Lévy Processes and Relativistic Quantum Dynamics

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Abstract: The traditional Gaussian framework (Wiener process as the "free noise", with the Laplacian as noise generator) is extended to encompass any infinitely divisible probability law covered by the Lévy–Khintchine formula. It implies a family of random environment models (of the fluctuating medium) governed by the generally non–Gaussian "free noises". Since the so called relativistic Hamiltonians $|\nabla|$ and $\sqrt{-\Delta + m^2} - m$ are known to generate such laws, we focus on them for the analysis of probabilistic phenomena, which are shown to be associated with the relativistic quantum propagation once an analytic continuation in time of the corresponding holomorphic semigroup is accomplished. The pertinent stochastic processes are identified to be spatial jump processes.

The Schrödinger equation and the generalized heat equation are connected by analytic continuation in time, known to take the Feynman–Kac (holomorphic semigroup) kernel into the Green function of the corresponding quantum mechanical problem. For $V = V(x), x \in R$, bounded from below, the generator $H = -2mD^2\Delta + V$ is essentially selfadjoint on a natural dense subset of L^2, and the kernel $k(x,s,y,t) = [\exp[-(t-s)H]](x,y)$ of the related dynamical semigroup is strictly positive. The quantum unitary dynamics $\exp(-iHt)$ is a final result of the analytic continuation.

As repeatedly emphasized [1, 2, 3, 4, 5], any temporal evolution that is analyzable in terms of a probability measure may be interpreted as a stochastic process. In view of the Born statistical interpretation postulate for quantum mechanics, the analytic continuation in time induces a class of probability measures, namely, consider $\rho(x,t) = |\psi(x,t)|^2$ as the density of a probability measure associated with a given solution $\psi(x,t)$ of the Schrödinger equation. Then, it is possible to address the problem of that stochastic dynamics which would be either (i) measure preserving or (ii) induce the time evolution of the measure proper. Keep in mind that the Schrödinger equation itself *is not* a genuine partial differential equation of probability theory; rather it is the Born postulate which embeds the unitary evolution problem into the probabilistic framework.

* Supported by the KBN research grant No 2 P302 057 07

A simple illustration of the analytic continuation in time is provided by considering the force–free propagation, where the formal recipe gives rise to the equations of motion (one should be aware that to execute a mapping for concrete solutions, the proper adjustment of the time interval boundaries is indispensable):

$$i\partial_t\psi = -D\triangle\psi \longrightarrow \partial_t\theta_* = D\triangle\theta_* \ ,$$
$$i\partial_t\overline{\psi} = D\triangle\overline{\psi} \longrightarrow \partial_t\theta = -D\triangle\theta \ , \tag{1}$$
$$it \rightarrow t \ .$$

Then

$$\psi(x,t) = \left[\rho^{1/2}\exp(iS)\right](x,t) = \int dx' G(x-x',t)\psi(x',0) \ ,$$
$$G(x-x',t) = (4\pi iDt)^{-1/2}\exp\left[-\frac{(x-x')^2}{4iDt}\right] \ , \tag{2}$$
$$\theta_*(x,t) = \int dx' k(x-x',t)\theta_*(x',0) \ ,$$
$$k(x-x',t) = (4\pi Dt)^{1/2}\exp\left[-\frac{(x-x')^2}{4Dt}\right] \ ,$$

The description in terms of the time adjoint pair of equations is not accidental and reflects the Markov property of probabilistic solutions of the associated Schrödinger problem: find an interpolation between the given pair of boundary (for the process on a finite fixed time interval) probability distributions.

Strictly positive semigroup kernels generated by Laplacians plus suitable potentials are very special examples in a surprisingly rich encompassing family. First of all, the concept of the "free noise", normally characterized by a Gaussian probability distribution appropriate to a Wiener process, can be extended to all infinitely divisible probability distributions via the Lévy–Khintchine formula. It expands our framework from continuous diffusion processes to jump or combined diffusion–jump propagation scenarios. All such (Lévy) processes are associated with strictly positive dynamical semigroup kernels.

