INFORMATION - Is There Anything Where Nobody Looks ?

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Random data collection on the **quantum abuse** (often heavy) of the **classical** terminology:

QUANTUM $\longrightarrow$ mechanics $\longleftarrow$ **CLASSICAL**

state of the system

measurement

groups

chaos

dissipation

Brownian motion

(non-commutative) geometry (commutative)

probability

communication

computing/computer

cryptography/algorithms

information theory

data processing

statistics

Cramér-Rao inequalities

Fisher information/state estimation

information and entropy

QUANTUM $\longrightarrow$ **entropy** and uncertainty $\longleftarrow$ **CLASSICAL**

An incomplete list of **quantum** political/quasi-religious parties, some *in statu nascendi*:

- Bohmians
- Everettians
- Consistent Historians
- Quantum Probabilists
- Spontaneous Collapseans
- Transactionalists
- Contextual Objectivists
- Einselectionists
- Entanglers/Teleporters ?
- Information Science Theoreticians ?
- Algorithmic/Cryptography Magicians?

Some **"APPETIZERS":** (feel provoked, that is right !)

(*) quant-ph/0205039; "Quantum Mechanics as **Quantum Information**" (C. A. Fuchs),
(**) "Quantum States: What the Hell are They ?" (Fuchs' home page),
(***) "How much **Information** in a State Vector ?" (Caves/Fuchs).

- **Why information ?** - "quantum states are states of knowledge, not states of nature" ≡ "the quantum state is solely an expression of information"; → *going sectarian, see above ?*

- "**Information about what ?** - nothing more than the potential consequences of our experimental interventions into the nature" (e.g. the measurement); ; → *then what about Hawking's "Wave Function of the Universe" ?*

- "The whole structure of quantum mechanics may be nothing more than the optimal method of reasoning and **processing information**"; → *then we must answer J. A. Smolin's question "Does Quantum Cryptography Imply Quantum Mechanics ?"*

⇓

- Plea: **"give an information theoretic reason !"** to everything.

DISREPUTABLE (?) AND POSSIBLY OUT-DATED VIEW:

back to state vectors

(*) (Penrose - "The Emperor's New mind")

- "When a system "has" (is in) a **state** $|\psi>$ there ought to be some **property** in (of) the system that corresponds to its "$|\psi>$"-ness"

(**) quant-ph/0312149; "A probabilistic and **information theoretic** interpretation of quantum evolutions" (Oppenheim/Reznik)

- "An isolated system is represented in QM by a **state vector** that conveys statistic predictions for measurement outcomes"

(***) "How much **Information** in a **State Vector** ?" (Caves/Fuchs) - again

INFORMATION THEORY START-UP

finite dimensional Hilbert space

quantum states $\equiv$ density operators

classical (Shannon) **entropy** $\longrightarrow$ quantum (von Neumann) **entropy**

- "**entropy** measures how much **uncertainty** there is in the state of a physical system"

$\Downarrow$

- **entropic** (also called information-theoretic) **uncertainty relations** for finite quantum systems (Deutsch, Maasen/Uffink)

(*) "Normal" way:

$\mu = (\mu_1, \mu_2, ..., \mu_N)$ probability measure on a system of N points,

e. g. $\sum_{j=1}^{N} \mu_j = 1$

Set: $S(\mu) = -\sum_{j=1}^{N} \mu_j \log \mu_j$

(base of the logarithm equals 2, but we recall that $\log b \cdot \ln 2 = \ln b$)

$0 \leq S(\mu) \leq \log N$

Shannon

$\downarrow$

von Neumann

Take a **finite** quantum system (with a finite dimensional Hilbert space, $\dim \mathcal{H} = N$).

Take ρ as the density operator with eigenvalues $\{p_1, p_2, ..., p_N\}$

Set $S(\rho) = -Tr(\rho \log \rho) = -\sum_{j=1}^{N} p_j \, \log p_j$

(Have a nice day, you may begin your "quantum information research".)

(**) "Abnormal" way: *(no nice day any longer !)*

1. Hilbert space dimension **infinite**, from the outset.

2. **We insist** on using the Shannon-type "classical entropy" in the manifestly quantum context, no mention (we are really sorry for that) of von Neumann and his quantum entropy.

3. The principal notion is the **information entropy** for (absolutely) continuous probability distributions and the **entropic uncertainty relations** for observables with continuous spectra (originally named "information-theoretic measures of uncertainty").

