The Schrödinger Problem

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1 The Schrödinger problem: microscopic dynamics from the input-output statistics

We invert the well developed strategy of studying dynamics in terms of probability densities and investigate the problem of the most likely microscopic propagation scenario, which is consistent with the given *a priori* (possibly phenomenological) input-output statistics data for the process taking place in a finite time interval. A subclass of solutions includes the familiar Smoluchowski diffusions.

It is clear that a stochastic process is any conceivable evolution which we can analyze in terms of probability. We shall be particularly interested in situations when the involved probability measures give rise to densities (probability distributions which are absolutely continuous with respect to the Lebesgue measure). In many branches of physics ranging from deterministic (the folk lore phrase "studying chaos with densities" pertains to the currently fashionable topic) to quantum problems, densities of probability measures do naturally arise. The quantum issue should receive a particular attention in connection with the Born statistical postulate, which implies that quantum theory deals with probability densities. However, quite generally the stochastic analysis is disregarded against the pragmatic viewpoint of deducing as many experimentally verifiable or rather falsifiable data as possible, even at the price of manipulating with the ill defined or not defined at all (safe bypassing of rather fundamental difficulties) probabilistic framework.

The main idea behind what we call the *Schrödinger problem* is an attempt to get an insight (in fact through modeling) into an unknown in detail physical process with a finite time of duration, in terms of random motions consistent with the prescribed input - output statistics i.e. the boundary distributions for repeatable single particle (sample) procedures. In less specific lore, we can simply look for a stochastic evolution (diffusion of probabilities) which interpolates between the boundary probability measures, in particular for the (invariant) measure preserving dynamics.

Given a dynamical law of motion (for a particle as example), in many cases one can associate with it (compute or approximate the observed frequency data) a probability distribution and various mean values. In fact, it is well known that inequivalent finite difference random motion problems may give rise to the same continuous approximant (e.g. the diffusion equation representation of discrete processes). The *inverse operation* of deducing the detailed (possibly individual, microscopic) dynamics, which either implies or is consistent with the given probability distribution (and eventually with its own time evolution) cannot thus have a unique solution.

For clarity of discussion, we shall confine our attention to continuous Markov processes, whose random variable $X(t), t \geq 0$ takes values on the real line R^1, and in particular can be restricted (constrained) to remain within the interval $\Lambda \subset R^1$, which may be finite or (semi-) infinite but basically an open set.

In the above input-output statistics context, let us invoke a probabilistic problem, originally due to Schrödinger : *given two strictly positive (on an open interval) boundary probability distributions $\rho_0(x), \rho_T(x)$ for a process with the time of duration $T \geq 0$. Can we uniquely identify the stochastic process interpolating between them?*

Perhaps unexpectedly in the light of our previous comments, the answer is known to be affirmative, if we assume the interpolating process to be Markovian. In fact, we get here a unique Markovian diffusion, which is specified by the joint probability distribution

$$m(A,B) = \int_A dx \int_B dy\, m(x,y) \tag{1}$$

$$\int dy\, m(x,y) = \rho_0(x)$$

$$\int dx\, m(x,y) = \rho_T(y)$$

where

$$m(x,y) = \Theta_*(x,0)\, k(x,0,y,T)\, \Theta(y,T) \tag{2}$$

and the two unknown (not necessarily Lebesgue integrable) functions $\Theta_*(x,0)$, $\Theta(y,T)$ come out as solutions of *the same sign* of the integral identities (1). Provided, we have at our disposal a bounded strictly positive integral kernel $k(x,s,y,t)$,
$0 \leq s < t \leq T$. Then:

$$\Theta_*(x,t) = \int k(0,y,x,t)\Theta_*(y,0)dy \tag{3}$$

$$\Theta(x,s) = \int k(s,x,y,T)\Theta(y,T)dy$$

and the sought for interpolation has a probability distribution $\rho(x,t) = (\Theta_*\Theta)(x,t), t \in [0,T]$.

2 Markov diffusions solving the Schrödinger problem: the role of natural boundaries

To have a definite Markov solution in hands, we must decide what is the most appropriate choice for the dynamical semigroup kernel in the above. Apparently it is the crucial step in the construction of any explicit random propagation consistent with the boundary measure data. The reader should be warned that the whole family of Levy processes and their perturbations (in the sense of Kato) is here allowed. The conventional Brownian diffusion and the equally conventional Poisson jump process are rather specialized examples in this context.