Remark: Apart from the wealth of physical phenomena described in terms of Gaussian stochastic processes, there is a number of physical problems where the Gaussian tool–box proves to be insufficient to provide satisfactory probabilistic explanations. Non–Gaussian Lévy processes naturally appear in the study of transient random walks when long–tailed distributions arise [7, 8, 9]. They are also found necessary to analyze fractal random walks [10], intermittency phenomena, anomalous diffusions, and turbulence at high Reynolds numbers [7, 12, 11].

Let us consider Hamiltonians of the form $H = F(\hat{p})$, where $\hat{p} = -i\nabla$ stands for the momentum operator and for $-\infty < k < +\infty$, $F = F(k)$ is a real valued, bounded from below, locally integrable function. Then, $\exp(-tH) = \int_{-\infty}^{+\infty}\exp[-tF(k)]dE(k)$, $t \geq 0$, where $dE(k)$ is the spectral measure of $\hat{p}$.

Most of our discussion will pertain to processes in one spatial dimension, and let us specialize the issue accordingly. Because $(E(k)f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{k} \exp(ipx)\hat{f}(p)dp$, where $\hat{f}$ is the Fourier transform of f, we learn that

$$[\exp(-tH)]\, f(x) = \left[\int_{-\infty}^{+\infty} \exp(-tF(k))dE(k)f\right](x) =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left[-tF(k)\right] \frac{d}{dk} \left[\int_{-\infty}^{k} \exp(ipx)\hat{f}(p)dp\right] dk = \tag{3}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-tF(k)) \exp(ikx)\hat{f}(k)dk = \left[\exp(-tF(p))\hat{f}(p)\right]^{\vee}(x)$$

where the superscript $\vee$ denotes the inverse Fourier transform.

Let us set $k_t = \frac{1}{\sqrt{2\pi}}[\exp(-tF(p))]^{\vee}$, then the action of $\exp(-tH)$ can be given in terms of a convolution: $\exp(-tH)f = f * k_t$, where $(f * g)(x) := \int_R g(x - z)f(z)dz$.

We shall restrict consideration only to those $F(p)$ which give rise to positivity preserving semigroups: if $F(p)$ satisfies the celebrated Lévy–Khintchine formula, then k_t is a positive measure for all $t \geq 0$. The most general case refers to a contribution from three types of processes: deterministic, Gaussian, and an exclusively jump process. We shall concentrate on the integral part of the Lévy–Khintchine formula, which is responsible for arbitrary stochastic jump features:

$$F(p) = -\int_{-\infty}^{+\infty} \left[\exp(ipy) - 1 - \frac{ipy}{1+y^2}\right] \nu(dy) \tag{4}$$

where $\nu(dy)$ stands for the so-called Lévy measure.

The disregarded Gaussian contribution would read $F(p) = p^2/2$. In this case we know in detail how the analytic continuation in time of the Laplacian generated holomorpic semigroup induces a mapping to a quantum mechanical (since the Schrödinger equation is involved) diffusion processes [5, 2, 3].

Our further attention will focus on two selected choices for the characteristic exponent $F(p)$, namely: $F_0(p) = |p|$ and $F_m(p) = \sqrt{p^2 + m^2} - m, m > 0$, where we have chosen suitable units so as to eliminate inessential parameters. (The relativistic Hamiltonian is better known in the form $\sqrt{m^2c^4 + c^2p^2} - mc^2$ where c is the velocity of light.)

The respective Hamiltonians (semigroup generators) H_0, H_m are pseudodifferential operators. The semigroup kernels k_t^0, k_t^m in view of the "free noise" restriction (no potentials, will be defined in below) are transition densities of the jump (Lévy) processes regulated by the corresponding Lévy measures $\nu_0(dy)$, $\nu_m(dy)$. It is instructive to notice that as in the case of Gaussian derivations

(1),(2), the pseudodifferential analog of the Fokker–Planck equation can be introduced. Namely, as a consequence of $[\exp(-tH)\overline{\rho}](x) = \overline{\rho}(x,t)$ and in view of the identification $F(p \to -i\nabla) := H$ we arrive at

$$F_0(p) \Longrightarrow \partial_t \overline{\rho}(x,t) = -|\nabla|\overline{\rho}(x,t) \tag{5}$$

or

$$F_m(p) \Longrightarrow \partial_t \overline{\rho}(x,t) = -\left[\sqrt{-\Delta + m^2} - m\right] \overline{\rho}(x,t) \tag{6}$$

respectively.

