maleev expansions, we put $\mathscr{S}_k^z \to -\mathscr{S}_k^z$, what implies

$$S_k^z \longrightarrow -S_k^z; \qquad \mathscr{S}_k^+ = a_k^*,$$
$$\mathscr{S}_k^- = a_k, \qquad \mathscr{S}_k^z = \tfrac{1}{2} - a_k^* a_k$$

and results in:

$$H_2^B = (\mu\mathscr{H} + \tfrac{1}{2}J(0)) \cdot \left(N - \sum_k n_k\right) - \tfrac{1}{2}\sum_{k,l} I_{kl} a_k^* a_l$$

$$H_4^B = -\tfrac{1}{2}\sum_{k,l} I_{kl} + \tfrac{1}{2}\sum_{k,l} I_{kl}(n_k + n_l) - \tfrac{1}{2}\sum_{k,l} I_{k,l} n_k n_l. \tag{6.23}$$

After the reordering of terms with respect to powers in which the operators a_k^*, a_k appear, we get:

$$E_0^B = \tfrac{1}{2}N\mu\mathscr{H} + \tfrac{3}{8}NJ(0) - \tfrac{1}{2}\sum_{n,l} I_{kl},$$

$$H_2^B = -(\mu\mathscr{H} + \tfrac{1}{2}J(0))\sum_k n_k - \tfrac{1}{2}\sum_{k,l} I_{kl} a_k^* a_l + \tfrac{1}{2}\sum_{k,l} I_{kl}(n_k + n_l),$$

$$H_B' = -\tfrac{1}{2}\sum_{k,l} I_{kl} n_k n_l, \qquad H_B'' = 0. \tag{6.24}$$

If compared with the H–P and D–M expansions, we have modified E_0^B and H_2^B, and no kinematical term at all.

Obviously, the Hamiltonian $H_B = E_B^0 + H_2^B + H_B'$ makes sense if employed in the theory of the Heisenberg ferromagnet in the form $\hat{P}_0 H_B \hat{P}_0$ only.

6.3. *The anisotropic crystal*

Let us consider the generalized Heisenberg model with the diagonal interaction exchange tensor and the anisotropy with respect to all three crystalographic coordinate axis:

$$I_{kl}^{\alpha\beta} = \delta_{\alpha\beta} I_{kl}^{\alpha}, \qquad \alpha, \beta = x, y, z$$
$$I_{kl}^z = I_{kl}, \qquad I_{kl}^x = \xi I_{kl}, \qquad I_{kl}^y = \eta I_{kl} \tag{6.25}$$

where ξ, η are the anisotropy parameters $|\xi|, |\eta| \leqslant 1$.

For the case of a ferromagnetic crystal, we additionally need $I_{kl} > 0$. Note that $I_{kl} < 0$ implies the antiferromagnetism.

The external magnetic field is again oriented along the z-axis $\mathscr{H} = (0, 0, \mathscr{H})$. Then the Hamiltonian of our spin system reads [9]:

$$H(I, \xi, \eta) = -\mu\mathscr{H}\sum_k S_k^z - \tfrac{1}{2}\sum_{kl} I_{kl}(\xi S_k^x S_l^x + \eta S_k^y S_l^y + S_k^z S_l^z\} \tag{6.26}$$

where we admit $-1 \leqslant \xi, \eta \leqslant 1$.

If to introduce the operators $S_k^{\pm} = S_k^x \pm \mathrm{i}S_k^y$, we get further:

$$H(I, \xi, \eta) = -\mu\mathscr{H}\sum_k S_k^z - \tfrac{1}{2}\sum_{kl} I_{kl}\{\tfrac{1}{2}(\xi + \eta)S_k^+ S_l^- + \tfrac{1}{4}(\xi - \eta)(S_k^- S_l^- + S_k^+ S_l^+) + S_k^z S_l^z\}. \tag{6.27}$$

Because of the translational invariance of the Hamiltonian (6.27), we can make a transition to the Fourier images of the operators $S_k^{\pm}$ in the momentum space:

$$S_k^{\pm} = (1/\sqrt{N})\sum_p S_p^{\pm}\exp(\pm \mathrm{i}kp), \qquad S_k^z = (1/\sqrt{N})\sum_p S_p^z \exp(-\mathrm{i}kp)$$

$$I_{kl} = I(k-l) = \sum_p I(p)\exp\{\mathrm{i}(k-l)p\} \tag{6.28}$$

getting in the place of $H(I, \xi, \eta)$:

$$H = -\mu\mathscr{H}\sqrt{N}S_0^z - \tfrac{1}{2}\sum_p I(p)\{\tfrac{1}{2}(\xi + \eta)S_p^+ S_p^- + \tfrac{1}{4}(\xi - \eta)(S_p^+ S_{-p}^+ + S_p^- S_{-p}^-) + S_p^z S_{-p}^z\} \tag{6.29}$$

where:

$$-\eta \leqslant \xi \leqslant \eta, \qquad 0 \leqslant \eta \leqslant 1$$

$$I(p) = I(0)\gamma_p, \qquad I(0) = I \cdot z, \qquad I_0 = \sum_p I(p) = 0$$

$$\gamma_p = (1/z)\sum_{\delta \neq 0}\exp(\mathrm{i}p\delta), \qquad |\gamma_p| \leqslant 1, \qquad \sum_p \gamma_p = 0 \tag{6.30}$$

and:

$$\gamma_{p \pm p_0} = -\gamma_p, \qquad \gamma_{p \pm 2p_0} = \gamma_p, \qquad \gamma_0 = -\gamma_{\pm p_0} = 1.$$

Here $p_0 = (\pi/a)\ (1, 1, 1)$ is the boundary momentum of the first Brillouin zone, and we explicitly assume the approximation of the nearest neighbour interaction: a parameter z denotes the number of the nearest neighbours of a single site.

From now on we shall restrict considerations to a special case of spin $s = \frac{1}{2}$ lattice. This restriction is justified both theoretically and experimentally [50, 51, 9] as a consequence of the fact that the majority of the physically interesting results rather weakly depends on a particular value of spin. Moreover the peculiarities of the spin kinematics are essential in case of the lowest spin values, while in the quasi-classical (quasi-Boson) limit $s \to \infty$ they are completely negligible.

In this place there is useful to employ the representation of spin operators through the Pauli matrices. We shall introduce:

$$S_k^+ = b_k^*, \qquad S_k^- = b_k, \qquad S_k^z = \tfrac{1}{2}\mathbf{1}_{\mathrm{F}}^k - b_k^* b_k.$$

Then:

$$S_p^+ = b_p^*, \qquad S_p^- = b_p, \qquad S_p^z = (\sqrt{N}/2)\delta_{p0} - \rho_p \tag{6.31}$$

where ρ_p is a Fourier image of the operator n_k:

$$\rho_p = (1/\sqrt{N}) \sum_k n_k \exp(-ikp) = (1/\sqrt{N}) \sum_q b_q^* b_{q+p}$$
$$\rho_p^* = \rho_{-p}. \tag{6.32}$$

As a consequence of the commutation relations:

$$[b_k, b_l^*]_- = (1 - 2n_k)\delta_{kl} \tag{6.33}$$
$$[b_k, b_l]_- = 0, \qquad b_k^2 = b_k^{*2} = 0$$

we get the following relations in the momentum space:

$$[b_p, b_{p'}^*]_- = (1/\sqrt{N})2S_{p-p'}^z = \delta_{pp'} - (2/\sqrt{N})\rho_{p-p'}$$
$$[b_p, \rho_{p'}]_- = (1/\sqrt{N})b_{p+p'}, \qquad [b_p^*, \rho_{p'}]_- = (-1/\sqrt{N})b_{p-p'}^*,$$
$$[\rho_p, \rho_{p'}]_- = 0 = [b_p, b_{p'}]_-. \tag{6.34}$$

By virtue of $b_k^2 = b_k^{*2} = 0$ and $n_k^2 = n_k$, $(S_k^z)^2 = \frac{1}{4}$, we get:

$$\sum_p b_{-p} b_{p+q} = 0, \qquad (1/\sqrt{N}) \sum_p \rho_{-p}\rho_{p+q} = \rho_q \tag{6.35}$$

for all k, q.

Introducing all these notions into (6.29) we receive the final form of the Hamiltonian:

$$H = E_0 + \sum_p \{A(p) b_p^* b_p + \tfrac{1}{2} C(p) \cdot (b_p^* b_{-p}^* + b_{-p} b_p)\} - \tfrac{1}{2} \sum_p I(p)\rho_p\rho_{-p} \tag{6.36}$$

where:

$$E_0 = -\tfrac{1}{2} N\mu\mathcal{H} - \tfrac{1}{8} NI(0), \qquad -\eta \leqslant \xi \leqslant \eta, \qquad 0 \leqslant \eta \leqslant 1,$$
$$A(p) = A^*(p) = A(-p) = \mu\mathcal{H} + \tfrac{1}{2} I(0)(1 - \tfrac{1}{2}(\xi + \eta)\gamma_p),$$
$$C(p) = C^*(p) = C(-p) = -\tfrac{1}{4} I(0)(\xi - \eta)\gamma_p.$$

Let us add that if $\xi = \eta$, the coefficient $C(p)$ obviously vanishes, and in that case E_0 becomes an energy of the ground state, while $A(p)$ an energy of the ideal gas of Bloch's magnons.

The general $\xi \neq \eta$ Hamiltonian (6.35) describes the magnetic properties of a few metals, and can be also applied to the quasi-spin formulation of the superfluidity problem for the nonideal lattice gas and the BCS model in the theory of superconductivity, compare [9, 52–56]. The one-particle dynamics generated by the Hamiltonian (6.35) leads to the following equations of motion:

$$ib_p^* = [b_p, H]_- = A(p)b(p) + C(p)b_{-p}^* + J_p \tag{6.37}$$

where:

$$J_p = \tfrac{1}{2}(J_p + J_{-p}^*) + \tfrac{1}{2}(J_p - J_{-p}^*)$$
$$J_p \pm J_{-p}^* = -(1/\sqrt{N}) \sum_q V_{pq}(x_\pm)\rho_q \cdot (b_{p-q} \pm b_{-p+q}^*)$$
$$x_+ = \xi, \qquad x_- = \eta \tag{6.38}$$

and a non-symmetric potential $V_{pq}(x)$ is given by:

$$V_{pq}(x) = I(0)(\gamma_q - x\gamma_{q-p}), \qquad V_{pq}(x) \neq V_{qp}(x). \tag{6.39}$$

In the Boson approximation the term $-x\gamma_{q-p}$ is neglected. Analogously for ρ_p:

$$i\dot{\rho}_p = [\rho_p, H]_- = (1/2\sqrt{N}) \sum_q W^{(+)}_{pq} b^*_{-q} b_{-q+p} + (1/4\sqrt{N}) \sum_q W^{(-)}_{pq} (b^*_{-q} b^*_{q-p} - b_q b_{-q+p}) \tag{6.40}$$

where:

$$W^{(\pm)}_{pq} = \pm \tfrac{1}{2}(\xi \pm \eta) V_{pq}(\pm 1) = \pm I(0) \tfrac{1}{2}(\xi \pm \eta)(\gamma_q \mp \gamma_{q-p}). \tag{6.41}$$

In the particular case of the Ising model $\xi = \eta = 0$, we get $V_{pq}(0) = I(0)\gamma_q$ for all p, and $i\dot{\rho}_p = 0$, $W^{(\pm)}_{pq} = 0$ so that all operators ρ_p are the integrals of motion.

Dyson's theory [48, 49] of the isotropic Heisenberg ferromagnet, which was further reproduced in terms of the Boson operators [57–64], is commonly treated as the most exact and appears as the standard theory in the low temperature domain. Its generalization onto the anisotropic case is immediate. Let us now reproduce from [9] what happens if the Dyson–Maleev expansions are used. Namely their translation to the momentum space reads:

$$b^*_p \longrightarrow a^*_p, \qquad b_p \to a_p - (1/\sqrt{N}) \sum_q \nu_{p-q} a_q, \qquad \rho_p \to \nu_p \tag{6.42}$$

where:

$$[a_p, a^*_{p'}]_- = \delta_{pp'}, \qquad [a_p, \nu_{p'}]_- = (1/\sqrt{N}) a_{p+p'},$$
$$\nu_p = \nu^*_{-p} = (1/\sqrt{N}) \sum_q a^*_q a_{q+p} \tag{6.43}$$

and the subsidiary condition, which reflects the requirement $b^2_k = b^{*2}_k = 0$ is imposed:

$$\sum_{qp} a^*_q \nu_{q+r-p} a_p = 0 \qquad \text{for all } r. \tag{6.44}$$

Then:

$$H_B = E_0 + \sum_p \{A(p) a^*_p a_p + \tfrac{1}{2} C(p)(a^*_p a^*_{-p} + a_{-p} a_p)\}$$
$$- \tfrac{1}{2} \sum_p I(p) \nu_p \nu_{-p} + (1/4\sqrt{N})(\xi + \eta) \sum_{pq} I(p) a^*_p \nu_{p-q} a_q$$
$$+ (1/4\sqrt{N})(\xi - \eta) \sum_{pq} I(p) a_{-p} \nu_{p-q} a_q + (1/2N) \sum_{pqr} C(p) \nu_{p-q} a_q \nu_{-p-r} a_r \tag{6.45}$$

In the model $\eta = \xi$, we get $C(k) = 0$ and then:

$$H_B = E_0 + \sum_p E^B_p a^*_p a_p - \tfrac{1}{2} \sum_p I(p) \nu_p \nu_{-p} + (1/2\sqrt{N}) \xi \sum_{pq} I(p) a^*_p \nu_{p-q} a_q. \tag{6.46}$$

Both Hamiltonians (6.45), (6.46) are obviously non-Hermitean and need an introduction of the new topology in the state space. However, if done, the eigenvalues of H_B are allowed to be negative and even complex.

If to make use of (6.30) the Hamiltonian (6.46) can be rewritten in the canonical Dyson's form of the symmetrized four-Boson interaction [48]:

$$H_{\mathrm{B}} = E_0 + \sum_p E_p^{\mathrm{B}} a_p^* a_p - (1/4N) \sum_{pqr} V_{pqr}(\xi) a_{r+q}^* a_{p-q}^* a_p a_r \tag{6.47}$$

where the Dyson's potential $V_{pqr}(\xi)$ is given by:

$$V_{pqr}(\xi) = [\gamma_q + \gamma_{p-r-q} - \xi(\gamma_{q+r} + \gamma_{p-q})] \cdot I(0). \tag{6.48}$$

In the case of the Ising model $\xi = \eta = 0$ we would get

$$V_{pqr}(0) = \gamma_q + \gamma_{p-r-q}.$$

The Hamiltonian (6.47) generates the following equations of motion: for the Boson operators a_p^*, a_p:

$$\mathrm{i}\dot{a}_p = [a_p, H_{\mathrm{B}}]_- = E_p^{\mathrm{B}} a_p + \frac{I(0)}{2N} \sum_{qr} \{\gamma_{p-r} + \gamma_{p-q} - \xi(\gamma_{p-r-q} + \gamma_p)\} a_{r+q-p}^* a_q a_r, \tag{6.49}$$

$$-\mathrm{i}\dot{a}_p^* = [a_p^*, H_{\mathrm{B}}]_- = E_p^{\mathrm{B}} a_p^* - (1/N) I(0) \sum_{qr} (\gamma_q - \xi\gamma_{p-q}) a_{p-q}^* a_{r+q}^* a_r$$

$$= E_p^{\mathrm{B}} a_p^* - (1/N) \sum_q V_{pq}(\xi) a_{p-q}^* \nu_{-q}. \tag{6.50}$$

The above equations are not mutually adjoint.
By taking into account:

$$\mathrm{i}\dot{\nu}_p = [\nu_p, H_{\mathrm{B}}]_- = (1/2\sqrt{N}) \xi I(0) \sum_q (\gamma_q - \gamma_{p-q}) a_{-q}^* \left\{ a_{-q+p} - (1/\sqrt{N}) \sum_r \nu_{-q+p-r} a_r \right\} \tag{6.51}$$

and combining it with (6.49), one gets one more equation of motion:

$$(\mathrm{i}\mathrm{d}/\mathrm{d}t - E_p^{\mathrm{B}}) \left\{ a_p - (1/\sqrt{N}) \sum_r \nu_{p-r} a_r \right\} = -(1/\sqrt{N}) \sum_q V_{pq}(\xi) \nu_q \left\{ a_{p-q} - (1/\sqrt{N}) \sum_r \nu_{p-q-r} a_r \right\}. \tag{6.52}$$

We have thus derived the complete Dyson images of the dynamical equations (6.37), (6.40) in the anisotropic lattice.

