Levy flights and Levy semigroups (May 2009)

Presentation · June 2021

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Levy flights and Levy semigroups

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Jan Łopuszański Memorial Symposium,
Wroclaw 29-30 May 2009

Close encounters with Jan Łopuszański (and Jan Rzewuski) – 30 May 1974

Eminent PhD advisors, during an official ceremony at Leopoldinum, Nov. 1997

Problem: read text in Latin ! Guess who is ready

More on linguistics (Acta Phys. Pol. XII, 87, (1953))

RELATIVISIERUNG DER THEORIE DER STOCHASTISCHEN PROZESSE

von JAN ŁOPUSZAŃSKI

Institut für theoretische Physik, Bolesław Bierut-Universität, Wrocław

(Eingegangen am 28. Oktober 1952)

Es werden die Grundintegralgleichungen der stochastischen Prozesse und zwar die Smoluchowski-Kolmogorow-Chapmanschen, so wie die, die Erhaltung der Wahrscheinlichkeit ausdrückenden, Integralgleichungen auf eine Lorentz-invariante Form gebracht. Es werden dann aus diesen ihnen äquivalente relativistische Differentialgleichungen abgeleitet. Aus der Diskussion der von Kolmogorow angesetzten Bedingungen geht die Unhaltbarkeit dieser Bedingungen im relativistischen Falle hervor. Es werden die Kolmogorowschen Gleichungen auf eine mit der Relativitätstheorie verträgliche Form gebracht und es wird gezeigt, dass diese mit den früher genannten relativistischen Differentialgleichungen übereinstimmen.

PHYSICAL REVIEW E VOLUME 51, NUMBER 5 MAY 1995

Schrödinger problem, Lévy processes, and noise in relativistic quantum mechanics

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Levy Processes and Relativistic Quantum Dynamics
Piotr Garbaczewski
31st Karpacz Winter School of Theoretical Physics, 1995

Cauchy noise and affiliated stochastic processes
Piotr Garbaczewski and Robert Olkiewicz
J. Math. Phys. 40, 1057, (1999)

Lévy flights and Lévy-Schrödinger semigroups

arXiv:0902.3536

Getting started: Brownian motion inspirations

$$\dot{x} = b(x,t) + A(t)$$

$$\langle A(s) \rangle = 0 \qquad \langle A(s)A(s') \rangle = \sqrt{2D}\delta(s-s')$$

$$\partial_t \rho = D\triangle\rho - \nabla\,(b\cdot\rho)\,.$$

Smoluchowski diffusion processes

$$b = \frac{f}{m\beta} = -\frac{1}{m\beta}\nabla V$$

stationary asymptotic regime

$$b = \boxed{b_* = u_* = D\nabla \ln \rho_*\,.}$$

Stationary pdf

$$\rho_*(x) = \exp\left([F_* - V(x)]/k_B T\right) \doteq \exp[2\Phi(x)]$$

$$\rho_*^{1/2} = \exp\Phi \text{ and } b = 2D\nabla\Phi$$

Becoming parabolic

$$\rho(x,t) \doteq \theta_*(x,t)\exp[\Phi(x)]\,.$$

$$\boxed{\partial_t\theta_* = D\Delta\theta_* - \mathcal{V}\theta_*}$$

$$\rho(x,t) \doteq \theta(x,t)\theta_*(x,t) = \int p(y,s,x,t)\rho(y,s)dy$$

$$\boxed{\partial_t\theta = -D\Delta\theta + \mathcal{V}\theta}$$

$$\theta = \theta(x) = \exp\Phi(x)$$

$$\theta \sim \rho_*^{1/2}$$

$$\mathcal{V}(x) = \frac{1}{2}\left(\frac{b^2}{2D} + \nabla b\right) \quad \boxed{= D\,\frac{\Delta\rho_*^{1/2}}{\rho_*^{1/2}}}\,.$$