SHANNON ENTROPY $\longrightarrow$ INFORMATION ENTROPY

Long **message** (n "entries"); an **"alphabet"** ($N << n$ "letters");

$\mu_j, 1 \leq j \leq N$ - probability of the $j-th$ "letter" , $\mu = (\mu_1, ..., \mu_N)$

$$\sum_1^N \mu_j = 1 \rightarrow \int \rho dx = 1$$
$$\Downarrow$$
$$S(\mu) = -\sum_1^N \mu_j \ln \mu_j \longrightarrow S(\rho) = -\int \rho(s) \ln \rho(s) ds$$

Pedestrian argument (no trace of rigor, basically somewhat controlled "wishful thinking"):

$0 \leq -\sum_1^N \mu_j \ln \mu_j \leq \ln N$

Take an interval of length L on a line and the partition/grating unit

$$\Delta s = L/N$$

Define: $\mu_j \doteq p_j \Delta s$ and notice that:

$$S(\mu) = -\sum_j (\Delta s) p_j \ln p_j - \ln(\Delta s)$$

Try either to keep L constant or make L large, however in both cases improve the coarse-graining precision $\Rightarrow$ the "alphabet" should be extended by new entries, since N needs to grow.

Well, let us fix L and allow N to grow, so that Δs decreases. Then:

$$0 \leq S(\mu) = -\sum_j (\Delta s) p_j \ln p_j - \ln L + \ln N \leq \ln N$$
$$\Downarrow$$
$$\ln(\Delta s) \ \leq \ -\sum_j (\Delta s) p_j \ln p_j \ \leq \ \ln L$$
$$\Downarrow$$
$$S(\rho) = -\int \rho(s) \ln \rho(s) ds$$

$S(\rho)$ is our **information entropy** for the probability measure on the interval L. Ultimately, in the infinite volume $L \rightarrow \infty$ and infinitesimal grating $\Delta s \rightarrow 0$ limits, the information entropy may be unbounded both from below and above. **Bad news ? Perhaps...**

INFORMATION ENTROPY (for pedestrians plus comments).

- Coherent state

$$\rho(x) = \frac{1}{[2\pi\sigma^2]^{1/2}} \exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right]$$
$$\Downarrow$$
$$\mathcal{S}(\rho) = \frac{1}{2}\ln(2\pi e\sigma^2)$$

- Coherent state for the harmonic oscillator

We choose the probability density in the form:

$$\rho(x,t) = \left(\frac{2\pi D}{\omega}\right)^{-1/2} \exp\left[-\frac{\omega}{2D}\,(x-q(t))^2\right]$$

where the classical harmonic dynamics with particle mass m and frequency ω is involved:

$q(t) = q_0\cos(\omega t) + (p_0/m\omega)\sin(\omega t)$
$p(t) = p_0\cos(\omega t) - m\omega q_0\sin(\omega t)$.

We readily get $d\mathcal{S}/dt = 0$, although $\rho = \rho(x,t)$ and the information entropy density $-(\rho\ln\rho)(x,t)$ show up a non-trivial time dependence.

- Free quantum dynamics for a Gaussian wave-packet

Take

$$\rho(x,t) = \frac{\alpha}{[\pi(\alpha^4 + 4D^2t^2)]^{1/2}} \exp\left(-\frac{x^2\alpha^2}{\alpha^4 + 4D^2t^2}\right) . \qquad (1)$$

In this case, the information entropy reads:

$$\mathcal{S}(t) = \frac{1}{2}\ln\left[e\pi\left\langle X^2\right\rangle(t)\right]$$

$$\left\langle X^2\right\rangle \doteq \int x^2\rho dx = (\alpha^4 + 4D^2t^2)/2\alpha^2$$

The information entropy ($\equiv$ the localization uncertainty) grows logarithmically with time.

Side **comment (i):**
For more general probability distributions $p(x)$ with a **fixed** variance σ we would have $S(p) \leq \frac{1}{2}\ln(2\pi e \sigma^2)$. $S(p)$ would become maximized if and only if p is a Gaussian: $p \rightarrow \rho$.

Side **comment (ii)**:
We shall address a general **time-dependent setting**. Before, by admitting $\sigma = \sigma(t)$, we gave a number of examples for time-dependent information entropy $S(\rho_t)$ (c.f. free quantum evolution, in the non-quantum context a good example is the free Brownian motion).