The theory of the previous section suggests to use the rigorous Boson expansions of the spin operators, which in case of lowest spins do not suffer of all peculiarities and difficulties of the Dyson–Maleev approach: notice for example that the operations of the Hermitean conjugation and the differentiation in time, do not commute.

In the rigorous approach the following substitutions are in order:

$$b_p^* \longrightarrow a_p^*, \qquad b_p \longrightarrow a_p, \qquad \rho_p \longrightarrow \nu_p$$
$$b_p^* = \hat{P}_0 a_p^* \hat{P}_0, \qquad b_p = \hat{P}_0 a_p \hat{P}_0, \qquad \rho_p = \hat{P}_0 \nu_p \hat{P}_0 \tag{6.53}$$

for all p, and the appropriate projector $\hat{P}_0$ in $\mathscr{F}_{\mathrm{B}}$.

Notice that to make an explicit use of the up to now performed calculations, we have changed the definition of S_k^z into $-S_k^z$.
Substituting (6.53) into (6.36), we get:

$$H = \hat{P}_0 H_{\mathrm{B}} \hat{P}_0$$
$$H_{\mathrm{B}} = E_0 + \sum_p \{A(p) a_p^* a_p + \tfrac{1}{2} C(p)(a_p^* a_p^* + a_{-p} a_p) - \tfrac{1}{2}\} \sum_p I(p) \nu_p \nu_{-p} \tag{6.54}$$

where:

$$\hat{P}_0 = :\exp\Big(-\sum_{k=1}^{N} a_k^* a_k\Big): + \sum_{k=1}^{N}\Big\{\hat{P}_0^k :\exp\Big(-\sum_{j\neq k}(a^*, a)_j\Big):\Big\}$$
$$\hat{P}_0^k = \mathbf{1}_F^k - :\exp(-a_k^* a_k): \tag{6.55}$$

and $\hat{P}_0$ projects in $\mathscr{F}_B$ onto the Hilbert space of the spin states of the lattice [17].

For $\eta = \xi$ we get in the place of (6.36) a simple analogue of (6.46)

$$H_B = E_0 + \sum_p E_p^B a_p^* a_p - \tfrac{1}{2}\sum_p I(p)\nu_p\nu_{-p} \tag{6.56}$$

differing from (6.48) by the absence of the term $\xi(2/\sqrt{N})\Sigma_{pq} I(p)a_p^*\nu_{p-q}a_q$. The canonical form of H_B is easily achieved:

$$H_B = E_0 + \sum_p E_p^B a_p^* a_p - (1/4N)\sum_{pqr} V_{pqr}(0)a_{r+q}^* a_{p-q}^* a_p a_r \tag{6.57}$$

what implies the following equations of motion for the operators a_p, a_p^*

$$i\dot{b}_p = \hat{P}_0\, i\dot{a}_p\hat{P}_0, \qquad i\dot{b}_p^* = \hat{P}_0\, i\dot{a}^*\hat{P}_0$$
$$i\dot{a}_p = [a_p, H_B]_- = E_p^B a_p - (1/2N)I(0)\sum_{qr}(\gamma_{p-r} + \gamma_{p-q})a_{r+q-p}^* a_q a_r \tag{6.58}$$
$$-i\dot{a}_p^* = [a_p^*, H_B]_- = E_p^B a_p^* - (1/N)I(0)\sum_{qr}\gamma_q a_{p-q}^* a_{r+q}^* a_r$$

and obviously, for ν_p:

$$\hat{P}_0\, i\dot{\nu}_p\hat{P}_0 = i\dot{\rho}_p$$
$$i\dot{\nu}_p = [\nu_p, H_B]_- = 0. \tag{6.59}$$

One can easily check that the equations (6.58) are mutually adjoint with respect to the Hermitean conjugation *.

Here, like in the Ising model (where $\xi = \eta = 0$ manifestly) all operators ν_p are the integrals of motion.

In addition, combining the first equation in (6.58) with (6.59) one can derive the following analogue of (6.52):

$$(id/dt - E_p^B)(a_p - (1/\sqrt{N})\sum_q \nu_{p-q}a_q) = -(1/\sqrt{N})\sum_q V_{pq}(0)\nu_q\Big(a_{p-q} - (1/\sqrt{N})\sum_r \nu_{p-q-r}a_r\Big) \tag{6.60}$$

where $V_{pq}(0) = I(0)\gamma_q = I(q)$.

Let us add that the extensions of the described methods onto the antiferromagnetic case are straightforward.

7. The atomic nucleus in the weak excitation limit

In the theory of atomic nuclei there was noticed that in many cases, the spectra of low-lying excited states are similar to these of the excited system of the weakly coupled quadrupole Bosons

[7, 18–26]. Such a Boson system can be thus used in the approximate description of the weakly excited atomic nucleus, when the anharmonic corrections, which correspond to the interaction of the quadrupole Bosons between themselves, are negligibly small.

On the other hand, there is well known that the Hamiltonians of the microscopic model of the nucleus, if expressed in the representation of the generalized quasi-spin operators, are in close relation with the previously discussed Heisenberg ferromagnet case. This suggests the use of Boson expansion methods, to get an approximation of the starting system by the set of collective Bosons which are responsible for its low-temperature properties.

In the spherical model of the nucleus, with the j–j coupling, each jth one-particle state is characterized by a collection $\alpha_k = (n, l, j, m)_k$ of quantum numbers. Because the magnetic parameter m plays a distinguished role in further considerations, we shall use the notation:

$$a_k = (n, l, j)_k \text{ so that } \alpha_k = (a, m_a)_k, \qquad k = 1, 2, \ldots$$

Let us define the operators of Fermion pairs, whose total angular momentum equals J, while its projection M:

$$\begin{aligned} A^*_{JM}(a, b) &= (1/\sqrt{2}) \sum_{m_a m_b} (j_a j_b m_a m_b \mid JM) b^*_\alpha b^*_\beta \\ B^*_{JM}(a, b) &= \sum_{m_a m_b} (j_a j_b m_a - m_b \mid JM)(-1)^{j_b - m_b} \cdot b^*_\beta b_\alpha . \end{aligned} \tag{7.1}$$

Here $(j_a j_b m_a m_b \mid JM)$ is the Clebsch–Gordan coefficient of the expansion.

Let us introduce the following abbreviations:

$$\alpha_k = (a_k, m_{a_k}) = (a, m_a)_k, \qquad A_{J_k M_k}(a_k, b_k) := A_k .$$

By the use of the sequence $k = 1, 2, \ldots$ of the operators (7.1) and their adjoints, one can construct the Hamiltonian of the nucleus in the microscopic model, as well as the transition operators between the nuclear states.

The operators (7.1) obey

$$\begin{aligned} [A_i, A^*_k]_- &= \delta^{(+)}_{ik} - 2 \sum_l \tfrac{1}{2}(1 + \hat{p}_i)\, \tfrac{1}{2}(1 + \hat{p}_k) Y(i, k, l) \cdot B_l \\ [B_l, A^*_i]_- &= 2 \sum_k \tfrac{1}{2}(1 + \hat{p}_i)\, \tfrac{1}{2}(1 + \hat{p}_k) Y(i, k, l) A^*_k \\ [B_i, B^*_l]_- &= \sum_k (1 - \hat{p}_i \hat{p}_k \hat{p}_l) Y(i, k, l) B_k \end{aligned} \tag{7.2}$$

where $\hat{p}_k$ is the permutation operator:

$$\begin{aligned} &\hat{p}_k f(k) = \hat{p}_k f(a_k, b_k, J_k, M_k) = -\Theta_k(a, b, J) \cdot f(b_k, a_k, J_k, M_k) \\ &\Theta(a, b, J) = (-1)^{j_a + j_b + J} \end{aligned} \tag{7.3}$$

and moreover:

$$\delta^{(+)}_{ik} = (1 + \hat{p}_i)\delta_{ik}(\tfrac{1}{2}) = \tfrac{1}{2}\, \delta_{J_i J_k} \delta_{M_i M_k} (\delta_{a_i a_k} \delta_{b_i b_k} - \Theta_i(a, b, J) \delta_{a_i b_k} \delta_{b_i a_k}) \tag{7.4}$$

while:

$$\begin{aligned} &(j_{a_k} j_{b_k} m_{a_k} m_{b_k} \mid J_k M_k) := c_k \\ &Y(i, k, l) = \sum_{(m)} c_i c_k c_l (-1)^{j_{a_k} - m_{a_k}} \cdot \delta_{\beta_i \beta_k} \delta_{\alpha_i \alpha_l} \delta_{\alpha_k \beta_l} . \end{aligned} \tag{7.5}$$

In the particular case of $J = M = 0$, the commutation relations (7.2) coincide with the commutation relations for the spin operators. Namely, in that case, for each $k = 1, 2, \ldots$ we have:

$$[A_k, A_k^*]_- = 1 - 2(1/\sqrt{2j_k + 1})B_k \tag{7.6}$$
$$[B_k, A_k^*]_- = (2/\sqrt{2j_k + 1})A_k^*$$

what allows to associate with each kth bifermion state a corresponding kth infinitesimal generator of the SU(2) group, according to:

$$S_k^+ = \sqrt{j_k + \tfrac{1}{2}}A_k, \qquad S_k^- = \sqrt{j_k + \tfrac{1}{2}}A_k^*$$
$$S_k^z = \tfrac{1}{2}(j_k + \tfrac{1}{2}) - (j_k + \tfrac{1}{2})(1/\sqrt{2j_k + 1}) \cdot B_k \tag{7.7}$$

this fact justifies the Boson translation of the theory at least in the simplest case $J = M = 0$, but motivates also the trials to get any extension of the Boson expansion methods onto more complex situations.

Now, as the interlude in the main stream of considerations let us study what happens in the weak coupling limit, called also a harmonic approximation, when higher excitations significantly influence the behaviour of the nucleus. In that case one can imagine the situation, when the compounds of the atomic nucleus though Fermions in the basic formalism, can behave like Bosons.

Following [7], let us assume to deal with the light nucleus, where the number n of the compounds is relatively small.

We know that the higher is the excitation level of the system (high temperature limit), the greater becomes the number $\Sigma_a (2j_a + 1)$ of the one-particle states which are mostly occupied by the single constituents of the nuclear system. Moreover, only the great values $j_a \gg 1$ become then significant. This corresponds to the weakening of the coupling forces between the nucleons, so that the weak coupling limit of the theory is approached.

Let us notice that for $j_a \gg 1$, we can use in the place of $Y(i, k, l)$ the corresponding asymptotic expression:

$$Y(i, k, l) \sim \underline{\delta_{bibk} \cdot \delta_{aiai} \cdot \delta_{akai}} \cdot (2j + 1)^{-1/2}$$
$$\times \sqrt{(2J_k + 1)(2J_i + 1)(2J_l + 1)} \cdot (J_k J_l M_k M_l \mid J_i M_i)(J_k J_l j_{ak} - j_{bi}, j_{al} - j_{ak} \mid J_i j_{al} - j_{bi}) \tag{7.8}$$

where:

$$(2j + 1)^3 \equiv (2j_{al} + 1)(2j_{ak} + 1)(2j_{bi} + 1).$$

By virtue of $j_a \gg 1$, (7.8) includes a small parameter $(2j + 1)^{-1/2} \sim [\Sigma_a (2j_a + 1)]^{-1/2}$. In consequence, the operator part of the right-hand side of the commutator $[A_1, A_2^*]_-$ can be neglected.

Moreover in case under consideration, the operators $B_{JM}(a, b)$ can be also neglected. If we denote by $|m)$ the ground state of our nucleus, and consider the expectation value of $B_{JM}(a, b)$ in this state, we get:

$$(m|b_\alpha^* b_\beta|m) \cong (-1)^{j_b - m_b}\delta_{\alpha\beta}n_\alpha \cong (-1)^{j_b - m_b}\delta_{\alpha\beta} \cdot n \cdot \left(1 \Big/ \sum_a (2j_a + 1)\right), \tag{7.9}$$

so that:

$$(m|B_{JM}(a, b)|m) \cong \sum_{m_a m_b} (j_a j_b m_a - m_b \mid JM)(-1)^{j_b - m_b}\delta_{\alpha\beta} n \cdot \left(1 \Big/ \sum_a (2j_a + 1)\right)$$
$$\cong (n/\sqrt{2j_a + 1}) \cdot \delta_{ab}\delta_{J0}\delta_{M0}. \tag{7.10}$$

Hence, in the approximation $\Sigma_a (2j_a + 1) \gg 1$, the operators $A_{JM}(a, b)$ can be considered as the ideal Bosons:

$$[A_i, A_k^*]_- = \delta_{ik}^{(+)}, \qquad \hat{p}_i A_i = A_i \tag{7.11}$$

while $B_{JM}(a, b) \cong 0$.

In consequence, in the place of the initial Hamiltonian describing the interaction of Fermion pairs, one can introduce an ideal Boson Hamiltonian, acting in the Hilbert space of the Boson states: no Pauli exclusion principle is observed. This is an example of the metamorphosis of Fermions into Bosons if higher excitations are essential. In that case, obviously, the Bosonic "phase" of the system significantly prevails.

In the above, the Pauli exclusion principle, standing for a defining property of the Fermionic behaviour, disappeared. Quite conversely, we shall now prove its appearance if to consider the weak excitation limit of the appropriately defined Boson system. The four-Fermion interaction Hamiltonian will be now in order.

In the microscopic theory of the weakly excited atomic nucleus, the collective excitation branches are connected with the pairing correlations inside the nucleus, which is assumed to consist of nearly the same number of protons and neutrons. Because the nucleons can occupy the same one-particle levels, differing by the isospin projection τ only in addition to the p–p and n–n correlations, the n–p ones should be taken into account.

We assume the correlations to be spin independent, so that the quantum numbers of interest are now $\alpha = (j, m, \tau)$.

Let there be given a four-Fermion interaction Hamiltonian [65]:

$$H = \sum_{\alpha} (\epsilon_\alpha - \mu) b_\alpha^* b_\alpha + \tfrac{1}{2} \sum_{\alpha\beta\gamma\sigma} b_\alpha^* b_\beta^* b_\gamma b_\sigma W(\alpha, \beta, \gamma, \sigma) \tag{7.12}$$

where μ is the chemical potential: in the ground state of the system when $W = 0$, μ equals to the Fermi energy ϵ_F.