Although the pseudodifferential generator of the semigroup implies that the Fokker–Planck equation is no longer exclusively differential but an integro-differential equation, each solution $\overline{\rho}(x,t)$ in the present case is nevertheless a solution of a partial differential equation of higher order. Specifically, the respective partial differential equations are of the second order, see[4]. Our two semigroups are holomorphic, hence we can replace the time parameter t by a complex one $\sigma = t + is$, $t > 0$ so that $\exp(-\sigma H) = \int_R \exp(-\sigma F(k))\, dE(k)$. Its action is defined by

$$[\exp(-\sigma H)]\, f = \left[(\hat{f} \exp(-\sigma F)\right]^{\vee} = f * k_\sigma \ . \tag{7}$$

Here, the kernel reads $k_\sigma = \frac{1}{\sqrt{2\pi}}[\exp(-\sigma F)]^{\vee}$. Since H is selfadjoint, the limit $t \downarrow 0$ leaves us with the unitary group $\exp(-isH)$, acting in the same way: $[\exp(-isH)]f = [\hat{f}\exp(-isF)]^{\vee}$, except that now $k_{is} := \frac{1}{\sqrt{2\pi}}[\exp(-isF)]^{\vee}$ in general is *not* a measure. In view of unitarity, the unit ball in L^2 is an invariant of the dynamics. Hence density measures can be associated with solutions of the Schrödinger pseudodiferential equations:

$$F_0(p) \Longrightarrow i\partial_t \psi(x,t) = |\nabla|\psi(x,t) \tag{8}$$

or

$$F_m(p) \Longrightarrow i\partial_t \psi(x,t) = \left[\sqrt{-\Delta + m^2} - m\right] \psi(x,t) \tag{9}$$

provided with the appropriate initial data functions $\psi(x,0)$.

An obvious consequence is that the corresponding partial differential equation of the second order takes on a familiar *relativistic* form

$$F_0(p) \Longrightarrow \Box\psi(x,t) := (-\Delta + \Delta_t)\psi(x,t) = 0 \tag{10}$$

while after setting $\psi(x,t) = \tilde{\psi}(x,t)\exp(imt)$, we arrive at the Klein–Gordon equation:

$$F_m(p) \Longrightarrow (\Box + m^2)\tilde{\psi}(x,t) = 0 \tag{11}$$

where the D'Alembert operator $\Box = -\Delta + \Delta_t$ replaces its Euclidean counterpart $-\Box_E$.

We have thus reached a point, at which our major question can be precisely stated:

What are the stochastic processes consistent with the probability measure dynamics $\rho(x,t) = |\psi(x,t)|^2$, determined by the pseudodifferential equations (8) and (9)?

We have chosen two rather special pseudodifferential counterparts of the Laplacian guided by two reasons: (i) their similarity on analytic grounds (the same criteria [13] for the existence of the bound state spectrum if summed with suitable potentials, (ii) the claim of Ref. [14] that the pertinent stochastic process in the mass $m > 0$ case actually displays the Markov property.

If the Markov property would hold true for the relativistic Hamiltonian generated dynamics, we would be able to repeat almost all steps of the previous, Schrödinger picture, quantum dynamics analysis [1, 2, 3]. However, the situation is not that simple, and the argument of [4] excludes the Markov property, in all nonstationary situations, in a flat contradiction with general statements by De Angelis [14].

Let us introduce some probabilistic notions, which will tell us how to work with pseudodifferential operators. We shall notice that for explicit computational purposes, the Cauchy generator $|\nabla|$ is much more suited than the $m > 0$ relativistic Hamiltonian. It is a real disadvantage when dealing with Lévy processes that rather limited number of concrete examples is available, in contrast to the wealth of the general theory.