We wish to discuss diffusive solutions only, and take for granted that the traditional Fokker-Planck equation sets rules of the game for the interpolating probability density. Then we look for the corresponding fundamental law of random displacements and choose the transition probability density for the Markovian diffusion process in the form (the so called h-transformation, invented long ago by Doob and Hille is here involved)

$$p(y,s,x,t) = k(y,s,x,t)\,\frac{\Theta(x,t)}{\Theta(y,s)} \tag{4}$$

with $s \leq t$. This transition density is required to come out from the forward Kolmogorov equation (e.g. the Fokker-Planck equation) as its fundamental solution ($p \to \delta(x-y)$ as $t \downarrow s$). For convenience we simplify the whole problem by utilising a diffusion constant $D > 0$ (this choice narrows slightly the allowed framework):

$$\partial_t p = D\triangle_x p - \nabla_x(bp) \tag{5}$$

$$\rho(x,t) = \int p(y,s,x,t)\,\rho(y,s)\,dy$$

with $\rho_0(x) = \rho(x,0)$ and the drift $b(x,t)$ given by :

$$b(x,t) = 2D\,\frac{\nabla\Theta}{\Theta}(x,t) \tag{6}$$

In addition we demand that the backward diffusion equation is solved by the same transition density (with respect to another pair of variables)

$$\partial_s p = -D\triangle_y p - b\nabla_y p \tag{7}$$

$$p = p(y,s,x,t)\,,\; s \leq t\,,\; b = b(y,s)$$

It implies that we deal here with a *unique* diffusion process, whose transition density is a *common* fundamental solution for both the backward and forward Kolmogorov equations.

To understand the rôle of the integral kernel $k(y,s,x,t)$ in (1)-(7) let us assume that $\Theta(x,t)$ is given in the form (drifts are gradient fields as a consequence):

$$\Theta(x,t) = \pm\exp\Phi(x,t) \Rightarrow b(x,t) = 2D\nabla\Phi(x,t) \tag{9}$$

$$x \in (r_1, r_2)$$

and insert (4) to the Fokker-Planck equation (5). Then, if $p(y,s,x,t)$ is to solve (5), the kernel $k(y,s,x,t)$ must be a fundamental solution of the generalised diffusion equation:

$$\partial_t\, k = D\triangle_x\, k - \frac{1}{2D}\Omega(x,t)\, k \tag{10}$$

$$k(y,s,x,t) \to \delta(x-y)\, \mathrm{as}\, t \downarrow s$$

$$\Omega(x,t) = 2D\big[\partial_t\Phi + \frac{1}{2}(\frac{b^2}{2D} + \nabla b)\big]$$

and to guarrantee (3), it must display the semigroup composition properties.

Notice that (4), (9) imply that the backward diffusion equation (7) takes the form of the adjoint to (10):

$$\partial_s\, k = -D\triangle_y\, k + \frac{1}{2D}\Omega(y,s)\, k \tag{11}$$

$$k = k(y,s,x,t)$$

If the process takes place in-between boundaries at infinity $r_1 = -\infty, r_2 = +\infty$, the standard restrictions on the auxiliary potential Ω (Rellich class) and hence on the drift potential $\Phi(x,t)$, yield the familiar Feynman-Kac representation of the fundamental solution $k(y,s,x,t)$ *common* for (10) and (11):

$$k(y,s,x,t) = \int \exp\big[-\frac{1}{2D}\int_s^t \Omega(X(u),u)du\big]\, d\mu[s,y \mid t,x] \tag{12}$$

which integrates $\exp[-(1/2D)\int_s^t \Omega(X(u),u)du]$ weighting factors with respect to the conditional Wiener measure i.e. along all sample paths of the Wiener process which connect y with x in time $t-s$. More elaborate discussion is necessary, if at least one of the boundary points is *not* at infinity.

Let us notice that the time independence of Ω is granted if either Φ is independent of time, or depends on time at most linearly. Then the standard expression $\exp[-H(t-s)](y,x)$ for the kernel k clearly reveals the involved semigroup properties, with $H = -D\triangle + (1/2D)\Omega(x)$ being the essentially self adjoint operator on its (Hilbert space) domain.