We know that in the nucleus, the interaction correlates the antiparallel magnetic momenta only, what results in the following form of the Hamiltonian:

$$H = \sum_{j, m>0} (\epsilon_j - \mu) N_{jm} + \tfrac{1}{2} \sum_{\substack{j, j', m, m'>0 \\ \sigma_1\sigma_2\tau_1\tau_2 = \pm 1/2}} b_{jm\tau_1}^* b_{j-m\tau_2}^* b_{j'-m'\sigma_1} b_{j'm'\sigma_2} \cdot V(j, j', m, m', \tau_1, \tau_2, \sigma_1, \sigma_2) \tag{7.13}$$

where:

$$N_{jm} = \sum_{\tau = \pm 1/2} (b_{jm\tau}^* b_{jm\tau} + b_{j-m\tau}^* b_{j-m\tau}).$$

Because terms differing by the sign at m give the identical counterparts to the total sum $\Sigma_{j, m>0} (\epsilon_j - \mu) N_{jm}$, we can in fact consider $N_{jm} = 2 \Sigma_\tau b_{jm\tau}^* b_{j'm\tau}$, compare e.g. [7, 65].

In [7] one can find still more simplified form of the Hamiltonian (7.13)

$$H = \sum_{j, m>0} (\epsilon_j - \mu) N_{jm} - G \sum_{\substack{j, j', m, m'>0 \\ \tau = 0, \pm 1}} A_{jm\tau}^* A_{j'm'\tau} \tag{7.14}$$

where:

$$A_{jm\tau}^* = (-1)^{j-m} \sum_{\tau_1, \tau_2 = \pm 1/2} C_{1/2\tau_1 1/2\tau_2}^{1\tau} b_{jm\tau_1}^* b_{j-m\tau_2}^* \tag{7.15}$$

and $C_{1/2\tau_1 1/2\tau_2}^{1\tau}$ denotes the Clebsch–Gordon coefficient.

To find the spectrum of (7.13) or (7.14), one must perform the appropriate diagonalization of the Hamiltonian, see e.g. [65].

Let us now study the low-temperature behaviour of the nucleus in the microscopic model. The discussion of the traditional Marumori approach can be found in [7], where the following form of the finite Marumori expansions of the bifermion operators was assumed:

$$(-1)^{m-j}A^*_{jm\tau} \longrightarrow c^*_{jm\tau}\Big(1-\sum_{\tau'} c^*_{jm\tau'}c_{jm\tau'}\Big)+\tfrac{1}{2}(-1)^{\tau}\sum_{\tau'} c^*_{jm\tau'}c^*_{jm-\tau'}c_{jm-\tau}(-1)^{\tau'}, \tag{7.16}$$

$$(-1)^{m-j}A_{jm\tau} \longrightarrow c_{jm\tau}$$

$$N_{jm} \longrightarrow 2\sum_{\tau} c^*_{jm\tau}c_{jm\tau}$$

$$\tau', \tau = 0, \pm 1.$$

Then the bosonized Hamiltonian includes the two- and four-Boson terms only, what seems to be encouraging at the first sight. Unfortunately, as we know from section 2 the Marumori expansion formula in the finite case, does not lead to the Hermitean Hamiltonian. Hence, any physical conclusions drawn from (7.16) may be incorrect: compare in this connection [7], where one diagonalizes the Boson Hamiltonian by the use of the nonunitary though linear and canonical, transformations. Obviously, one can use the infinite expansions, which preserve the Hermicity of the Hamiltonian and take account of the Pauli exclusion principle. However in that case, it is extremely difficult to make any explicit calculations with more than a few, lowest order expansion terms. Moreover, the infinite expansions were never proved to converge sufficiently quickly.

The above difficulties with the low-temperature limit of the Hamiltonians (7.13), (7.14) can be successfully overcome, if to use the rigorous Boson expansions of the multifermion operators from section 3.1.

Let us first notice that in the Hamiltonian (7.13) the second and fourth order terms only appear. We have here a clear splitting into the two pairs $\{b^*_{jm\tau_1}, b^*_{j-m\tau_2}\}$ and $\{b_{j'-m'\sigma_1}, b_{j'm'\sigma_2}\}$. Notice further that the bi-operators $b^*_{jm\tau_1}\cdot b^*_{j-m\tau_2}$ and $b_{j'-m'\sigma_1}\cdot b_{j'm'\sigma_2}$ commute to 0, and if multiplied, they produce a vanishing fourth-order term if any of triples in the sequence $\{jm\tau_1, j-m\tau_2, j'-m'\sigma_1, j'm'\sigma_2\}$ coincides with the other.

We define formally:

$$b^*_{jm\tau_1} = \beta^*_{k_1} \qquad b^*_{j-m\tau_2} = \beta^*_{k_2}$$
$$b_{j-m\tau_2} = \beta_{k_1} \qquad b_{jm\tau_2} = \beta_{k_2} \tag{7.17}$$

what gives us the kth two-dimensional (i.e. $\dim K = 2$) Fermi triple: $\{\beta^*_{ki}, \beta_{ki}, \Omega_F\}$ belonging to the family of the mutually commuting segments inside which the canonical anticommutation relations hold. If specified to the bifermion case, with β, β^* standing in the place of the generators b, b^*, the multifermion expansions of section 3.1 read: $\Omega_F \to \Omega_B$

$$\beta^*_{k_1}\beta^*_{k_2} = a^*_{k_1}a^*_{k_2}\epsilon_{k_1k_2}{:}\exp(-a^*_{k_1}a_{k_1} - a^*_{k_2}a_{k_2}){:}$$
$$\beta_{k_1}\beta_{k_2} = {:}\exp(-a^*_{k_1}a_{k_1} - a^*_{k_2}a_{k_2}){:}a_{k_1}a_{k_2}\epsilon_{k_2k_1} \tag{7.18}$$
$$\beta^*_{k_i}\beta_{k_i} = a^*_{k_i}{:}\exp(-a^*_{k_1}a_{k_1} - a^*_{k_2}a_{k_2}){:}a_{k_i}, \qquad i = 1, 2$$

where ϵ_{ij} is the two-point Levi–Civitta tensor: $(\epsilon_{ij}) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The finite Boson expansions read:

$$\beta^*_{k_1}\beta^*_{k_2} = \mathbf{1}_F a^*_{k_1}a^*_{k_2}\mathbf{1}_F\epsilon_{k_1k_2}$$
$$\beta_{k_1}\beta_{k_2} = \mathbf{1}_F a_{k_1}a_{k_2}\mathbf{1}_F\epsilon_{k_2k_1}$$
$$\beta^*_{k_i}\beta_{k_i} = \mathbf{1}_F a^*_{k_i}a_{k_i}\mathbf{1}_F. \tag{7.19}$$

Here ϵ_{ij} plays the role of the sign factor, while $\mathbf{1}_F$ restores the Pauli exclusion principle in the pure Boson scheme. Identities (7.19) hold only under the sign of the bilinear form: i.e. if multiplied from both sides by an antisymmetric tensor and summed over all the indices. The substitution of (7.19) into (7.14) gives the correct Boson expansion of the nuclear Hamiltonian:

$$H = \mathbf{1}_F H_B \mathbf{1}_F$$

$$H_B = \sum_{j,\, m>0} (\epsilon_j - \mu) 2 \sum_{\tau} a^*_{jm\tau} a_{jm\tau} + \tfrac{1}{2} \sum_{\substack{jj'mm'>0 \\ \sigma\tau}} V(j, j', m, m', \tau_1, \tau_2, \sigma_1, \sigma_2)$$

$$\times \epsilon_{(jm\tau_1)(j-m\tau_2)} \cdot \epsilon_{(j'm'\sigma_2)(j'-m'\sigma_1)} a^*_{jm\tau_1} a^*_{j-m\tau_2} a_{j'm'\sigma_1} a_{j'm'\sigma_2} \tag{7.20}$$

with $\sigma, \tau = \pm \frac{1}{2}$.

We have thus derived a very simple, purely Bosonic Hamiltonian for the microscopic model of the atomic nuclei, which exhibits all the essential properties, as e.g. the Pauli exclusion principle, of the starting one. For this Hamiltonian the diagonalization procedure can be performed on the pure Boson level.

The only difficulty arises here if to try to compare the correct expansion formula with this in the Marumori approach. Namely, the Bosons we have used to perform the construction are rather unphysical, as carrying the odd isospin quantum numbers. They are the ghost Bosons, which in the weak excitation limit behave like the physical Fermions. This situation can be easily improved. Let us assume:

$$V(j, j', m, m', \tau_1, \tau_2, \sigma_1, \sigma_2) = (-1)^{j-m+j-m'} \sum_{T\tau} C^{T\tau}_{1/2\tau_1\, 1/2\tau_2} C^{T\tau}_{1/2\sigma_2\, 1/2\sigma_1} \tag{7.21}$$

where $\tau = 0, \pm 1$, $T = 0, 1$. If to neglect the term with $T = 0$, we get, by the use of:

$$\sum_{\tau_1\tau_2} C^{T\tau}_{1/2\tau_1 1/2\tau_2} \epsilon_{(jm\tau_1)(j-m\tau_2)} a^*_{jm\tau_1} a^*_{j-m\tau_2} = c^{*T}_{jm\tau} \tag{7.22}$$

which, by virtue of the orthogonality and normalizability conditions for the Clebsch–Gordan coefficients, see e.g. [65], implies:

$$[c^T_{jm\tau}, c^{*T}_{jm\sigma}]_- = \delta_{\tau\sigma} \tag{7.23}$$

the following, physical, form of our ghost Hamiltonian:

$$H_B = \sum_{j,\, m>0} (\epsilon_j - \mu) 2 \sum_{T\tau} c^{*T}_{jm\tau} c^T_{jm\tau} + \tfrac{1}{2} \sum_{\substack{jmj'm'>0 \\ T\tau}} (-1)^{j-m+j'-m'} c^{*T}_{jm\tau} c^T_{j'm'\tau} \tag{7.24}$$

where the interaction term includes now the physical Bosons only: each one appearing in the place of the ghost pair. These are the nuclear analogues of the Cooper Bosons, which by virtue of $\mathbf{1}_F H_B \mathbf{1}_F = H$ still obey the Pauli exclusion principle.

We see that the explicit use of the orthogonality formulas for the Clebsch–Gordan coefficients, nearly unavoidable needs a two-component structure of the four-Fermion interaction Hamiltonian in the weak excitation limit. Namely, we have here the isospin 1 and isospin 0 "phases" of the weakly excited atomic nucleus.

Restricting considerations to the isospin 1 "phase" only, an obvious comparison with the Marumori approach is possible. It is quite clear that the Marumori formula, even in the finite case, proposes too many terms in the expansion for $A^*_{jm\tau}$ to get a correct equivalence with the starting four-Fermion Hamiltonian.

Let us add that in the second quantization approach to the four-Fermion interactions, one can also use the supplementary Boson field method [66] in which the quanta of the Boson field describe a collective motion of a group of Fermions. However, in that case, the Hamiltonian is not expressed in terms of Bosons only but needs the additional use of the new Fermions, which are constructed from the old Fermions and the supplementary Bosons.

8. The correspondence principle in quantum field theory: quantization of spinor fields

We shall not go beyond the framework of the conventional quantum field theory, and all the considerations are essentially based on its LSZ formulation [77, 78]. The basic assumption is here that any operator quantity characterising a given quantum system: the scalar field at the beginning, admits a decomposition into the power series expansions with respect to the normal ordered products of the free asymptotic fields. With the scalar quantum field $\phi(x) \to_{|t|\to\infty} \varphi(x)$ we associate an algebra of all operators:

$$:F(\varphi): = \sum_n (f_n, :\varphi^n:) = \sum_n \int \mathrm{d}x_1 \ldots \int \mathrm{d}x_n f_n(\boldsymbol{x}_n) :\varphi(x_1) \ldots \varphi(x_n): \tag{8.1}$$

$$\boldsymbol{x}_n = (x_1, \ldots, x_n), \qquad x \equiv (x, t), \qquad x_k \in M^4.$$

With the Fock representation of the CCR algebra (the asymptotic condition) we can introduce the coherent state domain for the operator algebra:

$$\{a^*, a, \Omega_B\}_k, \qquad K = \mathscr{L}^2(\mathbb{R}^3) \ni \alpha, \qquad (\alpha, \bar{\alpha}) = \int_{R^3} \mathrm{d}k \alpha(k)\bar{\alpha}(k) = \|\alpha\|^2$$

$$|\alpha) = \exp(-\tfrac{1}{2}\|\alpha\|^2) \exp(\alpha, a^*)\Omega_B$$

$$(\alpha|a(k)|\alpha) = \langle a(k)\rangle = \alpha(k). \tag{8.2}$$

If $\alpha, \bar{\alpha}$ are the classical Fourier amplitudes of $\varphi^c(x)$:

$$\alpha, \alpha \to a, a^* \Rightarrow \varphi^c(x) \to \varphi(x)$$

we get:

$$(\alpha|\varphi(x)|\alpha) = \langle \varphi(x)\rangle = \varphi^c(x)$$

$$(\alpha|:F(\varphi):|\alpha) = F(\varphi^c) = \sum_n (f_n, \varphi^{cn}) \tag{8.3}$$

what establishes a correspondence between a quantum and the classical level of the scalar field algebra, provided $h = c = 1$.

To restore correctly the corresponding quantum image, while the classical expressions are given, it is extremely useful to employ the so called functional representations of the CCR algebras, which arise in the theory of the functional power series, see e.g. [74–76, 27].

An introduction to the functional methods is given in the Appendix, and here we shall only quote, without any detailed explanations, the basic results.

Namely if we have an operator expression (8.1) and the corresponding classical image $F(\varphi^c) = \Sigma_n \int \mathrm{d}\boldsymbol{x}_n f_n(\boldsymbol{x}_n)\varphi^c(x_1) \ldots \varphi^c(x_n)$, it is sufficient to multiply $F(\varphi^c)$ by $\exp(\bar{\alpha}, \alpha)$ to get so called functional representation of the operator $:F(\varphi):$. The functional power series $F(\varphi^c)\exp(\bar{\alpha}, \alpha)$ play the rôle of the

operator $:F(\varphi)$; in the Bargmann space, which we denote by $\mathscr{F}_B = \mathscr{B}(\mathscr{L}^2(\mathbb{R}^3))$. This situation can be summarized in the correspondence principle; where the correspondence rule reads:

$$\{:F(\varphi):\} \longrightarrow \{F(\varphi^c)\}$$
$$(\alpha|:F(\varphi):|\alpha) = F(\varphi^c) \tag{8.4}$$

while the quantization rule:

$$\{F(\varphi^c)\} \longrightarrow \{:F(\varphi):\}$$
$$F(\varphi^c)\exp(\bar{\alpha}, \alpha) = :F(\varphi):(\bar{\alpha}, \alpha) \Rightarrow :F(\varphi):. \tag{8.5}$$

The pragmatists working in the domain of quantum field theory are strongly convinced that the quite satisfactory classical level for the field algebra associated with any Fermion (Dirac say) field is given in the framework of the Grassmann algebra. This algebra is built of the c-number like but anticommuting objects, what exhibits manifestly the Pauli exclusion principle, whose influence on the starting Fermion level is thus taken into account. There was even founded a complete theory of the anticommuting numbers in the functional-like integration and differentiation procedures, see e.g. [72, 74], to justify the use of anticommuting Schwinger sources [77–81, 74].

On the other hand, we have proved [27, 28] that one can always associate with any element of the Fermion field algebra, the corresponding c-valued functional power series with respect to the Fourier amplitudes of the Dirac spinors. According to the Klauder's prescription [73], one can even get the functions with respect to the free Dirac spinors in case of the quadratic forms at least. So it is rather surprising that no reasonable correspondence of the c-number classical level with the prospective Fermion level was established.