Side **comment (iii):**
Recall the Fourier transform for normalized Schrödinger wave functions, together with the notions of **position and momentum representation** wave packets.
Given an eigenfunction $\psi(x)$ of the energy operator, we denote $(\mathcal{F}\psi)(p)$ its Fourier transform. The corresponding probability densities follow:

$$\rho(x) = |\psi(x)|^2 \quad \text{and} \quad \tilde{\rho}(p) = |(\mathcal{F}\psi)(p)|^2.$$

Denote:

$$S_q = -\int \rho(x) \ln \rho(x) dx \qquad \text{and} \qquad S_p = -\int \tilde{\rho}(p) \ln \tilde{\rho}(p) dp$$

There holds the **entropic uncertainty relation** (Białynicki-Birula/Mycielski) between two forms (position and momentum respectively) of the information entropy:

$$S_q + S_p \geq (1 + \ln \pi)$$

Note:
(i) How to handle momentum entropies for systems confined to the interval or the half-line, c.f. "Canonical Quantization and Impenetrable Barriers" (P. G. + K. W., 2003), and Majernik/Richterek (1997; infinite well information entropies).
(ii) In case of more than one space dimension, an extra factor d (dimensionality) should precede $(1 + \ln \pi)$.

RANDOMNESS, ENTROPY AND INFORMATION - extras

An inherent feature of any **random phenomenon** is that a result of its observation cannot be predicted *a priori* (i.e. **before observation**)"

(*) If X is a **discrete random variable** taking values x_i with probabilities p_i, $i = 1, 2, ..., N$, the quantity

$$\mathcal{S}(X) = -\sum p_i \log p_i$$

is called the **Shannon entropy** of a discrete random variable **or** the entropy of the probability distribution $(p_1, ..., p_N)$.

- The logarithm **log** has base 2 $\rightarrow$ the unit of entropy is called a **bit** (binary digit)

- The natural logarithm **ln** has base e $\rightarrow$ the unit of entropy is called a **nat** (natural)

Note: If X takes infinitely many values $x_1, x_2, ...$ with probabilities $p_1, p_2, ...$, then the entropy $\mathcal{S}(X)$ is not necessarily finite.

(**) For a **continuous random variable** X with values in $x \in R^n$ and the probability density $\rho(x)$ one usually defines the entropy of a continuous random variable (called the differential entropy of X) as:

$$\mathcal{S}(X) = -\int_\Gamma \rho(x) \log \rho(x) dx$$

where $\Gamma \in R^n$ is the support set of X. One may also denote $\mathcal{S}(X) \doteq \mathcal{S}(\rho)$.

Note:

- In the **discrete** case, the entropy quantifies randomness in an *absolute* way.
- In the **continuous** case (there is **no** no smooth limiting passage from the discrete to continuous entropy), the entropy cannot work "as it is" as a measure of "global" randomness.
- However the difference $\mathcal{S}(\rho) - \mathcal{S}(\rho')$ of entropies characterizes the difference in randomness encoded in the functional form of ρ and ρ'.

INFORMATION ENTROPY DYNAMICS: pedestrian examples recalled

- Coherent state

$$\rho(x) = \frac{1}{[2\pi\sigma^2]^{1/2}} \exp\left[- \frac{(x-x_0)^2}{2\sigma^2}\right]$$
$$\Downarrow$$
$$\mathcal{S}(\rho) = \frac{1}{2}\ln(2\pi e\sigma^2)$$

- Coherent state for the harmonic oscillator; $D = \hbar/2m$

$$\rho(x,t) = \left(\frac{2\pi D}{\omega}\right)^{-1/2} \exp\left[-\frac{\omega}{2D}\,(x-q(t))^2\right]$$
$$\Downarrow$$
$$\sigma^2 = \frac{D}{\omega} \to \frac{d\mathcal{S}}{dt} = 0$$

- Free quantum dynamics for a Gaussian wave-packet

$$\rho(x,t) = \frac{\alpha}{[\pi(\alpha^4+4D^2t^2)]^{1/2}} \exp\left(-\frac{x^2\alpha^2}{\alpha^4+4D^2t^2}\right) .$$

$$\Downarrow$$
$$\sigma^2 \to \sigma^2(t) = \frac{\alpha^4+4D^2t^2}{2\alpha^2} \to \frac{d\mathcal{S}}{dt} = \frac{4D^2t}{\alpha^4+4D^2t^2}$$

- Squeezed state of the oscillator (atomic units)

$$\sigma^2 \to \sigma^2(t) = \frac{1}{2}\left(\frac{1}{s^2}\sin^2 t + s^2\cos^2 t\right)$$

- Non-quantum example: free Brownian motion; $D = k_BT/m\beta$

$$\sigma^2 \to \sigma^2(t) = 2Dt$$

DYNAMICS OF INFORMATION: Information entropy production

We consider **time-dependent** probability densities $\rho \doteq \rho(x,t)$

Take for granted that there holds (we consider space dimension one) :

(1) the **Fokker-Planck equation** for the diffusion-type process (best - Markovian):

$$\partial_t \rho = D \triangle \rho - \nabla \cdot (\rho b)$$

with a suitable (? !) forward drift $b = b(x,t)$ of the gradient form $b = \nabla \Phi$. D is a diffusion constant with dimensions of $\hbar/2m$ or $k_B T/m\beta$.