Schrödinger semigroups

$$\theta_*(t) = [\exp(-t\hat{H})\theta_*](0)$$

$$\boxed{\hat{H} = -D\Delta + \mathcal{V}}$$

$$k(y,s,x,t) = \left(\exp[-(t-s)\hat{H}]\right)(y,x) \quad = \int exp[-\int_s^t \mathcal{V}(X(u),u)du]\, d\mu[s,y \mid t,x]$$

$$\rho(x,t) \doteq \int p(y,s,x,t)\rho(y,s)dy$$

$$k(y,s,x,t) = p(y,s,x,t)\frac{\rho_*^{1/2}(y)}{\rho_*^{1/2}(x)} = p(y,s,x,t)\ \exp[\Phi(y) - \Phi(x)]$$

If $\rho_*(x)$ has the Gibbs form then $\Phi(y) - \Phi(x) = (1/2k_BT)[V(x) - V(y)]$

Lévy flights

$$E[\exp(ipX_t)] = \exp[-tF(p)]$$

$$F(p) = -\int_{-\infty}^{+\infty}[exp(ipy) - 1 - \frac{ipy}{1+y^2}]\nu(dy)$$

Lévy-Schrödinger semigroups

$$exp(-t\hat{H}) = \int_{-\infty}^{+\infty} exp[-tF(k)]dE(k)$$

$$[exp(-t\hat{H})f](x) = [exp(-tF(p))\tilde{f}(p)]^{\vee}(x)$$

Stable noise and its generator

$$F(p) = \lambda|p|^{\mu} \Rightarrow \hat{H} \doteq \lambda|\Delta|^{\mu/2}.$$

$\partial_t \rho = +D\Delta\rho.$

$$\partial_t \rho = -\lambda|\Delta|^{\mu/2}\rho$$

Cauchy noise

$$F(p) = \lambda|p| \rightarrow \hat{H} = F(\hat{p}) = \lambda|\nabla| \doteq \lambda(-\Delta)^{1/2}$$

Response to external potentials

Langevin scenario

$$\dot{x} = b(x) + A^{\mu}(t) \Longrightarrow \partial_t \rho = -\nabla(b \cdot \rho) - \lambda |\Delta|^{\mu/2} \rho$$

Lévy-Schrödinger semigroups

$$\hat{H}_{\mu} \doteq \lambda |\Delta|^{\mu/2} + \mathcal{V}$$

$$\exp(-t\hat{H}_{\mu})$$

$$\partial_t \theta_* = -\lambda |\Delta|^{\mu/2} \theta_* - \mathcal{V}\theta_* .$$

$$\partial_t \theta = \lambda |\Delta|^{\mu/2} \theta + \mathcal{V}\theta$$

Schrödinger's boundary data problem

$$\theta^*(x,t)\theta(x,t) = \rho(x,t)$$

$$\theta_*(x,t) = \rho(x,t) \exp[-\Phi(x)]$$

$$\theta(x) = \exp[\Phi(x)] = \rho_*^{1/2}(x)$$

$$\mathcal{V} = -\lambda \frac{|\Delta|^{\mu/2} \rho_*^{1/2}}{\rho_*^{1/2}}$$

$$b = 2D\nabla\Phi$$

$$\partial_t \rho = \theta \partial_t \theta^* = -\lambda (\exp \Phi) |\Delta|^{\mu/2} [\exp(-\Phi)\rho] - \mathcal{V} \cdot \rho$$