The reason is obvious: the Pauli exclusion principle is not still introduced, because by no means the classical spinors must give account of it. So, the exclusion principle on the classical level is an external and extremely artificial requirement which should be eventually imposed.

To solve the question of the c-number classical level for Fermion fields, we have previously [27, 28], developed the c-number language in the functional formulation of the quantum theory of the Fermi systems: the functional representations of the CAR algebra were invented there. Let us however begin from the chronologically earlier Klauder's proposal [73], whose short presentation will allow to understand better our arguments.

Let us consider a free, mass m scalar time-zero field,

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega}} [e^{ikx} a(k) + e^{-ikx} a^*(k)],$$

and the conjugate momentum

$$\pi(x) = \frac{-i}{(2\pi)^{3/2}} \int \sqrt{\frac{\omega}{2}}\, d^3k [e^{ikx} a(k) - e^{-ikx} a^*(k)].$$

The coherent states can be here defined by:

$$|f, g) = U(f, g)|0),$$

where the unitary operators U are given by:

$$U(f, g) = \exp\left\{ i \int [f(x)\phi(x) - g(x)\pi(x)]\, d^3x \right\} \tag{8.6}$$

and are labelled by two real, smooth test functions f, g. The overlap of two such states is given by:

$$(f, g \mid f', g') = \exp\left\{ -\tfrac{1}{4} \int [(k^2 + m^2)^{-1/4} |\tilde{f} - \tilde{f}'|^2 + (k^2 + m^2)^{1/4} |\tilde{g} - \tilde{g}'|^2 + 2\,\mathrm{i}(\tilde{f}^*\,\tilde{g}' - \tilde{g}^*\tilde{f}')]\,\mathrm{d}^3k \right\}. \tag{8.7}$$

Here $\tilde{f}$ denotes the Fourier transform of f. These states are so over-complete that diagonal matrix elements (the coherent state expectation values):

$$(f, g|G|f, g) = G(f, g) \tag{8.8}$$

of an operator polynomial uniquely define the operator. One can easily prove that the operator G is $G = :G(\pi, \phi)$: where the : . : denotes the normal ordering with respect to the creation and annihilation operators a^*, a,

From this place let us consider a nonrelativistic Fermion model, with the property that $\psi(x)|0) = 0$. It is equivalent to the assumption that $\psi(x)$ in fact represents the positive frequency portions of both the conventional $\psi(x)$ and $\psi^*(x)$.

In any case, let us define the basic states:

$$|x_1, \ldots, x_n)_{\mathrm{A}} = \psi^*(x_1) \ldots \psi^*(x_n)|0) \tag{8.9}$$

which are antisymmetric in the $\boldsymbol{x}_n$ variables, $\boldsymbol{x}_n = (x_1, \ldots, x_n)$. Now, following Friedrichs [86], let us introduce an ordering of points, $<$, in configuration space. We shall say that $x < y$ if (1) $x_1 < y_1$ (these are the first coordinates) or (2) if $x_1 = y_1$ and $x_2 < y_2$, or (3) if $x_1 = y_1$, $x_2 = y_2$, and $x_3 < y_3$. Next we introduce the ordering sign function, which we call the Friedrichs–Klauder function:

$$\sigma_n(\boldsymbol{x}_n) = \sigma(x_1, \ldots, x_n) = \pm 1 \tag{8.10}$$

given by the sign of the permutation P necessary to bring the arguments of σ to the "standard" order $x_{p_1} < x_{p_2} < \ldots < x_{p_n}$; if any two x's are equal, σ is defined to be zero.
With the aid of σ we define the symmetric vectors

$$|x_1, \ldots, x_n)_{\mathrm{S}} = \sigma(x_1, \ldots, x_n)|x_1, \ldots, x_n)_{\mathrm{A}} \tag{8.11}$$

which vanish, if any pair of arguments are equal. Note that, by virtue of considerations of section 3.1, any symmetric Fock space vector, admits a decomposition:

$$\begin{aligned} |x_1, \ldots, x_n) &= [\sigma_n(\boldsymbol{x}_n)]^2 |x_1, \ldots, x_n) + \{1 - [\sigma_n(\boldsymbol{x}_n)]^2\} |x_1, \ldots, x_n): \\ &= |x_1, \ldots, x_n)_{\mathrm{S}} + |x_1, \ldots, x_n)_{\mathrm{B}} \end{aligned} \tag{8.12}$$

to a part of which, we have restricted the considerations.

As the special example of the isomorphism E_n, let us see that the relation:

$$|x_1, \ldots, x_n)_{\mathrm{A}} = \sigma(x_1, \ldots, x_n)|x_1, \ldots, x_n)_{\mathrm{S}} \tag{8.13}$$

holds as well, by virtue of $\sigma_n^2 = 1$.

Armed with the symmetric states (8.11), we form:

$$\begin{aligned} |\chi) &= N \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^3\boldsymbol{x}_n \chi(x_1) \ldots \chi(x_n)|x_1, \ldots, x_n)_{\mathrm{S}} \\ \mathrm{d}^3\boldsymbol{x}_n &= \mathrm{d}^3x_1 \ldots \mathrm{d}^3x_n \end{aligned} \tag{8.14}$$

where N is the suitable normalization factor, and $\chi(x)$ denotes a complex, smooth c-number test function. These states play now the rôle of the Fermion "coherent states". Although these states are

not eigenstates of the Fermion field operator, they exhibit the essential property that:

$$(\chi|\psi^*(x)\psi(x)|\chi) = \chi^*(x)\chi(x) \tag{8.15}$$

which is a prototypical relation needed to derive classical images for the energy operator, which is a quadratic form.

We do not just wish to tilt at windmills and to advocate any pure c-number point of view, against the conventional Grassmann tools, especially because these last are widely spread and quite convenient for the explicit calculations (of propagators for example). We wish however to prove that the correspondence principle of the kind (8.15) can be established quite generally, by the use of the Boson expansion methods.

To get the Fock representation suitable for the description of a free Dirac field, we must start from the triples $\{a^*, a, \Omega_B\}_{\otimes^4 \mathscr{L}^2(\mathbf{R}^3)}$ and $\{b^*, b, \Omega_B\}_{\otimes^4 \mathscr{L}^2(\mathbf{R}^3)}$ exhibiting the number four of the internal degrees of freedom (spin and charge) in the theory. All results on the Boson expansions of the underlying Fermion operators hold here without any essential changes if compared with section 3.1, see e.g. [28, 76]. The standard construction

$$b^+ = \frac{1}{\sqrt{2}}\begin{bmatrix} b_1^* + ib_3^* \\ b_2^* + ib_4^* \end{bmatrix} \qquad b^- = \frac{1}{\sqrt{2}}\begin{bmatrix} b_1 + ib_3 \\ b_2 + ib_4 \end{bmatrix}$$
$$b^{*+} = \frac{1}{\sqrt{2}}\begin{bmatrix} b_1^* - ib_3^* \\ b_2^* - ib_4^* \end{bmatrix} \qquad b^{*-} = \frac{1}{\sqrt{2}}\begin{bmatrix} b_1 - ib_2 \\ b_3 - ib_4 \end{bmatrix} \tag{8.16}$$

(the analogous formulas for Boson operators) allows to get the quintets $\{b^{\pm}, b^{*\pm}, \Omega_B\}_{\oplus^4 \mathscr{L}^2(\mathbf{R}^3)}$ and $\{a^{\pm}, a^{*\pm}, \Omega_B\}_{\oplus^4 \mathscr{L}^2(\mathbf{R}^3)}$ with:

$$[b^+(f), b^{*-}(g)]_+ = (\bar{f}, g)\mathbf{1}_F = [b^-(f), b^{*+}(g)]_+$$
$$[a^+(f), a^{*-}(g)]_- = (\bar{f}, g)\mathbf{1}_B = [a^-(f), a^{*+}(g)]_- \tag{8.17}$$

the other (anti)commutators vanish.

With the Haag–LSZ expansion conjecture extended onto the case of Dirac fields ($\psi, \bar{\psi}$ are here the asymptotic free limits) we have for any elements of the field algebra the following operator expansion:

$$:\Omega(\psi, \bar{\psi}): = \sum_{nm} \frac{1}{n!m!}(\omega_{nm}, :\psi^n \bar{\psi}^m:) \tag{8.18}$$
$$= \sum_{nm} \frac{1}{n!m!} \sum_{\sigma\tau} \int d\mathbf{x}_n \int d\mathbf{y}_m \omega_{nm}^{\sigma\tau}(\mathbf{x}_n, \mathbf{y}_m) :\psi_{\sigma_1}(x_1) \dots \psi_{\sigma_n}(x_n)\bar{\psi}_{\tau_1}(y_1) \dots \bar{\psi}_{\tau_m}(y_m):$$

σ, τ are bispinor indices, and – denotes the Dirac conjugation of bispinors.

$:\Omega(\psi, \bar{\psi}):$ can be rewritten in the following form, resulting from the normal ordering of operators: the total antisymmetry of ω_{nm} in all $n + m$ variables is here essential:

$$:\Omega(\psi, \bar{\psi}): = \sum_{nm} \frac{1}{n!m!}(\omega_{nm}, :(\psi^+ + \psi^-)^n(\bar{\psi}^+ + \bar{\psi}^-)^m:) \tag{8.20}$$
$$= \sum_{nm} \frac{1}{n!m!}\left(\omega_{nm}, : \sum_k^n \binom{n}{k}(\psi^+)^k(\psi^-)^{n-k} \sum_l^m \binom{m}{l}(\bar{\psi}^+)^l(\bar{\psi}^-)^{m-l}:\right)$$
$$= \sum_{nmkl} \frac{1}{n!m!k!l!} \sum_{\mu\nu\sigma\rho} \int d\mathbf{x}_n \int d\mathbf{y}_m \int d\mathbf{z}_k \int d\mathbf{u}_l \omega_{nmkl}^{\mu\nu\rho\sigma}(\mathbf{x}_n, \mathbf{y}_m, \mathbf{z}_k, \mathbf{u}_l)$$
$$\times \psi_{\mu_1}^+(x_1) \dots \psi_{\mu_n}^+(x_n)\bar{\psi}_{\nu_1}^+(y_1) \dots \bar{\psi}_{\nu_m}^+(y_m)\psi_{\rho_1}^-(z_1) \dots \psi_{\rho_k}^-(z_k)\bar{\psi}_{\sigma_1}^-(u_1) \dots \bar{\psi}_{\sigma_l}^-(u_l)$$
$$= \sum_{nmkl} \frac{1}{n!m!k!l!}(\omega_{n+k, m+l}, (\psi^+)^n(\bar{\psi}^+)^m(\psi^-)^k(\bar{\psi}^-)^l)$$

where the operators $\psi^{\pm}, \bar{\psi}^{\pm}$ depend linearly, through Fourier transformations, on Fermion creation and annihilation operators $b^{\pm}, b^{*\pm}$, defined by (8.17):

$$[b_i^+(k), b_j^{*-}(p)]_+ = \delta_{ij}\delta(k-p)\mathbf{1}_F = [b_i^-(k), b_j^{*+}(p)]_+ \tag{8.21}$$

the other anticommutators vanish, and the indices $i, j = 1, 2$ denote the helicity states while μ, ν are the bispinor indices.

In (8.20) we have clearly distinguished the two groups of operators, $(\psi^+)^n(\bar{\psi}^+)^m$ and $(\psi^-)^k(\bar{\psi}^-)^l$ which involve respectively the $n+m$ point product of b^*'s and the $k+1$ point product of b's.

By virtue of considerations of section 3.1 we have here:

$$\begin{aligned} & b_{i_1}^+(k_1)\dots b_{i_n}^+(k_n)b_{j_1}^{*+}(p_1)\dots b_{j_m}^{*+}(p_m)b_{s_1}^-(q_1)\dots b_{s_k}^-(q_k)b_{t_1}^{*-}(r_1)\dots b_{t_l}^{*-}(r_l)\mathscr{F}_F \\ & \overset{a}{=} \sigma_{n+m}(\boldsymbol{k}_n, \boldsymbol{p}_m)\underset{\sim}{\sigma_{k+l}}(\boldsymbol{q}_k, \boldsymbol{r}_l)\mathbf{1}_F a_{i_1}^+(k_1)\dots a_{i_n}^+(k_n)a_{j_1}^{*+}(p_1)\dots a_{j_m}^{*+}(p_m) \\ & \quad \times a_{s_1}^-(q_1)\dots a_{s_k}^-(q_k)a_{t_1}^{*-}(r_1)\dots a_{t_l}^{*-}(r_l)\mathbf{1}_F\mathscr{F}_F, \end{aligned} \tag{8.22}$$

$$\sigma_n(\boldsymbol{k}_m)a_{i_1}^+(k_1)\dots a_{i_n}^+(k_n) := \sigma(i_1k_1, i_2k_2, \dots, i_nk_n)a_{i_1}^+(k_1)\dots a_{i_n}^+(k_n)$$

where $\overset{a}{=}$ means that the identity holds true only under the sign of the bilinear form, i.e. if integrated over all variables while multiplied from both sides by a suitable, antisymmetric $n+m+k+l$ point function; $\sigma_{k+l}(\boldsymbol{q}_k, \boldsymbol{r}_l) = \sigma(t_l r_l, \dots, t_1 r_1, s_k q_k, \dots, s_1 q_1)$ what means that $\sim$ reverses the order of variables. The operators $a^{\pm}$, $a^{*\pm}$ stand for the creation and annihilation operators of the fictitious subsidiary Boson field ψ^B whose weak excitation limit exhibits the correct physical properties of the Fermion field. This will be the mediating Boson level allowing to get a c-number classical image of the Fermion field algebra. In this place the Fermion Fock space $\mathscr{F}_F$ appears as a subspace $\mathbf{1}_F\mathscr{F}_B$ of the Boson Fock space $\mathscr{F}_B$. These are the representation spaces respectively for the triples $\{b^{\pm}, b^{*\pm}, \Omega_B\}$ and $\{a^{\pm}, a^{*\pm}, \Omega_B\}$. For clarity we shall restrict considerations to the two point product $\psi_\mu^+(x)\psi_\nu^+(y)$ where we immediately get:

$$\begin{aligned} \psi_\mu^+(x)\psi_\nu^+(y)\mathscr{F}_F \overset{a}{=} (1/2\pi)^3 \int \mathrm{d}k(\sqrt{2\omega_k})^{-1} \int \mathrm{d}p(\sqrt{2\omega_p})^{-1} \sum_{ij} v_\mu^{+i}(k)v_\nu^{+j}(p) \\ \times \exp\{\mathrm{i}(kx+py)\}\cdot\sigma_2(k,p)\mathbf{1}_F a_i^+(k)a_j^+(p)\mathbf{1}_F\mathscr{F}_F. \end{aligned} \tag{8.23}$$

Here again $\overset{a}{=}$ means the validity of (8.23) only under the sign of the bilinear form $(\omega_2, (\psi^+)^2)$.