The Lévy–Khintchine formula tells us that the action of the Hamiltonian $H = F(-i\nabla)$ on a function in its domain can be represented as follows:

$$(H\,f)(x) = -\int_R \left[f(x+y) - f(x) - \frac{y\,\nabla f(x)}{1+y^2} \right] \nu(dy) \tag{12}$$

It is important to observe that for the "free noise" processes whose semigroup generators are $|\nabla|$ and $\sqrt{-\Delta + m^2} - m$ we do know explicitly their kernels (transition probability densities) and the involved Lévy measures, as well as about the extension of the Feyman–Kac *path integral* construction of the semigroup kernels to these particular Lévy processes [13], in case of arbitrary space dimensions. Therefore we feel free to use the Feynman–Kac kernel notion instead of the semigroup kernel.

For the Cauchy process, whose generator is $|\nabla|$, we deal with a probabilistic classics:

$$\overline{\rho}(x,t) = \frac{1}{\pi}\frac{t}{t^2+x^2} \Longrightarrow k^0(y,s,x,t) = \frac{1}{\pi}\frac{t-s}{(t-s)^2+(x-y)^2} \tag{13}$$

$$0 < s < t$$

$$\langle \exp\left[ipX(t)\right] \rangle := \int_R \exp(ipx)\overline{\rho}(x,t)\,dx = \exp\left[-tF_0(p)\right] = \exp(-|p|t)$$

The characteristic function of $k^0(y,s,x,t)$ for y,s fixed, reads $\exp[ipy - |p|(t-s)]$, and the Lévy measure needed to evaluate the Lévy–Khintchine integral reads:

$$\nu_0(dy) := \lim_{t\downarrow 0}\left[\frac{1}{t}k^0(0,0,y,t)\right] dy = \frac{dy}{\pi y^2} \tag{14}$$

In the case of the relativistic generator $\sqrt{-\Delta + m^2} - m$, formulas determining the stochastic jump process are much less appealing:

$$\langle \exp[ipX(t)] \rangle := \exp[-tF_m(p)] = \exp\left[-t(\sqrt{p^2+m^2}-m)\right]$$

$$\overline{\rho}(x,t) = \frac{m}{\pi}\frac{t\exp(mt)}{\sqrt{x^2+t^2}} K_1(m\sqrt{x^2+t^2}) \tag{15}$$

$$[\exp(-(t-s)F_m(-i\nabla))]\,(x-y) = k^m(y,s,x,t) := \overline{\rho}(x-y,t-s)$$

$$\nu_m(dy) = \frac{m}{\pi|y|}K_1(m|y|)dy$$

where $K_1(z)$ is the modified Bessel function of the third kind of order 1.

We are interested in acting with the pseudodifferential generators $H = F(-i\nabla)$ on functions in the exponential form (recall the familiar Madelung procedure in the Gaussian case) $f(x,t) = \exp\Phi(x,t)$:

$$(H\exp\Phi)(x) = -\int_R \left[\exp\Phi(x+y) \;-\; \exp\Phi(x) \;-\; \frac{y\Phi'(x)\exp\Phi(x)}{1+y^2}\right]\nu(dy) =$$

$$= \exp\Phi(x)\int_R \left[\exp(\Phi(x+y)-\Phi(x)) \;-\; 1 \;-\; \frac{y\Phi'(x)}{1+y^2}\right]\nu(dy) \tag{16}$$

where $\Phi'(x) = \nabla\Phi(x)$. Since $(H\Phi)(x) = -\int_R[\Phi(x+y) - \Phi(x) - y\Phi'(x)/(1+y^2)]\nu(dy)$, we can make a safe rearrangement of (16):

$$(H\exp\Phi)(x) = \exp\Phi(x)\left[(H\Phi)(x) - \int_R(\exp\Phi_{xy} \;-\; 1 - \Phi_{xy})\nu(dy)\right] \tag{17}$$

$$\Phi_{xy} := \Phi(x+y) - \Phi(x)$$

In application to the pseudodifferential dynamics $i\partial_t\psi(x,t) = (H\psi)(x,t)$ with $\psi = \exp(R+iS)$, one easily derives [4] its implications for the real functions $\Theta = \exp(R+S)$ and $\Theta_* = \exp(R-S)$; plus a trivial extension from H to $H+V$ situations.