A discord

Processes induced by Cauchy noise

$$(|\nabla|f)(x) = -\frac{1}{\pi}\int \frac{f(z)-f(x)}{|z-x|^2}\,dz$$

Ornstein-Uhlenbeck-Cauchy process

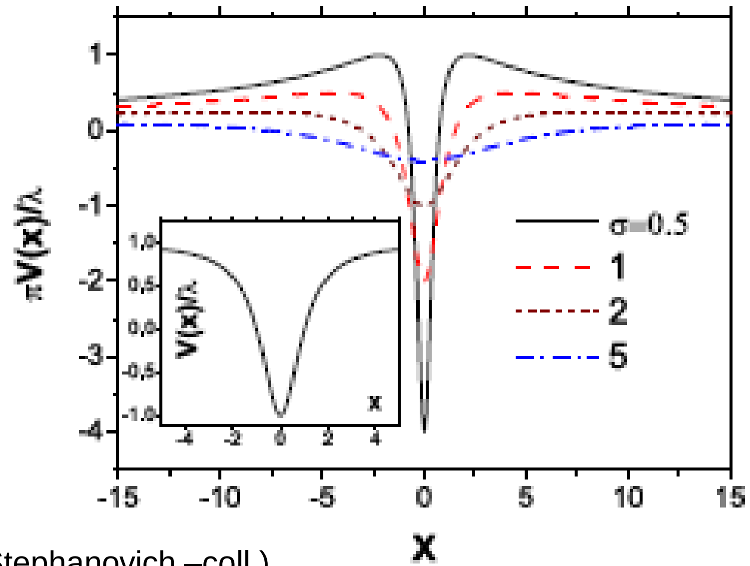
$$\partial_t \rho = -\lambda|\nabla|\rho + \nabla[(\gamma x)\rho] \qquad \rho_*(x) = \frac{\sigma}{\pi(\sigma^2+x^2)}$$

Invariant density vs semigroup potential

$$\mathcal{V} = -\lambda\,\frac{|\Delta|^{\mu/2}\,\rho_*^{1/2}}{\rho_*^{1/2}}$$

$$\mathcal{V}(x) = \frac{\lambda}{\pi}\left[-\frac{2}{\sqrt{a}} + \frac{x}{a}\ln\frac{\sqrt{a}+x}{\sqrt{a}-x}\right]$$

$$a = \sigma^2 + x^2$$



(V. Stephanovich –coll.)

Confined Cauchy process

Invariant density

$$\rho_*(x) = \frac{2}{\pi}\,\frac{1}{(1+x^2)^2}$$

Langevin scenario

$$\partial_t \rho_* = 0 = -\nabla(b\,\rho_*) - \gamma|\nabla|\rho_*$$

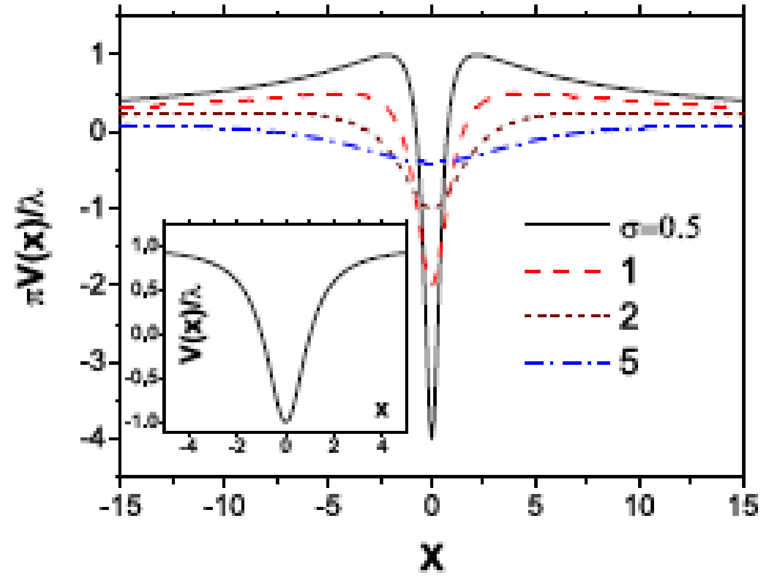
$$b(x) = -\gamma\,(x^2 + 3x)$$

semigroup route

$$\hat{H}_\mu \doteq \lambda|\Delta|^{\mu/2} + \mathcal{V}$$

$$|\Delta|^{1/2} \doteq |\nabla|$$

$$\mathcal{V}(x) = \lambda\frac{x^2-1}{x^2+1}$$



(V. Stephanovich –coll.)