To transform a product of the Fermion fields into a product of the Boson (spinor) fields, there is enough to construct the integral operator I_2 with the property:

$$\begin{aligned} [I_2(\psi^{B+})^2]^\mu{}_\nu(x,y)\mathscr{F}_F &\overset{a}{=} \sum_{\rho\tau} \int \mathrm{d}x' \int \mathrm{d}y' I^{\mu\nu\rho\tau}(x-x', y-y')\mathbf{1}_F\psi_\rho^{B+}(x')\psi_\tau^{B+}(y')\mathbf{1}_F\mathscr{F}_F \\ &\overset{a}{=} \psi_\mu^+(x)\psi_\nu^+(y)\mathscr{F}_F. \end{aligned} \tag{8.24}$$

With the use of the helicity basis:

$$\begin{aligned} & \sum_\mu v_\mu^{+i}(p)v_\mu^{*-j}(p) = \delta_{ij} = \sum_\mu v_\mu^{-i}(p)v_\mu^{*+j}(p) \\ & p = (\boldsymbol{p}, \omega_p) \end{aligned} \tag{8.25}$$

this purpose is realized if the following integral kernel of I_2 is defined:

$$I_2^{\mu\nu\rho\tau}(x-x', y-y') = \frac{1}{(2\pi)^4}\int dq \int dr \exp(-iqx' - iry')$$
$$\times \exp\{i(qx+ry)\}\sum_{ij}\sigma_2(q,r)\, v_\mu^{+i}(q)v_\rho^{*-i}(q)\, v_\nu^{+j}(r)v_\tau^{*-j}(r) \tag{8.26}$$

where $x, y \in M^4$, $q = (\boldsymbol{q}, q_0)$, and $\boldsymbol{q}, \boldsymbol{k}, \boldsymbol{p} \in \mathbb{R}^3$

$$\sigma_2(q,r) := \sigma_2(iq, jr). \tag{8.27}$$

Now (8.23) reads:

$$\psi_\mu^+(x)\psi_\nu^+(y)\mathscr{F}_F \overset{a}{=} \int dx' \int dy' \sum_{\rho\tau} I_2^{\mu\nu\rho\tau}(x-x', y-y')\frac{1}{(2\pi)^3}\int dk(\sqrt{2\omega_k})^{-1}$$
$$\times \int dp(\sqrt{2\omega_p})^{-1}\sum_{ij} v_\rho^{+i}(k)v_\tau^{+i}(p)\exp\{i(kx'+py')\}\mathbf{1}_F a_i^+(k)a_j^+(p)\mathbf{1}_F\mathscr{F}_F$$
$$\overset{a}{=} \int dx' \int dy' \sum_{\rho\tau} I_2^{\mu\nu\rho\tau}(x-x', y-y')\mathbf{1}_F\psi_\rho^{B+}(x')\psi_\tau^{B+}(y')\mathbf{1}_F\mathscr{F}_F$$
$$\overset{a}{=} \mathbf{1}_F[I_2(\psi^{B+})^2]_{\mu\nu}(x,y)\mathbf{1}_F\mathscr{F}_F, \tag{8.28}$$

where the superscript B means that $\psi^{B\pm}$, $\bar\psi^{B\pm}$ appear as the positive and negative respectively parts of the fictitious (as violating the assumptions of the spin-statistics theorem) spinor field, which obeys the Bose–Einstein statistics: Fermi operators are replaced by the Boson operators.

The generalization of (8.28) is obvious, and leads to the identity:

$$\psi_{\mu_1}^+(x_1)\ldots\psi_{\mu_n}^+(x_n)\bar\psi_{\nu_1}^+(y_1)\ldots\bar\psi_{\nu_m}^+(y_m)\psi_{\rho_1}^-(z_1)\ldots\psi_{\rho_k}^-(z_k)\bar\psi_{\sigma_1}^-(u_1)\ldots\bar\psi_{\sigma_l}^-(u_l)\mathscr{F}_F$$
$$\overset{a}{=} \mathbf{1}_F[I_{n+m}(\psi^{B+})^n(\bar\psi^{B+})^m\underset{\sim}{I_{k+l}}(\psi^{B-})^k(\bar\psi^{B-})^l]^{\mu\nu\rho\sigma}(\underset{\sim}{x_n}, \underset{\sim}{y_m}, \underset{\sim}{z_k}, \underset{\sim}{u_l})$$
$$= \int d\underset{\sim}{x'_n}\int d\underset{\sim}{y'_m}\int d\underset{\sim}{z'_k}\int d\underset{\sim}{u'_l}\sum_{\mu'\nu'\sigma'\rho'} I^{\mu\nu\rho\sigma\mu'\nu'\rho'\sigma'}((x-x')_n, (y-y')_m, (z-z')_{\underset{\sim}{k}}, (u-u')_{\underset{\sim}{l}})$$
$$\times \mathbf{1}_F\psi_{\mu_i'}^{B+}(x_1')\ldots\psi_{\mu_n'}^{B+}(x_n')\bar\psi_{\nu_i'}^{B+}(y_1')\ldots\psi_{\nu_m'}^{B+}(y_m')\psi_{\rho_i'}^{B-}(z_1')\ldots\psi_{\rho_k'}^{B-}(z_k')\bar\psi_{\sigma_i'}^{B-}(u_1')\ldots\bar\psi_{\sigma_l'}^{B-}(u_l')\mathbf{1}_F. \tag{8.29}$$

The sign $\sim$ in $\underset{\sim}{k+l}$ reverses the order of the $k+l$ variables. By virtue of (8.29), the following equivalence formula holds on the Fermion Fock space:

$$:\Omega(\psi,\bar\psi):\mathscr{F}_F = \mathbf{1}_F:\Omega^c(\psi^B,\bar\psi^B):\mathbf{1}_F\mathscr{F}_F = \sum_{nm}\frac{1}{n!m!}(\omega_{nm}^c, \mathbf{1}_F:\psi^{B_n}\bar\psi^{B_m}:). \tag{8.30}$$

Here the shorthand notation $\omega_{nm}^c = (\omega_{nm}I_n\underset{\sim}{I_m})$ is used for the coefficient functions.

We have thus proved that with the Fermion field algebra, one can associate a projection of the subsidiary Boson field algebra, so that on $\mathscr{F}_F$ both algebras coincide.

The pure Boson theory, obviously has its own classical image realized either by the use of the coherent state expectation values or by the use of the appropriate functional representations of the CCR algebra:

$$\Omega^c(\psi^c,\bar\psi^c) = :\Omega^c(\psi^B,\bar\psi^B):(\bar\alpha,\alpha)\cdot\exp[-(\bar\alpha,\alpha)] \tag{8.31}$$

where the classical fields $\psi^c, \bar{\psi}^c$ appear as a result of the replacement of the Boson operators $a^{*\pm}, a^{\pm}$ by the classical Fourier amplitudes $\alpha^{*\pm}, \alpha^{\pm}$ respectively. For more details see [76].

In consequence of this result, establishing the correspondence rule for Fermion fields by the mediation of the subsidiary Bosons, one can immediately formulate the quantization prescription for the classical spinors, what completes the derivation of the correspondence rule between spinors and Fermions. Namely, if one starts from the set of functionals of free Dirac (c-numbers) fields, then by the use of the functional representation of the CCR algebra:

$$\Omega^c(\psi^c, \bar{\psi}^c)\exp(\bar{\alpha}, \alpha) = :\Omega^c(\psi^B, \bar{\psi}^B):(\bar{\alpha}, \alpha) \Rightarrow :\Omega^c(\psi^B, \bar{\psi}^B): \tag{8.32}$$

one gets the corresponding set of Boson operators. Building the Pauli exclusion principle into the Boson structure through:

$$:\Omega^c(\psi^B, \bar{\psi}^B): \longrightarrow \mathbf{1}_F:\Omega^c(\psi^B, \bar{\psi}^B):\mathbf{1}_F \tag{8.32}$$

we have finished the job by observing that on the subspace $\mathscr{F}_F$ of the Boson Fock space $\mathscr{F}_B$ we have the identity:

$$\mathbf{1}_F:\Omega^c(\psi^B, \bar{\psi}^B):\mathbf{1}_F\mathscr{F}_F = :\Omega(\psi, \bar{\psi}):\mathscr{F}_F \tag{8.33}$$

defining the Fermion level of the theory.

Note that operators $\mathbf{1}_F:\Omega^c(\psi^B, \bar{\psi}^B):\mathbf{1}_F$ and $\Omega(\psi, \bar{\psi})$ possess exactly the same matrix elements between states from $\mathscr{F}_F$.

9. Plane pendulum in quantum field theory: lattice quantization of the Sine–Gordon system in two-dimensional space-time

A growing interest in soliton solutions of classical nonlinear equations, especially in connection with the Sine–Gordon equation [87, 88] and trials to understand what is the corresponding quantum Sine–Gordon system [44, 46, 89, 90, 92], succeeded in the rather involved and sophisticated correspondence between the so called "quantum soliton" of the Sine–Gordon system and the fundamental Fermion of the Thirring model.

The usual tool in this place was either the canonical quantization procedure or WKB approximation, or perturbation methods in application to the nonlinear equations. On the other hand, one a priori states that a correct quantum Sine–Gordon system is this with $:\sin\phi:$ on the right-hand side of the equation, what is believed to be a quantum map of the $\sin\phi$ appearing on the classical level. See in this connection [89, 90, 126], but also [92] where the massive Sine–Gordon quantum system $(\Box + m^2)\phi(x, t) = \lambda:\sin[\epsilon\phi(x, t) + \Theta]:$ was axiomatically studied and, for the price of the imaginary time, its connection with the classical statistical mechanics was established.

However up to the author's knowledge neither complete nor satisfactory quantization of the classical Sine–Gordon system in two space-time dimensions was proposed so far. The notion of "quantum soliton" is mostly introduced ad hoc and with no physical justification. The only exception in this connection are the Faddeev's papers [89], where a complete characterization of the classical phase space for the Sine–Gordon system was given, together with the semiclassical quantization prescription. We wish to perform here a somewhat naive and intuitive study of the quantization of Sine–Gordon 1-solitons in the lattice approximation of the system. This is a model study, where the

pecularities of the quantum description can be carefully investigated, forming thus the first step in the lattice quantization of the Sine–Gordon equation.

As we know, the linearity of the free field equations allows us to introduce the annihilation and creation operators, by which one can count up the number of particles in a state described by the c-number solution of the same free field equation. One of the most difficult problems in nonlinear field theories comes from the situation that it is not so easy to construct a clear-cut relationship between annihilation and creation operators, and the number of particles in the state described by the c-number soliton-type solution.

On the other hand, in the quantum world one can never control the behaviour of any system in terms of the continuous data functions. One should rather imagine a discrete set of the control points (the averaged experimental outcomes), which in the approximated sense can be eventually extrapolated to a continuous control curve.

The most instructive example, taking this fact into account, is the famous Toda lattice possessing the exactly known solutions, which in the continuous limit goes over to the so called KdV equation known from hydrodynamics [131].

In the traditional derivation of the quantum field theory of a free scalar field, the Hamiltonian (two space-time dimensions are taken for simplicity):

$$H = \int \mathrm{d}x\{\tfrac{1}{2}(\partial\phi/\partial t)^2 + \tfrac{1}{2}(\partial\phi/\partial x)^2 + \tfrac{1}{2}\mu^2\phi^2\}(x, t) \tag{9.1}$$

can be approximated on the finite linear lattice, by the (rescaled) Hamiltonian:

$$H = \Lambda \sum_s \{\tfrac{1}{2} p_s^2 + \tfrac{1}{2}(\nabla x_s)^2 + \tfrac{1}{2}\mu^2 x_s^2\} \tag{9.2}$$

with s enumerating the lattice sites.

The omission of the gradient part reduces the problem to its single site approximation by the linear chain of harmonic oscillators. In this approximation the quantization of the system lies in introducing the quantum oscillators in the place of the classical ones. To restore the complete quantum system we must here perform a translation to the quantum language of the neighbour interaction (gradient) term. One can make it according to [123]:

$$H = \Lambda \sum_s \left\{\frac{p_s^2}{2} + \frac{1}{2}\left(\frac{\mu^2}{\Lambda^2} + D(0)x_s^2\right)\right\} + \frac{1}{2}\sum_{s1 \neq s2} D(s_1 - s_2)x_{s1}x_{s2}. \tag{9.3}$$

Quite analogous procedure can be repeated in case of the Sine–Gordon system. The corresponding Hamiltonian:

$$H = \int \tfrac{1}{2}\{(\partial\phi/\partial t)^2 + (\partial\phi/\partial x)^2 + \lambda(1 - \cos\phi)\}(x, t) \tag{9.4}$$

is approximated on the linear lattice by:

$$H = \Lambda \sum_s \{\tfrac{1}{2}[\pi_s^2 + (\nabla\phi_s)^2] + \lambda(1 - \cos\phi_s)\} \tag{9.5}$$

where again the gradient term is in fact the interaction part of the Hamiltonian, and carries the nearest neighbour coupling. Its omission leaves us with the linear chain of the independent plane pendula, which was the root for the construction of the Scott's mechanical analog transmission line [87, 88] for

the Sine–Gordon pulse. The solitons can be observed experimentally in the chain, though a complete integrability of the lattice Sine–Gordon system was never proved, in contrast to the Toda lattice. We do not pretend here to get the exact solutions of the discrete problem. We wish rather to find a quantum image of 1-soliton solutions in the single-site approximation, together with the quantum term of the Hamiltonian which gives account of the classically observed long-range correlations between the nearest neighbors in the chain.

The quantization of the Sine–Gordon system in the single-site approximation lies obviously in the introduction of quantum pendula in the place of classical pendula: each one singly occupying a single lattice site. The quantum pendulum is a solvable problem [91], with a nondegenerate, positive set of eigenvalues and Mathieu functions as the eigenfunctions. Mathieu functions can be proved to constitute a complete orthonormal set in $\mathscr{L}^2(0, 2\pi)$ which is thus a Hilbert space of pendulary states. The quantum mechanical Hamiltonian though rather not admitting any reasonable number of particles representation, can be always considered in the matrix form:

$$H = \sum_j E_j f_j \otimes \bar{f}_j \tag{9.6}$$

where E_j are energy values while f_j denotes the jth eigenfunction. We need rather the knowledge of the energy spectrum than of the particular operator (creation and annihilation operators can be here easily constructed in the tensor product form) structure.

Following the preliminary formulation [35], we shall now perform the lattice quantization of the Sine–Gordon 1-solitons. They are the solutions of the equation:

$$\Box\phi(x, t) = (\partial^2/\partial x^2 - \partial^2/\partial t^2)\phi(x, t) = m^2 \sin \phi(x, t) \tag{9.7}$$

which are of the form:

$$\phi(x, t) = 4 \tan^{-1} \exp(\pm m(x - vt)/\sqrt{1 - v^2}). \tag{9.8}$$

The energy $E = 8m/\sqrt{1 - v^2}$ and momentum of the soliton can be easily calculated: $P = 8mv/\sqrt{1 - v^2}$, $c = \hbar = 1$.

The approximation of the 1-soliton pulse on the linear lattice is given immediately, if with each site (the spacing is a) to associate a corresponding characteristic function

$$\Delta_s \leftrightarrow \chi_s(x) = \begin{cases} 1 & x \in \Delta_s \\ 0 & x \notin \Delta_s \end{cases},$$

so that:

$$\{\Delta_s\}_{s=0,\pm1,\pm2,\ldots}, \Delta_s \cap \Delta_t = \emptyset \text{ for } s + t, \bigcup_{s=1}^{\infty} \Delta_s \subseteq \mathbb{R}^1, \mu(\Delta_s) = a \ll 1,$$

$$\phi_s(t) = \frac{1}{a} \int_{R^1} dx \chi_s(x) \phi(x, t)$$

$$\mathscr{L}_s(t) = \frac{1}{a} \int_{R^1} dx \{\tfrac{1}{2}[\phi_x^2 - \phi_t^2 + 2m^2(1 - \cos\phi)\}(x, t) \cdot \chi_s(x)$$

$$\mathscr{H}_s(t) = \frac{1}{a} \int_{R^1} dx \{\tfrac{1}{2}[\phi_x^2 + \phi_t^2 + m^2(1 - \cos\phi)\}(x, t) \cdot \chi_s(x). \tag{9.9}$$

Having given the energy density of the 1-soliton $\mathcal{H}(x, t) = \mathcal{H}(\phi)(x, t)$ one can establish a position of the energy centre of the pulse at the initial instant of time $t = 0$. Let us assume that this particular point y, the collective variable (see e.g. Christ's and Gervais' articles in [126]), belongs to the 0th site which is identified with the $s = 0$ interval $y \in \Delta_0$.