Remark : Experience [15, 2] with the Gaussian (standard Laplacian generated) noise proves that the Madelung substitution $\psi(x,t) = \exp[R(x,t)+iS(x,t)]$ would associate with the Schrödinger equation a pair of time adjoint generalised diffusion equations where the Feynman–Kac potential (time dependent in the general case) equals $\frac{1}{2mD}[2Q(x,t) - V(x)]$. Here $Q(x,t) = 2mD^2\frac{\Delta\rho^{1/2}}{\rho^{1/2}}(x,t)$ and $V(x)$ is taken as an external conservative force potential. Let us emphasize that $V(x)$ actually *was* the Feynman–Kac potential of the dynamical semigroup prior to the analytic continuation in time procedure. The mapping $V(x) \to 2Q(x,t) - V(x)$ is an effect of the analytic continuation in time, as manifested on the level of the associated [2, 3] Feynman–Kac kernels.

In view of (17), the pseudodifferential Schrödinger equation $i\partial_t\psi(x,t) = H\psi(x,t)$ implies the following time evolution of the Madelung exponents:

$$\partial_t R = HS - \int_R [\exp(R_{xy}) \sin S_{xy} - S_{xy}] \, d\nu(y)$$

$$\partial_t S = -HR + \int_R [\exp(R_{xy}) \cos S_{xy} - 1 - R_{xy}] \, d\nu(y) \tag{18}$$

where $H = F(-i\nabla)$.

By employing (17) with respect to $\rho^{1/2} = \exp(R)$, we arrive at:

$$Q := \frac{H\rho^{1/2}}{\rho^{1/2}} = HR - \int_R [\exp(R_{xy}) - 1 - R_{xy}] \, d\nu(y) \tag{19}$$

and hence:

$$\partial_t S = -Q + \int_R \exp(R_{xy}) \, [\cos(S_{xy}) - 1] \, d\nu(y) \tag{20}$$

The same procedure can be repeated for $\Theta = \exp(R+S)$ and $\Theta_* = \exp(R-S)$, which implies:

$$\partial_t \Theta = H\Theta + \Theta \left[-2Q + \int_R \exp(R_{xy})[-\sin S_{xy} + \cos S_{xy} + \exp(S_{xy}) - 2] \, d\nu(y) \right]$$

$$\partial_t \Theta_* = -H\Theta_* + \Theta_* \left[2Q - \int_R \exp(R_{xy}) \, [\sin S_{xy} + \cos S_{xy} + \exp(-S_{xy}) - 2] \, d\nu(y) \right] \tag{21}$$

In contrast to the Gaussian case [15, 2], equations (21) do not take the form of a time adjoint pair, unless some additional restrictions are imposed on the Madelung exponent $S(x,t)$ (notice that we have restored time dependence, skipped before for convenience). An obvious demand is $S(x+y,t) = S(x,t)$ for all y, t, and any fixed x. But then, equations (21) would manifestly refer to the *stationary* (measure preserving) random dynamics, governed by the pair of equations:

$$\partial_t \Theta = H\Theta - 2Q\Theta$$

$$\partial_t \Theta_* = -H\Theta_* + 2Q\Theta_* \tag{22}$$

which are mutually time adjoint. Hence they would fall into the Schrödinger problem framework[2, 4] with a trivial implication that the measure preserving process is Markovian. This however cannot be a property of the "free" dynamics since we need external potentials to secure stationarity. Let us therefore make an essential amelioration by performing the previous analysis for the case $i\partial_t\psi = (H+V)\psi$ with $V = V(x)$. Then, the stationary system of equations would take the form:

$$\partial_t \Theta = H\Theta - (2Q+V)\Theta \tag{23}$$

$$\partial_t \Theta_* = -H\Theta_* + (2Q+V)\Theta_*$$

which upon substituting $S(x,t) = Et$, where E is a constant, yields a pseudodifferential version of the Sturm–Liouville problem:

$$H\rho^{1/2}(x) - \left[2\frac{H\rho^{1/2}}{\rho^{1/2}} + V(x) - E\right]\rho^{1/2}(x) = 0 \tag{24}$$

$$\Downarrow$$

$$V(x) - E = -\frac{H\rho^{1/2}(x)}{\rho^{1/2}(x)}$$

to be solved (for a chosen value of E) with respect to the square root of the probability density $\rho(x)$, once the external force potential $V(x)$ is selected.