Confined Cauchy family

$$\rho_*(x) = \frac{\Gamma(\alpha)}{\sqrt{\pi}\Gamma(\alpha - 1/2))} \frac{1}{(1+x^2)^\alpha} \qquad \alpha > 1/2$$

Let us consider

$$\rho_*(x) = \frac{16}{5\pi} \frac{1}{(1+x^2)^4}$$

$$\mathcal{V} = -\lambda \frac{|\Delta|^{\mu/2} \rho_*^{1/2}}{\rho_*^{1/2}}$$

$$\mathcal{V}(x) = \frac{\gamma}{2(1+x^2)} (x^4 + 6x^2 - 3)$$

$$b(x) = -\frac{\gamma}{\rho_*(x)} \int (|\nabla|\rho_*)(x)\, dx$$

$$\partial_t \rho = -\nabla(b \cdot \rho) - \lambda |\Delta|^{\mu/2} \rho$$

$$b(x) = -\frac{\gamma x}{16} (5x^6 + 21x^4 + 35x^2 + 35)$$

$$\partial_t \rho_* = 0 = -\nabla(b\, \rho_*) - \gamma |\nabla| \rho_*$$

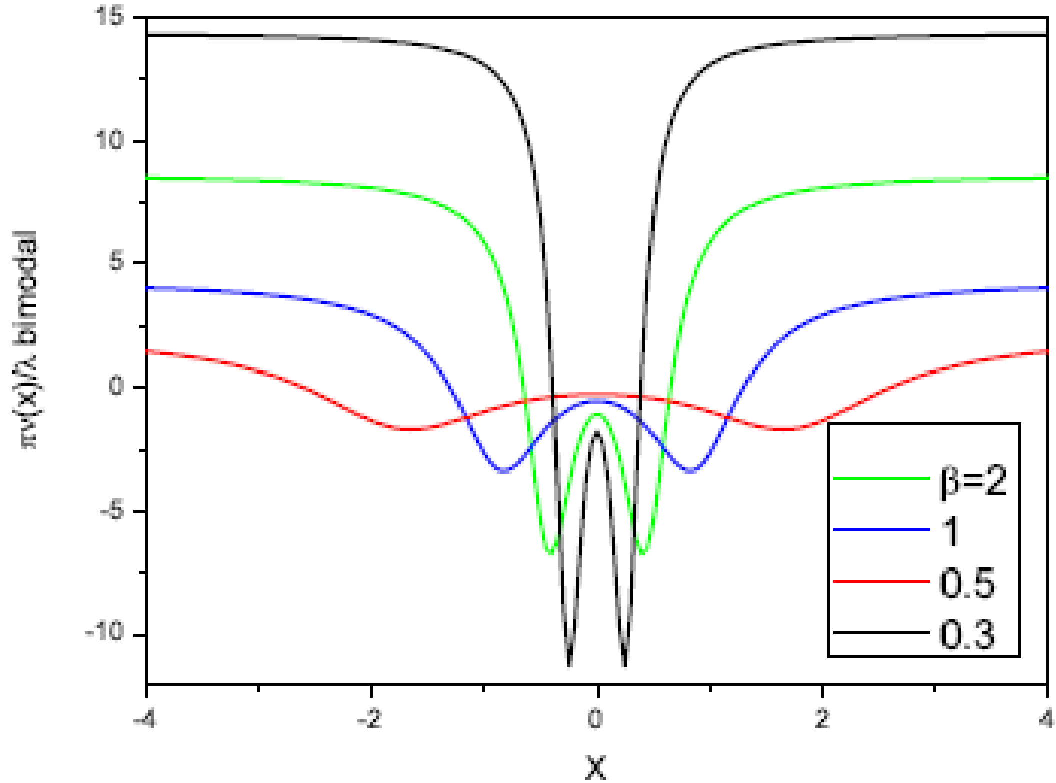
$$\partial_t \rho_* = 0 = -\nabla(b\,\rho_*) - \gamma|\nabla|\rho_*$$

$$\rho_* = \frac{\beta^3}{\pi(x^4 - x^2\beta^2 + \beta^4)}$$

$$b = -\gamma x^3,\ \beta = \lambda/\gamma$$

$$\mathcal{V} = -\lambda\,\frac{|\Delta|^{\mu/2}\,\rho_*^{1/2}}{\rho_*^{1/2}}$$

$$v(x) = \frac{\lambda}{\pi}\sqrt{x^4 - \beta^2 x^2 + \beta^4}\int_{-\infty}^{\infty}\frac{dy}{y^2}\left[\frac{1}{\sqrt{(x+y)^4 - \beta^2(x+y)^2 + \beta^4}} - \frac{1}{\sqrt{x^4 - \beta^2 x^2 + \beta^4}}\right]$$



(V. Stephanovich –coll.)