In consequence, the 1-soliton is completely described by the following collection of the initial data:

$$\phi_s(0) = \varphi(y + sa) = \varphi_s(y), \qquad \pi_s(0) = \dot{\phi}_s(0) = \pi(y + sa) = \pi_s(y)$$

$$E = \int_{R^1} dx \mathcal{H}(\phi)(x, 0) \cong \sum_s \mathcal{H}(\phi)_s(y) = \sum_s \mathcal{H}_s(y). \tag{9.10}$$

Here $\varphi_0(y) = \varphi(y)$, $\pi_0(y) = \pi(y)$ and all the data tend to $\varphi(y)$ and $\pi(y)$ respectively with $|y| \to \infty$.

From now on, we shall simplify considerations by omitting the collective variable y in all the formulas $\varphi_s(y) = \varphi_s$, $\pi_s(y) = \pi_s$, $E \cong \Sigma_s \mathcal{H}_s$. The uniform motion rule $\phi(x - vt) = \phi(x, t)$ which holds on the continuous level is now approximated by the following motion rule of the set of the initial data:

$$\varphi_s(t) = \varphi_s(y, t) = \varphi(y - vt + sa) \underset{\Delta t = na/v}{\Rightarrow} \varphi_s(t) = \varphi_{s-n}$$

$$\pi_s(t) = \pi_s(y, t) = \pi(y - vt + sa) \Rightarrow \pi_s(t) = \pi_{s-n} \tag{9.11}$$

what is simply the shift of the data along the chain, following from the influence of the neighbour coupling, implied by the gradient term. We have thus separated on the classical level the nonlinear geometry (shape) of the solution from the fully linear dynamics. Let us add that a similar procedure can be repeated also in case of the n-soliton solutions where the number n of collective variables is necessary, for more details see [95].

Let now the quantum chain be given, where in the single site approximation a sequence of independent quantum pendula appears, together with a corresponding single-site basis. We shall try to translate the classical data and motion rule to the quantum lattice.

Let us begin from the question of statistics. Because each site of the lattice is occupied by a single quantum pendulum whose spectrum is positive and nondegenerate, if we pretend to describe the line of quantum pendula, the Pauli exclusion principle should govern its behaviour: the occupation number of each (s, n)th state of the lattice is either 1 or 0. s, n means that the nth energy level of the quantum pendulum is occupied at the sth site.

In consequence the single component Fermions should appear on the quantum level. Because, as we know from the previous section there is not immediate to have a reasonable correspondence between the classical and Fermion level, we shall formulate all the results for the subsidiary mediating Bosons and then, in the sense of the weak excitation limit, the transition to the final Fermion variables will be performed.

Let us denote by E_0, E_1 the energies of the two lowest stationary levels of the quantum pendulum. We assume to have mapped each plane pendulum, whose energy $\mathcal{H}_s$ does not exceed $E_{\min} = E_1 - E_0$, into a non-excited, hence occupying the ground state E_0, quantum pendulum. This receipt is motivated by the naive hope that such energies cannot be quantized, and play in the theory the rôle of an unessential noise. Now, the question of interest there becomes an energy sharing in-between the quantum pendula of the net energy E of 1-soliton, which we consider as the net in the sense of the renormalization by a subtraction of the ground state energy from the total energy at each site of the

lattice. We expect E to be approximated by the sum of quantized portions $E_{(s,n)}$

$$\sup_{\{(s,n)\}} \sum_{(s,n)} \{E_{(s,n)} - E_0\} \cong E. \tag{9.12}$$

Because E is a macroscopic value, the equality in fact holds. The 1-soliton pulse has a finite energy value, what if combined with the requirement (9.12) clearly requires at most finite number of quantum pendula to be simultaneously excited. Note that the free field techniques, especially the Fock space methods, can be used by virtue of this argument: Fock space vectors are the linear combinations of states which in a finite number of entries only (the product states) differ from the vacuum state.

With each single lattice site let us now associate the subsidiary Boson field ϕ_s, whose lattice Fourier expansion:

$$\phi_s = \frac{1}{\sqrt{V}} \sum_k \left\{ a_k^* \exp\left(\frac{ik\pi}{V} s\right) + a_k \exp\left(-\frac{ik\pi}{V} s\right) \right\} = \phi_s^+ + \phi_s^- \tag{9.13}$$

allows to introduce the corresponding creation and annihilation operators:

$$[a_k, a_l^*]_- = \delta_{kl}$$
$$[a_k, a_l]_- = 0, \qquad a_k \Omega_B = 0 \qquad \text{for all } k. \tag{9.14}$$

In the above the normalization constant $V = \dim\{(s, n)\}$ for the set of pairs realizing the supremum in (9.12), and k enumerates the finite set of degrees of freedom (energy levels of pendula reproducing the 1-soliton pulse). The quantum numbers k are defined by the initial 1-soliton data, if to define the appropriate correspondence rule; by the use of the coherent state methods:

$$|f) = \exp\left\{ \sum_k f_k^+ a_k^* \right\} \Omega_B \cdot \exp\{-\|f\|^2 (1/2)\}$$
$$f_s = \frac{1}{\sqrt{V}} \sum_k \left\{ f_k^+ \exp\left(\frac{ik\pi}{V} s\right) + f_k^- \exp\left(-\frac{ik\pi}{V} s\right) \right\}. \tag{9.15}$$

Let us notice that putting $f_s = \varphi_s$, we get

$$(\varphi|\phi_s|\varphi) = \varphi_s. \tag{9.16}$$

We can expect the existence of the proper $\hat{\Phi}_s$ such that $(\varphi|\hat{\Phi}_s|\varphi) = \pi_s$ however for this purpose we must realize the solitary dynamics in the quantum chain.

In the single-site approximation the form of the energy operator immediately follows:

$$H_s = \sum_{s'} a_s^* \left\{ \frac{1}{V} \sum_k \epsilon_k \exp\left[i \frac{k\pi}{V} (s - s') \right] \right\} a_{s'}, \qquad H = \sum_s H_s \tag{9.17}$$

where ϵ_k must still be properly defined.

A total energy operator for the 1-soliton reads then:

$$H = \sum_k a_k^* a_k \epsilon_k. \tag{9.18}$$

The solitary evolution rule on the classical level implies:

$$\varphi_s(t) = \varphi(y + sa - vt) \overset{t = na/v}{\Rightarrow} \varphi_s(t) = \varphi_{s-n}$$
$$\varphi_{s-n} = \frac{1}{\sqrt{V}} \sum_k \left\{ \varphi_k^+ \exp\left(-\frac{ik\pi}{V} n\right) \exp\left(\frac{ik\pi}{V} s\right) + \varphi_k^- \exp\left(\frac{ik\pi}{V} n\right) \exp\left(-\frac{ik\pi}{V} s\right) \right\}$$

so that:

$$\varphi_k^+\left(t=\frac{na}{v}\right)=\varphi_k^+\exp\left(-\frac{ik\pi}{V}n\right). \tag{9.19}$$

In consequence, for $t = na/v$

$$H(t)=\sum_k a_k^*(t)a_k(t)\epsilon_k \tag{9.20}$$

implies:

$$(\varphi|H(t)|\varphi)=(\varphi|H|\varphi)\cong E. \tag{9.21}$$

On the other hand:

$$H=\sum_k H_k \tag{9.22}$$

and $(\varphi|H_k|\varphi)=(\varphi|H_k(t=na/v)|\varphi)$.

If now to require simultaneously:

$$\mathcal{H}_s=\frac{2}{\sqrt{V}}\sum_k \mathcal{H}_k\cos\frac{k\pi s}{V},\qquad H_s=\frac{2}{\sqrt{V}}\sum_k H_k\cos\frac{k\pi s}{V} \tag{9.23}$$

then the correspondence rule $\mathcal{H}_s=(\varphi|H_s|\varphi)$ establishes the following connection between the classical and quantum energy data:

$$E_k-E_0=\mathcal{H}_k/\varphi_k^+\varphi_k^-. \tag{9.24}$$

To get a quantum image of the 1-soliton evolution there is useful to know that, if the quantum gradient term is taken in the form [128]:

$$D_{nm}^2=D^2(n-m)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k\cdot k^2\exp[ik(m-n)]$$

$$a\longrightarrow 0\Rightarrow D^2\longrightarrow -\nabla^2\delta(x-y) \tag{9.25}$$

then an immediate quantum lattice analogue of the space translation operator can be given:

$$P=-i\sum_{nm}\Pi_n D_{nm}\phi_m,\qquad \Pi_n=-\frac{i}{\sqrt{V}}\sum_k\left\{a_k\exp\left(\frac{ik\pi}{V}n\right)-a_k^*\exp\left(-\frac{ik\pi}{V}n\right)\right\}\cdot k_0 \tag{9.26}$$

$$D_{nm}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k\cdot k\exp[ik(m-n)],\qquad [\Pi_n,\phi_m]_-=-i\delta_{nm}$$

so that:

$$\exp(iP_n)\cdot\phi_m\exp(-iP_n)=\phi_{m-n} \tag{9.27}$$

$$\exp(iPvt)\cdot\phi_m\exp(-iPvt)=\phi_m(t)=\phi_{m-vt}\underset{t=na/v}{\Rightarrow}\phi_{m-n}.$$

Obviously, in the sense of the correspondence principle (9.27) is the quantum image of the 1-soliton evolution rule, what seems at the first sight to contradict the ordinary expectations that an energy operator H should play this role rather $(\varphi|\phi_{m-n}|\varphi)=\varphi_{m-n}$.

Let us in this place prove that the energy operator of the just constructed "quantum soliton" cannot be a correct generator of the solitary time translations. For this purpose there is enough to notice that H obeys the restrictions of the Borchers theorem [132]:

Given a one parameter group $U_t = \exp(-iHt)$ where the generator $H \geq c > -\infty$. Denote $F_t = U_t F U_t^{-1}$ for any operator F. If there is a pair of projectors E, F such that for $|t| < \epsilon$, $EF_t = 0$, then for any $t \in \mathbb{R}^1$, $EF_t = 0$.

Let us remark that the solitary evolution rule:

$$t \longrightarrow t - na/v \Rightarrow \varphi_s(t) \longrightarrow \varphi_{s-n}(t) \tag{9.28}$$

can be equivalently described by the motion of the localization volumes (sites) while the 1-soliton not evolving at all:

$$\Delta_s \longrightarrow \Delta_{s-n} \Rightarrow \varphi_s(t) \longrightarrow \varphi_{s-n}(t). \tag{9.29}$$

In the operator language it needs to associate with each Δ_s a corresponding projector E_{Δ_s}.

Let us consider the three sites $\Delta_s, \Delta_{s-1}, \Delta_{s-2}$. Then, obviously $E_{\Delta_s} \cdot E_{\Delta_{s-2}} = 0$, and one needs at least finite time interval $|t| \geq \epsilon$ to get $E_{\Delta_{s-2}} U_t E_{\Delta_s} U_t^{-1} \neq 0$, where for $|t| < \epsilon$, $0 = E_{\Delta_{s-2}} U_t E_{\Delta_s} U_t^{-1}$ holds. In consequence, for neither time t we can get the required transition:

$$\Delta_s \longrightarrow \Delta_s^t = \Delta_{s-2} \tag{9.30}$$

if the positive evolution operator is used. In the connection let us notice that the correct evolution operator $-Pv$ for the quantum image of our 1-soliton is manifestly not positive.

The above arguments justify, in a little bit sophisticated way, the independence of the single sites of the lattice for all times, like this appearing if the gradient term is absent in the Hamiltonian. Does it at all exclude the long range correlations for any class of positive Hamiltonians?.

The above considerations suggest that together with the collective shift operator P, one should introduce a collective velocity operator $\dot{Q}$, which in case of 1-solitons is not proportional to P; as in that case $P\dot{Q}$ would be λP^2, and hence positive† if $\lambda > 0$.

The last step in our considerations is now to make a transition to Fermion variables, which should appear by virtue of the built-in Pauli exclusion principle. The most convenient here, though obviously not unique, tool seems to employ the weak excitation limit concept of the previous sections, and then the map:

$$\begin{aligned} &\mathbf{1}_F \phi_s \mathbf{1}_F = \Psi_s \\ &\mathbf{1}_F H \mathbf{1}_F = H_F \\ &\mathbf{1}_F P \mathbf{1}_F = P_F \end{aligned} \tag{9.31}$$

which realizes the translation of our construction of the "quantum soliton" to the Fermion language.

10. Appendix: Functional integration (eq. differentiation) methods in quantum field theory

Here we shall concentrate on a particular domain of applications of the path integral methods in quantum theory, namely on the theory of the so called functional power series and arising in it the theory of functional representations of the canonical algebras: CCR and CAR respectively. The

† In the particular case under consideration, the velocity operator should be a constant of motion, and its 1-soliton eigenvalue v enters $-P\dot{Q}$.

review of topics is based on the papers [74–76, 27–29], see also [72, 73]. Hilbert spaces of functional power series were introduced into physics by V. Fock, as early as 1934, and were investigated since that time by many authors, including in this number quite abundant mathematical literature of this subject [72].

We are especially interested in their application in connection with Fock space methods, with special account of Fermions and Bosons.

Let us denote by K a Hilbert space defined by the scalar product $\langle \alpha, \beta \rangle$, and let us assume that there exists an involution $\alpha \to \bar{\alpha}$ in $K \ni \alpha, \beta$, satisfying $\bar{\bar{\alpha}} = \alpha$ and $\overline{\langle \alpha, \beta \rangle} = \langle \bar{\alpha}, \bar{\beta} \rangle$.

We can define in this case a bilinear form in K as $(\alpha, \beta) := \langle \alpha, \bar{\beta} \rangle$. The scalar product, bilinear form and involution in K induce the corresponding notions in $\mathcal{H}_n = K^{\otimes n}$:

$$\langle f_n, g_n \rangle, (f_n, g_n) := \langle f_n, \bar{g}_n \rangle, \qquad f_n \longrightarrow \bar{f}_n \tag{10.1}$$
$$(\bar{\bar{f}}_n = f_n, \overline{\langle f_n, g_n \rangle} = \langle \bar{f}_n, \bar{g}_n \rangle; f_n, g_n \in \mathcal{H}_n).$$

Consider now the space $\mathcal{F}$ of sequences $\{f_n\}_{n \in N}$ ($f_n \in \mathcal{H}_n$, $n = 0, 1, \ldots; \mathcal{H}_0 = \mathbb{C}$) satisfying the condition

$$\|f\| = \sqrt{\sum_n \|f_n\|^2} < \infty \tag{10.2}$$

where $\|f_n\| = \sqrt{\langle f_n, f_n \rangle}$

$$\mathcal{F} = \{f = \{f_n\}_{n \in N}: \quad f_n \in \mathcal{H}_n, \quad \|f\| < \infty\} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n. \tag{10.3}$$

An arbitrary element f_n of $\mathcal{H}_n = K^{\otimes n}$ can be represented as a sum of terms which are invariant with respect to the irreducible representations of the symmetry group, acting in the nth tensor product of Hilbert spaces. With the help of the Young's idempotents Y_n, we can write:

$$f_n = \sum_Y Y_n f_n \tag{10.4}$$

where from the various Y_n of particular interest for us will be the two:

$$S_n = \frac{1}{n!} \sum P_n, \qquad A_n = \frac{1}{n!} \sum (-1)^p P_n \tag{10.5}$$

being the symmetrizing and antisymmetrizing operators. The sums are extended over all permutations P_n of n elements. Young's operators have their duals Y_n^{d} among themselves, e.g. $S_n = A_n^{\mathrm{d}}$, $A_n = S_n^{\mathrm{d}}$.