(2) By introducing:

$$u(x,t) = D \nabla \ln \rho(x,t)$$

we can write

$$v(x,t) = b(x,t) - u(x,t)$$

$$\Downarrow$$

$$\partial_t \rho = -\nabla(v\rho)$$

i.e. the **continuity equation.**

Now the **information entropy**, typically is **not** a conserved quantity.

$$\mathcal{S}(t) = -\int \rho(x,t) \, \ln \rho(x,t) \, dx$$

$$\Downarrow$$

(with boundary restrictions that $\rho, v\rho, b\rho$ vanish at spatial infinities or finite interval borders)

$$\frac{d\mathcal{S}}{dt} = \int [\rho \, (\nabla \cdot b) + D \cdot \frac{(\nabla \, \rho)^2}{\rho}] \, dx$$

Remembering that $v = b + u$ and $u = D\nabla \ln \rho$, we have:

$$\frac{d\mathcal{S}}{dt} = \int [\rho\,(\nabla \cdot b) + D \cdot \frac{(\nabla\,\rho)^2}{\rho}]\,dx$$

$$\Updownarrow$$

$$D\dot{\mathcal{S}} \doteq D\,\langle \nabla \cdot b \rangle + \left\langle u^2 \right\rangle = -\,\langle v \cdot u \rangle$$

$$\Updownarrow$$

$$D\dot{\mathcal{S}} = \left\langle v^2 \right\rangle - \langle b \cdot v \rangle$$

$$\Downarrow$$

"Thermodynamic" formalism

Set formally (adjusting dimensional constants):

$$b = \frac{F}{m\beta}$$

Exploit $j \doteq v\rho$ and $F = -\nabla V$ and set $D = k_B T/m\beta$.
Notice that:

$$\frac{d\mathcal{S}}{dt} = \frac{d\mathcal{S}_{prod}}{dt} - \frac{d\mathcal{Q}}{dt}$$

where:

$$\frac{d\mathcal{S}_{prod}}{dt} \doteq \frac{1}{D}\left\langle v^2 \right\rangle \geq 0$$

stands for the **information entropy production**, while:

$$\frac{d\mathcal{Q}}{dt} \doteq \frac{1}{D}\int \frac{1}{m\beta} F \cdot j\,dx = \frac{1}{D}\,\langle b \cdot v \rangle$$

may be interpreted as the **heat dissipation rate**.
Note:

$$k_B T \dot{\mathcal{Q}} = \int F \cdot j\,dx$$

Furthermore, assume that $V = V(x)$ does not depend on time and define:

$$j = \rho D F_{th}$$

with:

$$k_B T F_{th} = F - k_B T \nabla \ln \rho \doteq -\nabla \Psi$$

With

$$k_B T F_{th} = F - k_B T \nabla \ln \rho \doteq -\nabla \Psi$$

consider:

$$\Psi = V + k_B T \ln \rho$$

$$\Downarrow$$

$$\langle \Psi \rangle = \langle V \rangle - T\, \mathcal{S}'$$

where $\mathcal{S}' \doteq k_B \mathcal{S}$.

Minor surprise:
(1) $< \Psi >$ stands for the**Helmholtz free energy**
(2) $< V >$ stands for the (mean) **internal energy**

$$\Downarrow$$

($\rho V v$ needs to vanish at the integration volume boundaries).

$$\left\langle \dot{\Psi} \right\rangle = -k_B T \int F_{th} \cdot j \, dx = -(m\beta) \left\langle v^2 \right\rangle = -k_B T \, \frac{d\mathcal{S}_{prod}}{dt} \leq 0$$

As long as there is an information entropy production, the "Helmholtz free energy" decreases as a function of time towards its minimum. If there is none, the "Helmholtz free energy" reamins constant.

Note: In the above there was no explicit phase-space input nor reference to the standard statistical mechanics/thermodynamics. The temperature T is an artifice as well.

CONCLUSION: *Is there anything where nobody looks ?*

?

Thank you for attention.