Hint: Targeted stochasticity, Eliazar and Klafter, J. Stat. Phys, (2003)

Lévy-Driven Langevin Systems: Targeted Stochasticity

$$X(dt) = \underbrace{-f(X(t))\,dt}_{\text{Drift}} + \underbrace{L(dt)}_{\text{Driver}}$$

1. **Evolution:** *What is the Fokker–Planck equation governing the evolution of the pdf of the system's state?*

2. **Steady state:** *In steady state, what is the connection between the system's drift function f, driving noise, and stationary pdf?*

3. **Reverse engineering:** *Given a "target" pdf p, can we "tailor design" a drift function f so that the system's stationary pdf would equal the desired "target" pdf p?*

We have given tentative answers as a byproduct of the previous discussion

Hint: Levy (Cauchy) processes in confining potentials (2009)

$$\partial_t \rho = -\nabla \left(-\frac{\nabla V}{m\beta} \rho \right) - \lambda |\Delta|^{\mu/2} \rho$$

$$\partial_t \rho = \theta \partial_t \theta^* = -\lambda (\exp \Phi) |\Delta|^{\mu/2} [\exp(-\Phi)\rho] - \mathcal{V} \cdot \rho$$

A discord and its analysis

(i) choose a functional form of $V(x)$ and thus the drift of the Langevin-type process, — Done

(ii) infer an invariant density ρ_* that is compatible with the fractional Fokker-Planck equation — Done

$$\exp[\Phi(x)] = \rho_*^{1/2}(x)$$

(iii) given ρ_*, deduce the corresponding Feynman-Kac (e.g. dynamical semigroup) potential $\mathcal{V}$ — Done

$$\mathcal{V} = -\lambda \frac{|\Delta|^{\mu/2} \rho_*^{1/2}}{\rho_*^{1/2}}$$

(iv) use $\mathcal{V}$ and verify whether the "topologically induced dynamics" is affine to Langevin equation with Lévy noise — incomplete

(v) check an asymptotic behavior of $\rho(x,t)$ in both scenarios — incomplete

(vi) repeat the procedure in reverse by starting from (iii) and then deduce the drift for the Langevin equation with Lévy noise — Done

Hint: PG and RO, J .Math. Phys. 40, 1057, (1999)

Corollary 2:

a) The Schrödinger boundary-data and interpolation problem (3)–(6) admits a class of unique solutions in terms of Markov stochastic processes, for each concrete choice of the (Feynman–Kac) kernel function that is determined by the Cauchy generator plus a locally bounded, positive and measurable potential function.

(b) The pertinent processes are of the jump-type and arise as suitable limits of *step* processes. In particular, the uniform in time $t \in [0,T]$ convergence in distribution to the perturbed Cauchy process X_t^V is established, when the potential function is bounded.

Remark 1: The developed techniques can be used to investigate the existence issue (including that of the step process approximation) of more general jump-type processes, in particular those related to the quantum evolution with relativistic Hamiltonians.[5,30]

Remark 2: In the present paper, to simplify calculations and to make formulas more transparent, we have considered processes associated with the Cauchy generator (and thus with the α-stable symmetric process as a major tool) in space dimension 1. A glance at the construction of solutions of the Schrödinger problem makes clear that the previous limitations are inessential. In fact, we could consider any $\alpha \in (0,2)$-symmetric stable processes on R^n, for arbitrary $n \geq 1$, and secure the strict positivity and joint continuity in space variables of the corresponding transition density. Such properties for $n \geq 2$ and for potentials from the Kato class $K_{n,\alpha}$ were established in a very recent publication, Ref. 31, Theorems 3.3 and 3.5. However, an issue of sample path properties and of step-process approximations must be settled separately.