We shall make one more step narrowing the scope of our discussion by admitting diffusions (1)-(7) whose drift fields are time-independent:$\partial_t b(x,t) = 0$ for all x. We know that both the free Brownian motion and the Brownian motion in a field of force in the Smoluchowski approximation, belong to this class of processes. We know also that the boundary value problems for the Smoluchowski equation have a profound physical significance, albeit the attention paid to various cases is definitely unbalanced in the literature. It is then interesting to observe that the situation we encounter in connection with (1)-(7) is very specific from the point of view of Feller's classification of one-dimensional diffusions encompassing effects of the boundary data. Our case is precisely the Feller diffusion respecting (confined between) the *natural boundaries*. An equivalent statement is that boundary points r_1, r_2 are *inaccessible* barriers for the process i.e. there is *no* positive probability that any of them can be reached from the interior of (r_1, r_2) within a *finite* time for all $X(0) = x \in (r_1, r_2)$.

In the mathematical literature a clear distinction is made between the backward and forward Kolmogorov equations. The backward one defines the *transition density* of the process, while the forward (Fokker-Planck) one determines the *probability distribution* (density) of diffusion. With a given backward equation one can usually associate the whole family of forward (Fokker-Planck) equations, whose explicit form reflects the particular choice of boundary data. This fundamental distinction seemingly evaporates in our previous discussion (1)-(11), but it is by no means incidental. In fact, according to Feller: in order that there exists one and only one (homogeneous ; $p(y,s,x,t) = p(t-s;y,x)$) process satisfying $-\partial_t u = D\triangle u + b\nabla u$ in a finite or infinite interval $r_1 < x < r_2$ it is necessary and sufficient that both boundaries are inaccessible (the probability to reach either of them within a finite time interval must be zero). A general feature of the inaccessible boundary problems is that the density of diffusion vanishes at the boundaries:$\rho(r_1) = 0 = \rho(r_2)$.

The standard (unrestricted) Brownian motion on R^1 is the most obvious example of diffusion with natural boundaries. It is not quite trival to construct explicit examples, if one of the boundaries is not at infinity. The classic example of diffusion on the half-line with natural boundaries at 0 and $+\infty$ is provided by the so called Bessel process. As mentioned before, diffusions with inaccessible barriers might have drifts which are unbounded on (r_1, r_2). Hence, our discussion definitely falls into the framework of diffusion processes with singular drift fields, which is *not* covered by standard monographs on stochastic processes.

We skip the standard details concerning the probability space, filtration and the process adapted to this filtration and notice that a continuous random process $X(t), t \in [0,T]$ with a probability measure P is called a process of the diffusion type if its drift $b(x)$ obeys:

$$P[\int_0^T \mid b(X(t)) \mid dt < \infty] = 1 \tag{13}$$

and, given the standard Wiener process (Brownian motion) $W(t)$, the integral identity (D constant and positive)

$$X(t) = \int_0^t b(X(s))\, ds + \sqrt{2D}W(t) \tag{14}$$

holds true P-almost surely (except possibly on sets of P-measure zero). It means that $W(t) = (1/\sqrt{2D}[X(t) - \int_0^t b(X(s))ds]$ is a standard Wiener process with respect to the probability measure P of the process $X(t)$.

For diffusions with natural boundaries, we remain within the regularity interval of $b(X(t))$ for all (finite) times, and (13) apparently is valid. Therefore, the standard rules of the stochastic Itô calculus can be adopted to relate the Fokker-Planck equation (7) with the natural boundaries to the diffusion process $X(t)$, which "admits the stochastic differential"

$$dX(t) = b\big(X(t)\big)\, dt + \sqrt{2D} dW(t) \tag{15}$$

$$X(0) = x_0 \, , \, t \in [0,T]$$

for all (finite) times. The weak (in view of assigning the density $\rho_0(x)$ to the random variable $X(0)$) solution of (15) is thus well defined.