This problem has its Gaussian counterpart in the study of the measure preserving dynamics [2] , and in the present context it can be solved by invoking those potentials for the original pseudodifferential Schrödinger equation, for which the bound states (i.e., stationary solutions) have granted the existence status. The relevant analysis has been carried out in the studies of the relativistic stability of matter [13]. In addition we know that in the stationary case, the Feynman–Kac path integral generalization to Lévy semigroup kernels is available.

However, the Markov property cannot [4] automatically be attributed to the nonstationary dynamics, as described by (21).

The probability density $\rho(x,t)$ (respectively $\overline{\rho}(x,t)$) was a fundamental entity in our previous considerations: in fact, providing the time evolution of the probability measure for the whole time interval of interest, so that the transition probability densities could be sought for [4].

In the Gaussian case we dealt with the temporal evolution of the probability density given in its traditional Fokker–Planck form appropriate for Markov diffusion processes [2, 3]. In connection with the pseudodifferential ("free noise") dynamics, we address an obvious extension of the previous notion to a class of jump processes. We shall extend the usage of the name Fokker–Planck equation to any first order in time differential equation determining the space–time properties of $\rho(x,t)$ or $\overline{\rho}(x,t)$.

Let us investigate the time developement of $\overline{\rho}(x,t) = \theta(x,t)\theta_*(x,t)$, where $\theta(x,t)$, $\theta_*(x,t)$ come out as solutions of the temporally adjoint pair of equations of the form

$$\partial_t \theta = H\theta - V\theta \tag{25}$$

$$\partial_t \theta_* = -H\theta_* + V\theta_*$$

with a Feynman–Kac potential V. Then, in view of (17) and $\theta = \exp(R+S)$, $\theta_* = \exp(R-S)$, we get an evolution equation for the probability density:

$$\partial_t \overline{\rho}(x,t) = \theta_*(x,t)\,(H\theta)(x,t) \; - \; \theta(x,t)\,(H\theta_*)(x,t) = \tag{26}$$

$$\int_R \left[-\theta_*(x,t)\theta(x+y,t) + \theta(x,t)\theta_*(x+y,t) + 2\overline{\rho}(x,t)\nabla S(x,t)\frac{y}{1+y^2}\right] d\nu(y)$$

Following the traditional recipes when dealing with Lévy measures [6], let us consider an open neighborhood of the origin $|\epsilon| \ll 1$. Instead of integrating over all possible jump sizes, let us integrate over jumps of size $|y| > \epsilon > 0$). The removal of this lower bound as $\epsilon \to 0$ will eventually amount to evaluating the principal value of the integral. In case $\epsilon > 0$, we can safely remove the compensating term including $y/(1+y^2)$ from the integral, and restrict considerations to the contribution from the first two terms only.

Our purpose is to establish a connection with the conventional theory of jump stochastic processes, as developed in [16]. Integrating over a Borel set $A \subset R$, $x \in A$ we get:

$$\int_A dx \int_{|y|>\epsilon} [-\theta_*(x,t)\theta(x+y,t) + \theta(x,t)\theta_*(x+y,t)]\, d\nu(y) =$$

$$\int_R dx \int_{|y|>\epsilon} \chi_A(x) \left[-\overline{\rho}(x,t)\, \frac{\theta(x+y,t)}{\theta(x,t)} + \overline{\rho}(x+y,t)\, \frac{\theta(x)}{\theta(x+y)}\right] d\nu(y) = \quad (27)$$

$$\int_R dx\, \overline{\rho}(x,t) \int_{|y|>\epsilon} \frac{\theta(x+y,t)}{\theta(x,t)} [\chi_A(x+y) - \chi_A(x)]\, d\nu(y)$$

where we interchanged the order of integrations, and made appropriate adjustments of integration variables ($x \to x-y$ and $y \to -y$), while exploiting the property $d\nu(-y) = -d\nu(y)$ of Lévy measures; $\chi_A(x)$ is an indicator function of the Borel set $A \subset R$, equal to 1 when $x \in A$ and 0 otherwise.