Let us now introduce the inversion operators P_{ik} in $\mathcal{H}_n$ interchanging the indices of the kth and ith element of the tensor product $K^{\otimes n}$. Let further E_n be a bounded operator in $\mathcal{H}_n$ satisfying the relations:

$$E_n^* = E_n, \qquad E_n^3 = E_n$$
$$P_{ik} E_n = - E_n P_{ik}. \tag{10.6}$$

It follows from (10.6) that E_n^2 is a projector, which realizes the following decomposition of $\mathcal{H}_n$:

$$\mathcal{H}_n = \mathcal{H}^1{}_n \oplus \mathcal{H}_n^2$$
$$\mathcal{H}^1{}_n = E_n^2 \mathcal{H}_n, \qquad \mathcal{H}_n^2 = (1 - E_n^2) \mathcal{H}_n. \tag{10.7}$$

If to take into account the Young's decompositions:

$$\mathcal{H}_n = \sum_Y Y_n \mathcal{H}^1{}_n \oplus \sum_Y Y_n \mathcal{H}_n^2 \tag{10.8}$$

then by the basic property of E_n:

$$Y_n\mathscr{H}_n^1 = Y_nE_n^2\mathscr{H}_n, \qquad Y_n\mathscr{H}_n^2 = Y_n(1-E_n^2)\mathscr{H}_n$$
$$E_n^2Y_n = Y_nE_n^2. \tag{10.9}$$

E_n is an automorphism of $\mathscr{H}_n$ consisting of the isomorphisms $Y_n^d\mathscr{H}_n^1 \leftrightarrow Y_n\mathscr{H}_n^1$ what implies:

$$A_n\mathscr{H}_n^1 \leftrightarrow S_n\mathscr{H}_n^1$$
$$E_nf_n^{1A} = f_n^{1S}, \qquad E_n^2f_n^{1A} = f_n^{1A}. \tag{10.10}$$

We are basically interested in the E_n's possessing the additional property

$$E_n^2A_n\mathscr{H}_n = A_n\mathscr{H}_n \Rightarrow A_n\mathscr{H}_n^2 = 0 \tag{10.11}$$

which allows to consider the map:

$$A_n\mathscr{H}_n \leftrightarrow S_n\mathscr{H}_n^1, \qquad E_nf_n^A = f_n^{1S}, \qquad E_n^2f_n^A = f_n^A. \tag{10.12}$$

The simplest examples of E_n are found in [27].
The operator:

$$E_n = \sum_{i_1\dots i_n} e_{i_1}\otimes\cdots\otimes e_{i_n}\epsilon_{i_1\dots i_n}\bar{e}_{i_1}\otimes\cdots\otimes\bar{e}_{i_n} \tag{10.13}$$

defined by the eigenfunctions $\{e_{i_1}\otimes\cdots\otimes e_{i_n}\}$ and the eigenvalues $\epsilon_{i_1\dots i_n}$. E_n is an example if K is separable, and $\{e_{i_1}\otimes\cdots\otimes e_{i_n}\}$ is an orthonormal set in $\mathscr{H}_n$ corresponding to the orthonormal set $\{e_i\}$ in K. In addition we need $\epsilon_{i_1\dots i_n}$ to be the totally antisymmetric Levi–Civitta tensor with $\epsilon^2_{i_1\dots i_n} = 1$ for $i_r \neq i_s$ $(r, s = 1, 2, \dots, n; r \neq s)$ and $\epsilon_{i_1\dots i_n} = 0$ otherwise.

Another example is the operator with the integral kernel:

$$E_n(\boldsymbol{x}_n, \boldsymbol{y}_n) = \sigma(\boldsymbol{x}_n)\delta(x_1-y_1)\dots\delta(x_n-y_n) \tag{10.14}$$

which in case of the greater than one number of the internal degrees of freedom in theory goes over to:

$$E_n^{ij}(\boldsymbol{x}_n, \boldsymbol{y}_n) = \sigma(i_1x_1, \dots, i_nx_n, j_1y_1, \dots, j_ny_n)\cdot\delta(x_1-y_1)\dots\delta(x_n-y_n)\delta_{i_1j_n}\dots\delta_{i_nj_n}. \tag{10.15}$$

In the above $\sigma(x_1, \dots, x_n) = \sigma(\boldsymbol{x}_n)$ is the previously introduced Friedrichs–Klauder sign function.

Above study of the symmetry structure of the nth tensor product is immediately generalized onto $\mathscr{F}$, so that in the symmetric and antisymmetric cases we have:

$$\mathscr{F}^S = \bigoplus_{n=0}^{\infty} S\mathscr{H}_n = \bigoplus_{n=0}^{\infty}\{S\mathscr{H}_n^1 \oplus S\mathscr{H}_n^2\} = \mathscr{F}^{1S}\oplus\mathscr{F}^{2S}$$
$$\mathscr{F}^A = \bigoplus_{n=0}^{\infty} A\mathscr{H}_n = \bigoplus_{n=0}^{\infty}\{A\mathscr{H}_n^1 \oplus A\mathscr{H}_n^2\} = \mathscr{F}^{1A} \tag{10.16}$$
$$\mathscr{F}^{iS} = \bigoplus_{n=0}^{\infty} S\mathscr{H}_n^i, \qquad \mathscr{F}^{iA} = \bigoplus_{n=0}^{\infty} A\mathscr{H}_n^i \qquad (i = 1, 2)$$
$$\mathscr{F}^{2A} = 0 \Rightarrow \mathscr{F}^A \leftrightarrow \mathscr{F}^{1S}.$$

Let us consider now the corresponding Hilbert spaces of functional power series (generating functionals) which are defined as mappings V: $K \ni \vec{\alpha} \to V(\vec{\alpha}) \in \mathbb{C}$ in the following way:

$$B^{S}(K) = B^{S} = \left\{ V^{S}\colon V^{S}(\bar{\alpha}) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} (v_n^{S}, \bar{\alpha}^n), \qquad \|V^{S}\| < \infty \right\}$$

$$B^{A}(K) = B^{A} = \left\{ V^{A}\colon V^{A}(\bar{\alpha}) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} (E_n v_n^{A}, \bar{\alpha}^n), \|V^{A}\| < \infty \right\} \tag{10.17}$$

$$\|V^{S}\|^2 = \sum_{n=0}^{\infty} \|v_n^{S}\|^2, \qquad \|V^{A}\|^2 = \sum_{n=0}^{\infty} \|v_n^{A}\|^2, \qquad v_n \in \mathscr{H}_n.$$

The first definition is the conventional one in the symmetric case, the second definition is the generalization to the antisymmetric case. In this last connection see also [73].

Given a series

$$V(\bar{\alpha}) = \sum_{n} \frac{1}{\sqrt{n!}} (v_n, \bar{\alpha}^n),$$

we construct the series:

$$(g_m, v^{(m)})(\bar{\alpha}) = \sum_{n} \frac{\sqrt{(m+n)!}}{n!} (v_{m+n}, g_m \bar{\alpha}^n) \tag{10.18}$$

obtained from $V(\bar{\alpha})$ by differentiating each term in this series m times according to the formula:

$$\left(g, \frac{\mathrm{d}}{\mathrm{d}\bar{\alpha}}\right) V(\bar{\alpha}) = \lim_{\epsilon \to 0} \epsilon^{-1} \{ V(\bar{\alpha} + \epsilon g) - V(\bar{\alpha}) \} \tag{10.19}$$

where $\bar{\alpha}, g \in K$, $\epsilon \in \mathbb{R}$.

One can prove [75] that (10.18) is equal to the mth derivative of $V(\bar{\alpha})$ so that we can interchange the summation and differentiation:

$$(g_m, \mathrm{d}^m/\mathrm{d}\bar{\alpha}^m) V(\bar{\alpha}) = (g_m, V^{(m)})(\bar{\alpha}). \tag{10.20}$$

We have here the isomorphism of B^{A} and B^{1s}:

$$B^{1s} = \left\{ V^{1s}\colon V^{1s}(\bar{\alpha}) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} (v_n^{1s}, \bar{\alpha}^n), \|v^{1s}\| < \infty \right\}. \tag{10.21}$$

Thus $B^{A} \subset B^{S}$ and we can specialize the results proved in the symmetric case, to the antisymmetric case by putting everywhere $v_n^{1s} = E_n v_n^{A}$. The operator E_n can be omitted in all scalar products due to the properties (10.10) which make $\langle v_n, g_n \rangle$ invariant with respect to E_n if at least one of the elements in $\langle v_n, g_n \rangle$ belongs to $\mathscr{H}_n^1$. Indeed:

$$v_n' = E_n v_n, \qquad g_n' = E_n g_n, \qquad v_n \in \mathscr{H}_n \Rightarrow \langle v_n', g_n' \rangle = (\bar{g}_n, E_n^2 v_n) = \langle v_n, g_n \rangle.$$

In particular:

$$\langle E_n v_n^{A}, E_n g_n^{A} \rangle = \langle v_n^{A}, g_n^{A} \rangle, \qquad \langle E_n v_n^{1s}, E_n g_n^{1s} \rangle = \langle v_n^{1s}, g_n^{1s} \rangle. \tag{10.22}$$

This shows that the theory of Hilbert spaces of functional power series, which is originally developed for pure symmetric coefficient functions v_n works as well in the antisymmetric case, where all scalar products containing symmetric functions, can be replaced by the corresponding scalar products of antisymmetric functions.

The scalar product in the Hilbert space of functional power series can be written formally in

several alternative ways, which explicitly exploit path integration methods:

$$\langle V, V'\rangle = \sum_n \langle v_n, v_n'\rangle = V(\mathrm{d}/\mathrm{d}\bar{\alpha})\bar{V}'(\bar{\alpha})|_{\bar{\alpha}=0}$$
$$= \int V(\bar{\alpha})\bar{V}'(\alpha)\exp(-\|\alpha\|^2)\,\mathrm{d}(\alpha/\sqrt{\pi}) = \int \varphi(x)\bar{\psi}(x)\,\mathrm{d}(x/\sqrt{2\pi}) \tag{10.23}$$

where:

$$\int \mathrm{d}\left(\frac{\alpha}{\sqrt{\pi}}\right) = \int \mathrm{d}\left(\frac{x}{\sqrt{2\pi}}\right)\int \mathrm{d}\left(\frac{y}{\sqrt{2\pi}}\right); \quad \bar{\alpha} = \frac{1}{\sqrt{2}}(x+\mathrm{i}y), \qquad \bar{\alpha}\in K, \qquad x, y \in K_0 \tag{10.24}$$

$$\int \mathrm{d}\left(\frac{x}{\sqrt{2\pi}}\right)V(x) = \lim_{n\to\infty}\int \frac{\mathrm{d}x_1}{\sqrt{2\pi}}\ldots\int \frac{\mathrm{d}x_n}{\sqrt{2\pi}}\,V\left(\sum_1^n x_i e_i\right)$$
$$x \in K_0.$$

K is the Hilbert space defined at the beginning of this section, and K_0 is the corresponding real Hilbert space $(x = (\alpha+\bar{\alpha})/\sqrt{2} = \bar{x}\in K_0,\ (x, x') = \langle x, x'\rangle = \frac{1}{2}(\alpha+\bar{\alpha}, \alpha'+\bar{\alpha}'),\ \|\alpha\|^2 = \frac{1}{2}(\|x\|^2+\|y\|^2)$, $\{e_i\}$ is an orthonormal set in K_0: $(e_i, e_k) = \langle e_i, e_k\rangle = \delta_{ik}$ and $K_0 \ni x = \Sigma\, x_i e_i,\ x_i \in \mathbb{R}$).

In the last expression in (10.23):

$$\varphi(x) = \sum_n \mathcal{J}(v_n, x^n) = \int A(x, \bar{\alpha})V(\alpha)\exp(-\|\alpha\|^2)\,\mathrm{d}(\alpha/\sqrt{\pi}), \tag{10.25}$$

where v_n are the coefficients of the expansions for $V(\bar{\alpha})$ (in the same way $\psi(x)$ corresponds to $G(\bar{\alpha})$), $J(v_n, x^n)$ are the Hermite functionals defined by:

$$J(v_n, x^n) = \frac{1}{\sqrt{n!}}\exp(-\tfrac{1}{4}\|x\|^2)\exp\left\{-\tfrac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}x}, \frac{\mathrm{d}}{\mathrm{d}x}\right)\right\}(v_n, x^n) \tag{10.26}$$

and $(\mathrm{d}/\mathrm{d}x, \mathrm{d}/\mathrm{d}x) = (\mathbf{1}_{11}, \mathrm{d}^2/\mathrm{d}x^2)$ where $\mathbf{1}_{11}$ is the unit operator in K_0. The kernel $A(x, \bar{\alpha})$ of the integral in (10.25) is the generalization

$$A(x, \bar{\alpha}) = \exp\{-\tfrac{1}{4}\|x\|^2 - \tfrac{1}{2}(\bar{\alpha}, \bar{\alpha}) + (x, \bar{\alpha})\} \tag{10.27}$$

of the Bargmann operator [72] to countably dimensional spaces. The Hilbert space of the functional power series can be considered as a carrier space of the algebra of operators in it, defined by the double functional power series:

$$A(\bar{\alpha}, \alpha) = \sum_{n,m}\frac{1}{\sqrt{n!m!}}(a_{nm}, \bar{\alpha}^n\alpha^m) \tag{10.28}$$

whose action as operators from $\mathcal{B}(K)$ into $\mathcal{B}(K)$ is given in accordance with:

$$(AV)(\bar{\alpha}) = V'(\bar{\alpha}) = \sum_n \frac{1}{\sqrt{n!}}\left(\sum_k (a_{nk}, v_k), \bar{\alpha}^n\right)$$
$$= A\left(\bar{\alpha}, \frac{\mathrm{d}}{\mathrm{d}\bar{\gamma}}\right)V(\bar{\gamma})|_{\bar{\gamma}=0} = \int A(\bar{\alpha}, \gamma)V(\bar{\gamma})\exp(-\|\gamma\|^2)\,\mathrm{d}\left(\frac{\gamma}{\sqrt{\pi}}\right). \tag{10.29}$$

To establish a connection of the developed above formalism with the Fock space, let us first notice

that (10.29) induces the following multiplication law for operators:

$$(AB)(\bar{\alpha}, \alpha) = \sum_{nm} \frac{1}{\sqrt{n!m!}} \left(\sum_k (a_{nk}, b_{km}), \bar{\alpha}^n \alpha^m \right)$$
$$= A\left(\bar{\alpha}, \frac{\mathrm{d}}{\mathrm{d}\bar{\gamma}}\right) B(\bar{\gamma}, \alpha)|_{\bar{\gamma}=0} = \int A(\bar{\alpha}, \gamma) B(\bar{\gamma}, \alpha) \exp(-\|\gamma\|^2)\, \mathrm{d}\left(\frac{\gamma}{\sqrt{\pi}}\right) \tag{10.30}$$

so that introducing the double series:

$$\mathscr{A}(\bar{\alpha}, \alpha) = \sum_{nm} \frac{1}{\sqrt{n!m!}} (A_{nm}, \bar{\alpha}^n \alpha^m) \tag{10.31}$$

connected with (10.28) by the relation:

$$A(\bar{\alpha}, \alpha) = \mathscr{A}(\bar{\alpha}, \alpha) \exp(\bar{\alpha}, \alpha) \tag{10.32}$$

we can express the multiplication law (10.30) in terms of $\mathscr{A}(\bar{\alpha}, \alpha)$

$$(A_1 A_2)(\bar{\alpha}, \alpha) = \exp(\bar{\alpha}, \alpha) \cdot \mathscr{A}_1(\bar{\alpha}, \alpha)(*)\mathscr{A}_2(\bar{\alpha}, \alpha) = :\mathscr{A}_1\left(\bar{\alpha}, \frac{\mathrm{d}}{\mathrm{d}\bar{\alpha}}\right)::\mathscr{A}_2\left(\bar{\alpha}, \frac{\mathrm{d}}{\mathrm{d}\bar{\alpha}}\right): \exp(\bar{\alpha}, \alpha) \tag{10.33}$$

where:

$$\mathscr{A}_1(\bar{\alpha}, \alpha)(*)\mathscr{A}_2(\bar{\alpha}, \alpha) = :\mathscr{A}_1\left(\bar{\alpha}, \alpha + \frac{\mathrm{d}}{\mathrm{d}\bar{\alpha}}\right):\mathscr{A}_2(\bar{\alpha}, \alpha) = \mathscr{A}_1(\bar{\alpha}, \alpha) \exp\left\{\left(\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\alpha}, \frac{\vec{\mathrm{d}}}{\mathrm{d}\bar{\alpha}}\right)\right\} \mathscr{A}_2(\bar{\alpha}, \alpha) \tag{10.34}$$

and $:\mathscr{A}(\bar{\alpha}, \mathrm{d}/\mathrm{d}\bar{\alpha}):$ means that in the series (10.31) α is replaced by $\mathrm{d}/\mathrm{d}\bar{\alpha}$ in such a way that $\mathrm{d}/\mathrm{d}\bar{\alpha}$ stands always to the right of $\bar{\alpha}$ (similarly in $:\mathscr{A}(\bar{\alpha}, \alpha + (\mathrm{d}/\mathrm{d}\bar{\alpha}):$).