For stochastic differential equations of the form (15), the explicit Wiener noise input, because of (9) implies that irrespective of whether natural boundaries are at infinity or not, the Cameron-Martin-Girsanov method of measure substitutions which parallel transformations of drifts, is applicable. Even though the drifts are generally unbounded on (r_1, r_2) and the original theory is essentially based on the boundedness demand. It is basically due to the fact that the probabilistic Cameron-Martin formula relating the probability measure P_X of $X(t)$ with the Wiener measure P_W (strictly speaking it is the Radon-Nikodym derivative of one measure with respect to another) reduces to the familiar Feynman-Kac formula (with the multiplicative normalisation). The problem of the existence of the Radon-Nikodym derivative (and this of the absolute continuity of P_X with respect to P_W, which implies that sets of P_W-measure zero are of P_X-measure zero as well) is then replaced by the standard functional analytic problem of representing the semigroup operator kernel via the Feynman-Kac integral with respect to the conditional Wiener measure.

The Feynman-Kac formula is casually viewed to encompass the unrestricted (the whole of R^n) motions, however it is known to be *localizable*, and its validity extends also to finite and semi-infinite subsets of R^1 (R^nmore generally) as demonstrated in the context of the statistical mechanics of continuous quantum systems. More specifically, it refers to the Dirichlet boundary conditions for self-adjoint Hamiltonians, which ensure their essential self-adjointness (to yield the Trotter formula).

Let us emphasize the importance of (15), and of the Itô differential formula induced by (15) for smooth functions of the random variable $X(t)$. Its *first* consequence is that given $p(y,s,x,t)$, for any smooth function of the random variable the forward time derivative in the conditonal mean can be introduced (we bypass in this way the inherent non-differentiability of sample paths of the process)

$$lim_{\triangle t \downarrow 0} \frac{1}{\triangle t} \Big[\int p(x,t,y,t+\triangle t) f(y,t+\triangle t) dy - f(x,t) \Big] = (D_+ f(X(t),t) \quad (16)$$

$$= (\partial_t + b\nabla + D\triangle) f(X(t),t)$$

$$(D_+(X)(t) = b(x,t) \;\; X(t) = x$$

so that the second forward derivative associates with our diffusion the local field of accelerations:

$$(D_+^2 X)(t) = (D_+ b)(X(t),t) = (\partial_t b + b\nabla b + D\triangle b)(X(t),t) = \nabla \Omega(X(t),t) \quad (17)$$

with the (auxiliary potential $\Omega(x,t)$ introduced before in the formula (10). Since we have given $\rho(x,t)$ for all $t \in [0,T]$, the notion of the backward transition density $p_*(y,s,x,t)$ can be introduced as well

$$\rho(x,t) p_*(y,s,x,t) = p(y,s,x,t) \rho(y,s) \quad (18)$$

which allows to define the backward derivative of the process in the conditional mean (it is quite illuminating to appply this discussion in case of the most traditional Brownian motion)

$$lim_{\triangle t\downarrow 0} \frac{1}{\triangle t}[x - \int p_*(y, t-\triangle t, x, t) y dy] = (D_- X)(t) = b_*(X(t), t)$$

$$= [b - 2D\nabla ln\rho](X(t), t) \tag{19}$$

$$(D_- f)(X(t), t) = (\partial_t + b_*\nabla - D\triangle) f(X(t), t)$$

Apparently, the validity of (17) extends to $(D_-^2 X)(t)$ as well, and there holds

$$(D_+^2 X)(t) = (D_-^2 X)(t) = \partial_t v + v\nabla v + \nabla Q = \nabla \Omega \tag{20}$$

$$v(x,t) = \frac{1}{2}(b + b_*)(x,t)\,,\; u(x,t) = \frac{1}{2}(b - b_*)(x,t) = D\nabla ln\rho(x,t)$$

$$Q(x,t) = 2D^2\, \frac{\triangle \rho^{1/2}}{\rho^{1/2}}$$

Clearly, if b and ρ are time-independent, then (20) reduces to the identity

$$v\nabla v = \nabla(\Omega - Q) \tag{21}$$

while in case of constant (or vanishing) current vélocity v, the acceleration formula (21) reduces to

$$0 = \nabla(\Omega - Q) \tag{22}$$

which establishes a very restrictive relationship between the auxiliary potential $\Omega(x)$ (and hence the drift $b(x)$) and the probability distribution $\rho(x)$ of the stationary diffusion. The pertinent random motions have their place in the mathematically oriented literature.