In the present case we deal with a Markov process with transition probability densities given for arbitrary time instants: $\overline{\rho}(x,t) = \int_R p(y,s,x,t)\overline{\rho}(y,s)dy$, $s < t$. By invoking the standard wisdom about jump Markov processes [16], and exploiting $\lim_{t\downarrow s} p(y,s,A,t) = \chi_A(y)$, for any Borel set $A \subset R$ away from $(-\epsilon, +\epsilon)$, we can define the jump process running with jumps of size $|y| > \epsilon > 0$. It should be viewed as an approximation of the original stochastic process governed by (26), with the initial data $\overline{\rho}(x,0)$ common for both:

$$\partial_t \overline{\rho}_\epsilon(A,t) = \int_R q(x,t,A)\, \overline{\rho}_\epsilon(x,t)\, dx + \langle v\rangle_A(t) \int_{|y|>\epsilon} \frac{y}{1+y^2} d\nu(y) \qquad (28)$$

where

$$q(x,t,A) := \lim_{u\downarrow t} \frac{1}{u-t}\, [p(x,t,A,u) - \chi_A(x)] =$$

$$\int_{|y|>\epsilon} \frac{\theta(x+y,t)}{\theta(x,t)} [\chi_A(x+y) - \chi_A(x)]\, d\nu_(y), \qquad (29)$$

$$\langle v\rangle_A(t) := \int_A \overline{\rho}(x,t)\, [2\nabla S(x,t)]\, dx$$

Here $q(x,t,A) \geq 0$ for all x which are *not* in A, in agreement with [16]. We have also introduced a pseudodifferential counterpart of the current velocity field $v(x,t) = 2\nabla S(x,t)$, previously attributed to diffusion processes where the probability conservation law (a continuity equation in another lore) $\partial_t \rho = -\nabla(v\rho)$ plays the rôle of the Fokker–Planck equation.

Notice, that in the particular case of $\theta(x,t) \equiv 1$ for all x,t, and $V = 0$, Eq.(26) reduces to the "free noise" situation covered by the traditional Fokker–Planck equations. Then, $q(t,x,A) = \int_{|y|>\epsilon} [\chi_A(x+y) - \chi_A(x)] d\nu(y)$, while $R = -S$, $\overline{\rho} = \exp(2R) = \theta_*$ implies $\langle v \rangle_A(t) = -\overline{\rho}(x,t)|_a^b$ where $[a,b] := A \subset R$.

Now, let us address the Fokker–Planck equation for the pseudodifferential–Schrödinger dynamics case, which we consider in the form analogous to (26); see also (1) for comparison:

$$i\partial_t \psi = H\psi + V\psi \tag{30}$$
$$i\partial_t \overline{\psi} = -H\overline{\psi} - V\overline{\psi}$$

We re-emphasize that to define the probability density $\rho(x,t) = |\psi(x,t)|^2$ one actually employs solutions of the time adjoint pair of Schrödinger equations.

In view of (30), the pseudodifferential continuity equation follows:

$$\partial_t \rho(x,t) = -i\left[\overline{\psi}(x,t)(H\psi)(x,t) - \psi(x,t)(H\overline{\psi})(x,t)\right] = \tag{31}$$

$$-i\int_R \left[-\overline{\psi}(x,t)\psi(x+y,t) + \psi(x,t)\overline{\psi}(x+y,t) + 2i\rho(x,t)\nabla S(x,t)\frac{y}{1+y^2}\right] d\nu(y)$$