For $\mathscr{A}(\bar{\alpha}, \alpha) = 1$, ($A_{00} = 1$, $A_{10} = A_{01} = A_{11} = \cdots = 0$) we obtain the unit operator $\mathbf{1}_B$ with the kernel

$$\mathbf{1}_B(\bar{\alpha}, \alpha) = \exp(\bar{\alpha}, \alpha) = \sum_n \frac{1}{n!} (\bar{\alpha}^n, \alpha^n)$$
$$(\mathbf{1}_B V)(\bar{\alpha}) = V(\bar{\alpha}). \tag{10.35}$$

Note that the coefficients in the expression (10.31) for the unit operator are of the form $A_{nm} = \delta_{nm} \mathbf{1}_{mm}$ where $\mathbf{1}_{mm}$ is the unit operator in $K^{\otimes m}$: $(\mathbf{1}_{mm}, \bar{\alpha}^m \alpha^m) = (\bar{\alpha}^m, \alpha^m)$.

The next simple operators after unity are the operators represented by double series (10.32) where $\mathscr{A}(\bar{\alpha}, \alpha)$ is a first order polynomial. There are two such independent operators: the annihilation operator $a(f)$ and the creation operator $a(g)^*$ given by the kernels:

$$a(f)(\bar{\alpha}, \alpha) = \exp(\bar{\alpha}, \alpha)(\alpha, \bar{f})$$
$$a(g)^*(\bar{\alpha}, \alpha) = \exp(\bar{\alpha}, \alpha) \cdot (f, \bar{\alpha}) \tag{10.36}$$

of certain operator valued elements of K, which we shall denote by a and a^*.

One easily derives the commutation relations:

$$[a(f), a(g)^*]_-(\bar{\alpha}, \alpha) = (\bar{f}, g) \exp(\bar{\alpha}, \alpha) \Rightarrow [a(f), a(g)^*]_- = (\bar{f}, g)\mathbf{1}_B \tag{10.37}$$
$$[a(f), a(g)]_-(\bar{\alpha}, \alpha) = 0$$

and:

$$\{a(f_1)^* \ldots a(f_n)^*\}(\bar{\alpha}, \alpha) = \exp(\bar{\alpha}, \alpha) \cdot \{(\bar{\alpha}, f_1) \ldots (\bar{\alpha}, f_n)\} \tag{10.38}$$
$$\{a(f_1) \ldots a(f_n)\}(\bar{\alpha}, \alpha) = \exp(\bar{\alpha}, \alpha) \cdot \{(\bar{f}_1, \alpha) \ldots (\bar{f}_n, \alpha)\}.$$

Furthermore:

$$:\mathcal{A}(a^*, a):(\bar{\alpha}, \alpha) = \exp(\bar{\alpha}, \alpha)\mathcal{A}(\bar{\alpha}, \alpha) = A(\bar{\alpha}, \alpha) \tag{10.39}$$

which enables us to express every operator with a kernel $A(\bar{\alpha}, \alpha)$ of the form (10.31), (10.32) in terms of creation and annihilation operators. In (10.39) $:\mathcal{A}(a^*, a):$ is defined by the equation:

$$:\mathcal{A}(a^*, a): = \sum_{nm} \frac{1}{\sqrt{n!m!}} (A_{nm}, a^{*n} a^m). \tag{10.40}$$

Similarly, one can express every element of $\mathcal{B}(K)$ in terms of creation operators:

$$V(a^*)(\bar{\alpha}, 0) = \sum_n \frac{1}{\sqrt{n!}} (v_n, a^{*n})(\bar{\alpha}, 0) = V(\bar{\alpha}). \tag{10.41}$$

In particular, if $v_n = f^n/\sqrt{n!}$, $f \in K$ we obtain the set of so called principal vectors or coherent states:

$$\exp(f, \bar{\alpha}) = \exp(f, a^*)\,(\bar{\alpha}, 0). \tag{10.42}$$

Evidently $a(f)v_0 = 0$, $v_0 \in \mathbb{C}$ and f_0 (if normalized) is to be considered the vacuum state.:

$$V(\bar{\alpha}) = (V(a^*)v_0)(\bar{\alpha}), \qquad \exp(f, \bar{\alpha}) = (\exp(f, a^*) \cdot v_0)(\bar{\alpha}) \tag{10.43}$$

what shows that each element of $\mathcal{B}(K)$ is obtained from the vacuum by repeated application of creation operators. Together with relations (10.39) they establish the connection between the Hilbert space $\mathcal{B}(K)$ of functional power series and the Fock space $\mathcal{F}_B$.

The triple $\{a^*, a, \Omega_B = v_0, |v_0| = 1\}$ we call a functional representation of the CCR (canonical commutation relations) algebra $\mathcal{B}(K) = \mathcal{F}_B$.

If now to define more complex operators:

$$\begin{aligned} b(f)^*(\bar{\alpha}, \alpha) &= \sum_n \frac{1}{n!} (\bar{\alpha}^{1+n} E_{1+n}, f E_n \alpha^n) \\ b(f)(\bar{\alpha}, \alpha) &= \sum_n \frac{1}{n!} (\bar{f}\bar{\alpha}^n E_n, E_{1+n}\alpha^n) \end{aligned} \tag{10.44}$$

one gets the following formulas, see e.g. [25, 26]

$$\begin{aligned} (v_k, b^k)(\bar{\alpha}, \alpha) &= \sum_n \frac{1}{n!} (v_{\underset{\sim}{k}} \bar{\alpha}^n E_n, E_{k+n}\alpha^{k+n}) \\ (v_k, b^{*k})(\bar{\alpha}, \alpha) &= \sum_n \frac{1}{n!} (\bar{\alpha}^{k+n} E_{k+n}, v_k E_n \alpha^n) \\ ((v_k, b^{*k})(w_i, b^i))(\bar{\alpha}, \alpha) &= \sum_n \frac{1}{n!} (\bar{\alpha}^{k+n} E_{k+n} v_k w_{\underset{\sim}{i}} E_{i+n} \alpha^{i+n}) \end{aligned} \tag{10.45}$$

where $w_{\underset{\sim}{i}}$ denotes the element w_i with inversed order of indices. The calculation of $b \cdot b^*$ gives:

$$\begin{aligned} b(f)\left(\bar{\alpha}, \frac{\mathrm{d}}{\mathrm{d}\bar{\gamma}}\right) b(g)^{**}(\bar{\gamma}, \alpha)|_{\bar{\gamma}=0} &= \sum_n \frac{1}{n!} \left(\bar{f}\bar{\alpha}^n E_n, E_{1+n} \frac{\mathrm{d}^{1+n}}{\mathrm{d}\bar{\gamma}^{1+n}}\right) \sum_m \frac{1}{m!} (\bar{\gamma}^{1+m} E_{1+m}, g E_m \alpha^m)|_{\bar{\gamma}=0} \\ &= \sum_{n!} \frac{n+1}{n!} (\bar{f}\bar{\alpha}^n E_n A_{1+n}, E_{1+n}^2 g E_n \alpha^n) = \sum_n \frac{n+1}{n!} (\bar{f}\bar{\alpha}^n E_n A_{1+n}, g E_n \alpha^n) \\ &= (\bar{f}, g) \sum_n \frac{1}{n!} (\bar{\alpha}^n, E_n^2 \alpha^n) - \sum_n \frac{1}{n!} (\bar{\alpha}^{1+n} E_{1+n}, g\bar{f} E_{1+n} \alpha^{1+n}) \end{aligned} \tag{10.46}$$

so that the canonical anticommutation relations for b and b^* follow at once:

$$[b(f), b(g)^*]_+(\bar{\alpha}, \alpha) = (\bar{f}, g)\mathbf{1}_F(\bar{\alpha}, \alpha)$$
$$[b(f), b(g)]_+(\bar{\alpha}, \alpha) = 0$$
$$\mathbf{1}_F(\bar{\alpha}, \alpha) = \sum_n \frac{1}{n!}(\bar{\alpha}^n, E_n^2 \alpha^n) \Rightarrow [b(f), b(g)^*]_+ = (\bar{f}, g)\mathbf{1}_F$$
$$[b(f), b(g)]_+ = 0. \tag{10.47}$$

Together with the identity $b(f)v_0 = 0$ we have defined the triple $\{b^*, b, \Omega_B = v_0, |v_0| = 1\}$ which we call to generate a functional representation of the CAR (canonical anticommutation relations) algebra. The corresponding carrier space for the representation is selected from $\mathcal{B}(K) = \mathcal{F}_B$ by the use of its unit operator $\mathcal{F}_F = \mathbf{1}_F \mathcal{F}_B \subset \mathcal{F}_B$.

A formulation of functional formulas in the operator language is here immediate, by analogy with the symmetric case. Any vector from $\mathcal{F}_F$ can be generated by the use of series of normal products of creation and annihilation operators. The functional representation formula in the general case reads:

$$:\mathcal{A}(b^*, b):(\bar{\alpha}, \alpha) = \sum_{nm} \frac{1}{\sqrt{n!m!}}(A_{nm}, b^{*n} b^m)(\bar{\alpha}, \alpha) = \sum_{nm} \frac{1}{\sqrt{n!m!}} \sum_k \frac{1}{k!}(\bar{\alpha}^{n+k} E_{n+k}, A_{n\underset{\sim}{m}} E_{m+k} \alpha^{m+k}) \tag{10.48}$$

where $\underset{\sim}{m}$ indicates that the order of the corresponding indices has been reversed.

All the important results of section 3.1 were proved by the use of functional representations.

One can here easily check that really:

$$:F(b^*, b): = :\exp(-(a^*, a)) \cdot \sum_{nm} \frac{1}{\sqrt{n!m!}} \sum_k \frac{1}{k!}(\sigma_{n+k} f_{n\underset{\sim}{m}} \sigma_{m+k}, a^{*k+n} a^{k+m}): \tag{10.49}$$

and further:

$$:F(b^*, b): = :\exp(-(a^*, a)) \cdot F^c(a^*, a): \tag{10.50}$$

where $f^c_{nm} = \sigma_n f_{n\underset{\sim}{m}} \sigma_m$ is a symmetric function with respect to permutations of variables inside groups(n) and $(\underset{\sim}{m})$ respectively, but antisymmetric with respect to permutations from (n) into (m) and conversely.

The Projection Theorem:

$$\mathbf{1}_F :F^c(a^*, a):\mathbf{1}_F \mathcal{F}_F = :F(b^*, b):\mathcal{F}_F \tag{10.51}$$

needs then the proof of the following identity:

$$(:F(b^*, b):V)(\bar{\alpha}) = \sum_{nm} \frac{1}{\sqrt{n!m!}} \sum_k \frac{\sqrt{(k+m)!}}{k!}(\bar{\alpha}^{k+n}, \sigma_{k+n} f_{n\underset{\sim}{m}} v_{k+m}) \tag{10.52}$$

$$= (\mathbf{1}_F :F^c(a^*, a):V)(\bar{\alpha}) = \sum_{nm} \frac{1}{\sqrt{n!m!}} \sum_k \frac{\sqrt{(k+m)!}}{k!}(\bar{\alpha}^{k+n}, \sigma^2_{k+n} \sigma_n f_{n\underset{\sim}{m}} \sigma_m \sigma_{k+m} v_{k+m})$$

needing in fact a simpler identity (a sequence of them):

$$(\bar{\alpha}^{k+n}, \sigma^2_{k+n} \sigma_n f_{n\underset{\sim}{m}} \sigma_m \sigma_{k+m} v_{k+m}) = (\bar{\alpha}^{k+n}, \sigma_{k+n} f_{n\underset{\sim}{m}} v_{k+m}). \tag{10.53}$$

In this place one must notice that the integrations symbolized by the sign of the bilinear form $(\ ,\ ,\ )$

include a non-zero counterpart from these symmetry group decomposition terms of functions which are totally symmetric in the group $k+n$ of variables.

Denoting $A(n, m)$ the antisymmetrization operator of the product $\sigma_n \cdot \sigma_m$ of sign functions, we immediately have:

$$\begin{aligned}\sigma_n f_{n\underset{\sim}{m}} \sigma_m &= \mathrm{Sym}\{\sigma_n \sigma_m f_{n\underset{\sim}{m}}\} + \text{other decomposition terms} \\ &= f_{n\underset{\sim}{m}} A(n, m)\sigma_n \sigma_m + \text{o.d.t.}\end{aligned} \tag{10.54}$$

In consequence:

$$f_{n\underset{\sim}{m}} v_{k+m} = \{\sigma_{k+n}^2 A(n, m)\sigma_n \sigma_m \sigma_{k+m}\} f_{n\underset{\sim}{m}} v_{k+m} + \text{o.d.t.} \tag{10.55}$$

Quite analogously:

$$\sigma_{k+n} f_{n\underset{\sim}{m}} v_{k+m} = \{\sigma_{k+n} A(n, m)\sigma_n \sigma_m \sigma_{k+m}\}\sigma_{k+n} f_{n\underset{\sim}{m}} v_{k+m} + \text{o.d.t.} \tag{10.56}$$

which proves (10.52) and hence (10.51).

As an application of the Projection Theorem one can prove that operators $\mathbf{1}_F a(f)\mathbf{1}_F, \mathbf{1}_F a(f)^* \mathbf{1}_F$ satisfy the canonical anticommutation relations in $\mathscr{F}_F = \mathbf{1}_F \mathscr{F}_B$.

More details can be found in [76].

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