Let us notice that (20) allows to transform the Fokker-Planck equation (7) into the familiar continuity equation, so that the diffusion process $X(t)$ admits a recasting in terms of the manifestly *hydrodynamical* local conservation laws (we adopt here the kinetic theory lore)

$$\partial_t \rho = -\nabla(\rho v) \tag{23}$$

$$\partial_t v + v\nabla v = \nabla(\Omega - Q)$$

$$\rho_0(x) = \rho(x,0)\,,\; v_0(x) = v(x,0)$$

which form a closed (in fact, Cauchy) nonlinearly coupled system of differential equations, strictly equivalent to the previous (7), (17).

In view of the natural boundaries (where the density $\rho(x,t)$ vanishes), the diffusion respects a specific ("Euclidean looking") version of the Ehrenfest theorem:

$$E[\nabla Q] = 0 \Rightarrow \tag{24}$$

$$\frac{d^2}{dt^2} E[X(t)] = \frac{d}{dt} E[v(X(t),t)] = E[(\partial_t v + v\nabla v)(X(t,t)] = E[\nabla\Omega(X(t),t)]$$

Notice that the auxiliary potential of the form $\Omega = 2Q - V$ where V is any Rellich class representative, defines drifts of Nelson's diffusions for which $E[\nabla Q] = 0 \Rightarrow E[\nabla \Omega] = -E[\nabla V]$ and the "standard looking" form of the second Newton law in the mean arises.

At this point it seems instructive to comment on the essentially hydrodynamical features (compressible fluid/gas case)of the problem (23), where the "pressure" term ∇Q might look annoying from the traditional kinetic theory perspective. Although (23) has a conspicuous Euler form, one should notice that if the starting point of our discussion would be a typical Smoluchowski diffusion (7), (15) whose drift is given by the Stokes formula (i.e. is proportional to the external force $F = -\nabla V$ acting on diffusing molecules), then its external force factor is precisely the one retained from the original Kramers phase-space formulation of the high friction affected random motion. In the Euler description of fluids and gases, the very same force which is present in the Kramers (or Boltzmann in the traditional discussion) equation, should reappear on the right-hand-side of the local conservation law (momentum balance formula) (23). Except for the harmonic oscillator example, in view of (10) it is generally *not* the case in application to diffusion processes. As it appears, the validity of the stochastic differential representation (15) of the diffusion (5) implies the validity of the hydrodynamical representation (23) of the process. It in turn gives a distinguished status to the auxiliary potential $\Omega(x,t)$ of (10)-(12). We encounter here a fundamental problem of what is to be interpreted by a physicist (observer) as the *external force field manifestation* in the diffusion process. Should it be dictated by the drift form following Smoluchowski and Kramers, or rather by $\nabla \Omega$ entering the evident (albeit "Euclidean looking") second Newton law, respected by the diffusion ?

In the standard derivations of the Smoluchowski equation, the deterministic part (force and friction terms) of the Langevin equation is postulated. What however, if the experimental data pertain to the local conservation laws like (23), and there is no direct (experimental) access to the microscopic dynamics ?

If the field of accelerations $\nabla \Omega$ is taken as the primary defining characteristics of diffusion we deal with, then we face the problem of deducing all drifts, and hence diffusions, which give rise to the same acceleration field, and thus form a class of *dynamically equivalent diffusions.*

Let us analyze the *second consequence* of the unattainability of the boundaries, which via (13) gives rise to (15). On the same footing as in case of (13), we have satisfied another probabilistic identity:

$$P\left[\int_0^T b^2(X(t))\, dt < \infty\right] = 1 \tag{25}$$

For a diffusion $X(t)$ with the differential (15), we know that (25) is a sufficient and necessary condition for the absolute continuity of the measure $P = P_X$ with respect to the Wiener measure P_W. Since, for any (Borel) set A, $P_W(A) = 0$ implies $P_X(A) = 0$, the Radon-Nikodym theorem applies and densities of these measures can be related. It is worthwhile to mention the demonstration due to

Fukushima that the mutual absolute continuity (the previous implication can be reversed) holds true for most measures we are interested in.