Our next step is a repetition of the procedures behind (27), which implies:

$$\partial_t \rho(x,t) = \int_R \left[2\mathcal{I}\left[\psi(x,t)\overline{\psi}(x+y,t)\right] + 2\rho(x,t)\nabla S(x,t)\frac{y}{1+y^2}\right] d\nu(y)$$

$$\Downarrow$$

$$\int_A dx \int_{|y|>\epsilon} 2\mathcal{I}\left[\psi(x,t)\overline{\psi}(x+y,t)\right] =$$

$$\int_R dx \int_{|y|>\epsilon} \chi_A(x) 2\rho^{1/2}(x,t)\rho^{1/2}(x+y,t)\sin\left[S(x,t) - S(x+y,t)\right] d\nu(y) =$$

$$\int_R \rho(x,t)dx \int_{|y|>\epsilon} \frac{\rho^{1/2}(x+y)}{\rho^{1/2}(x)} \sin\left[S(x+y,t) - S(x,t)\right]\left[\chi_A(x+y) - \chi_A(x)\right] d\nu(y) \tag{32}$$

where $\mathcal{I}[f(x,t)]$ stands for an imaginary part of a complex function $f(x,t)$. So, a counterpart of (28) reads:

$$\partial_t \rho_\epsilon(A,t) = \int_R q(x,t,A)\rho_\epsilon(x,t)dx + \langle v \rangle_A(t) \int_{|y|>\epsilon} \frac{y}{1+y^2} d\nu(y) \tag{33}$$

where, however

$$q(x,t,A) := \int\limits_{|y|>\epsilon} \mathcal{I}\left[\frac{\psi(x+y,t)}{\psi(x,t)}\right] [\chi_A(x+y) - \chi_A(x)]\, d\nu(y) \tag{34}$$

no longer can be derived from transition probability densities of the process, as in the previous discussion, because in general our process is *not* Markovian [4]. At least in the case of nonstationary dynamics, the only transition probability density which is at our disposal connects an initial instant of the evolution with any later one. In fact, we might even not be sure that $q(x,t,A)$ is a well defined probabilistic object, because of the presence of $\sin[S(x+y,t) - S(x,t)]$ in the integrand. At this point an observation of [14] helps. Namely, in view of the identity:

$$\int\limits_R dx \int\limits_{|y|>\epsilon} |\psi(x+y,t)\psi(x,t)|\, [\chi_A(x+y) - \chi_A(x)]\, d\nu(y) \;=\; 0 \tag{35}$$

valid for Borel sets $A \subset R$, which are away from $(-\epsilon, +\epsilon)$, we can always pass from (32) to the rearranged form of (33):

$$q(x,t,A) = \int\limits_{|y|>\epsilon} \left[\left|\frac{\psi(x+y,t)}{\psi(x,t)}\right| + \mathcal{I}\left[\frac{\psi(x+y,t)}{\psi(x,t)}\right]\right] [\chi_A(x+y) - \chi_A(x)]\; d\nu_(y) \tag{36}$$

implying that $q(x,t,A)$ is positive for all x which are *not* in A, as should be the case [16].

In fact, our Fokker–Planck equations involve exclusively the integral term on their right–hand–side:

$$\partial_t \overline{\rho}_\epsilon(A,t) = \int\limits_R \overline{q}(x,t,A)\overline{\rho}_\epsilon(x,t)dx \tag{37}$$

$$\partial_t \rho_\epsilon(A,t) = \int\limits_R q(x,t,A)\rho_\epsilon(x,t)dx$$

where an overbar distinguishes between probabilistic quantities characterising different families of stochastic jump processes before and after an analytic continuation in time of the given holomorphic semigroup. respectively. Let us emphasize that the above simplification occurs only in the $|y| > \epsilon > 0$ jumping size regime. The real rôle of the two spurious, in the present regime, terms is to compensate the divergent contributions from the Lévy measure when the principal value integral $\epsilon \longrightarrow 0$ limit is considered; then the *standard* jump process theory does not apply. Anyway, those two terms are irrelevant for any $\epsilon > 0$, irrespectively of how small ϵ is.

More detailed analysis and a number of extensions of the described formalism can be found in the original publication [4], while a discussion of the Gaussian (Wiener measure generated) case and this of the Schrödinger interpolation problem in Refs. [1, 2, 15, 3, 17].

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