In the notation (12), the conditional Wiener measure $d\mu[s,y\mid t,x]$ gives rise to the familiar heat kernel, if we set $\Omega = 0$ identically. It in turn induces the Wiener measure P_W of the set of all sample paths, which originate from y at time s and terminate (can be located) in the Borel set A after time $t-s$:

$$P_W[A] = \int_A dx \int d\mu[s,y\mid t,x] = \int_A d\mu \tag{26}$$

where, for simplicity of notations, the $(y, t-s)$ labels are omitted and $\int d\mu[s,y\mid t,x]$stands for the standard path integral expression for the heat kernel.

Having defined an Itô diffusion $X(t)$, (5), (15) with the natural boundaries, we are interested in the analogous (with respect to (26)) path measure P_X

$$P_X[A] = \int_A dx \int d\mu_X[s,y\mid t,x] = \int_A d\mu_X \tag{27}$$

The absolute continuity $P_X \ll P_W$ implies the existence of the strictly positive Radon-Nikodym density, which we give in the Cameron-Martin-Girsanov form

$$\frac{d\mu_X}{d\mu}[s,y\mid t,x] = \exp\Big[\int_s^t \frac{1}{2D} b(X(u))dX(u) - \frac{1}{2}\int_s^t \frac{1}{2D}[b(X(u))]^2 du\Big] \tag{28}$$

Notice that the standard normalisation appears, if we set $D = 1/2$ which implies $D\triangle \rightarrow \frac{1}{2}\triangle$ in the Fokker-Planck equation.

On account of our demand (9) and the Itô formula for $\Theta(X(t),t)$ we have

$$\frac{1}{2D}\int_s^t b(X(t))dX(t) = \Phi(X(t),t) - \Phi(X(s),s) - \int_s^t du\,[\partial_t\Phi + \frac{1}{2}\nabla b](X(u),u) \tag{29}$$

so that, apparently

$$\frac{d\mu_X}{d\mu}[s,y\mid t,x] = \exp[\Phi(X(t),t) - \Phi(X(s),s)]\,\exp\,[-\frac{1}{2D}\int_s^t \Omega(X(u),u)du] \tag{30}$$

with $\Omega = 2D\partial_t\Phi + D\nabla b + (1/2)b^2$ introduced before in (10), by means of the substitution of (4) in the Fokker-Planck equation.

In case of natural boundaries at infinity, the connection with the Feynman-Kac formula (12) is obvious, and we have

$$P_X[A] = \int_A \frac{d\mu_X}{d\mu} d\mu = \int_A dx \int \frac{d\mu_X}{d\mu}[s,y\mid t,x] d\mu[s,y\mid t,x] \tag{31}$$

where the second integral refers to the path integration of the Radon-Nikodym density with respect to the conditional Wiener measure.

In the context of (31) and (12) we can safely assert that the pertinent processes ($X(t)$ and $W(t)$) have *coinciding sets of sample paths.* The stochastic process "realizes" them merely (via sampling) with a probability distribution (frequency) different from this for the Wiener process $W(t)$.

The situation drastically changes, if we wish to exploit the "likelihood ratio" formulas (28), (30) for diffusions confined between the unattainable (natural) boundaries, at least one of which is *not* at infinity. In view of the absolute continuity of P_X with respect to P_W, we must be able to select a subset of Wiener paths which coincide with these admitted by the process $X(t)$, except on sets of measure zero (both with respect to P_X and P_W).

3 Brownian motion and Smoluchowski diffusions

A mathematical idealisation of the individual Brownian particle dynamics, in case of the free evolution in the high friction regime, is provided by the configuration space (Wiener) projection of the phase space (Ornstein-Uhlenbeck) process. One deals then with the stochastic differential equation

$$dX(t) = \sqrt{2D}\, dW(t) \tag{32}$$

$$X(0) = x_0 \in R^3,\ t \in [0,T],\ D > 0$$

which is a symbolic expression representing an ensemble of possible instantaneous values (sample locations in space), generated by the random noise W(t) according to a definite statistical law. Eq. (32) is known (via the stochastic Itô calculus) to imply the Kolmogorov equation for the transition probability density (heat kernel here) i.e. a fundamental law of random displacements of the process, which gives rise to the Fokker-Planck (heat) equation for the time developement of the probability distribution of diffusing particles

$$\partial_t \rho = D \triangle \rho \tag{33}$$

$$\rho(x,0) = \rho_0(x)$$

Then, $\rho(x,t)$ is the probability distribution of the random variable $X(t)$, given the distribution $\rho_0(x)$ of its initial values $X(0)$ in R^3.

By introducing the (irrotational, $rot v = 0$) local velocity field

$$v = -D\frac{\nabla \rho}{\rho} \Rightarrow \partial_t \rho = -\nabla(\rho v) \tag{34}$$

for all conceivable choices of the smooth function $\rho_0(x)$ the heat equation, if combined with the assumption (34), inevitably gives rise to the local conservation law (the momentum balance equation in the kinetic theory lore)

$$\partial_t v + (v\nabla)v = -\frac{1}{m}\nabla Q \tag{35}$$

$$Q = 2mD^2\,\frac{\triangle \rho^{1/2}}{\rho^{1/2}}$$

$$v_0 = -D \frac{\nabla \rho_0}{\rho_0}$$

where m stands for the hitherto absent (albeit included in the definition of the diffusion constant D via the fluctuation-dissipation theorem) mass parameter of diffusing particles, while the potential Q is recognized to have the standard functional form of the familiar de Broglie-Bohm "quantum potential", except for the opposite sign.

In case of an arbitrary non-symmetric distribution $\rho_0(x)$ we have fulfilled the following property, which is maintained in the course of the diffusion process $(X(t) \in R^3)$:

$$\frac{1}{m} \partial_i Q = \frac{1}{\rho} \sum_{j=1}^{3} \partial_j P_{ij} \tag{36}$$

$$P_{ij} = D^2 \rho \partial_i \partial_j \ln \rho$$

where $\nabla \equiv (\partial_1, \partial_2, \partial_3)$ and $i, j, = 1, 2, 3$. Apparently, $P_{ij} = \delta_{ij} D^2 \rho \triangle ln \rho$ in the totally isotropic case. The unconventional "pressure" term $(-\frac{1}{m}\nabla Q)$ in (35) is a distinctive characteristic of all diffusions derivable (via conditioning as example) from the Brownian motion proper and is a collective, statistical ensemble measure of momentum transfer per unit of time and per unit of volume: *away* ($-\nabla Q$ corresponds to the conventional Brownian propagation with the obvious tendency of a particle to leave the area of the higher concentration) or *towards* ($+\nabla Q$) the infinitesimal surrounding of the given spatial location $x \in R^3$ at time t, in the very same rate.

The conventional Brownian dynamics is a very special solution of the general Cauchy problem composed of the mass conservation law (33) and the momentum balance equation (35) with the initial data $\rho_0(x), v_0(x)$ in principle unrelated, in contrast to the assumption (34). Then, we arrive at the rich family of Markovian diffusions, all of which are the *descendants* of the Brownian motion, the Brownian motion itself included.

To be more specific, let us consider the boundary probability distributions $\rho_0(x) = \rho(x, 0), \rho_T(x) = \rho(x, T)$ for a stochastic diffusion process in R^3, confined to the time interval $[0, T] \ni t$. We realise that the dynamical semigroup operator $\exp(tD\triangle)$ provides us with the ***probabilistic semigroup transition mechanism***, in the sense that the strictly positive semigroup (heat in our case) kernel is given:

$$h(y, 0, x, t) = (4\pi Dt)^{-1/2} \exp[-\frac{(x-y)^2}{4Dt}] = [\exp(tD\triangle)](y, x) \tag{37}$$

Following Schrödinger, we ask for the joint probability distribution $m(x, y) = \theta_*(x, 0)\, h(x, 0, y, T)\, \theta(y, T)$ whose marginals $\int dx\, m(x, y) = \rho_T(y)$, $\int dy\, m(x, y) = \rho_0(x)$ coincide with the previously prescribed boundary data for the random propagation in the interval $[0, T]$. It is clear, that for arbitrarily chosen (not necessarily disjoint) areas A and B in R^3, the probability to find in B a particle which originated from A at time 0 and was subject to the random (Brownian, e.g. Wiener) perturbations in the whole run of duration T, reads $m(A, B) = \int_A dx \int_B dy\, m(x, y)$.

With the data $\theta_*(x)$ and $\theta_T(x)$ we can construct respectively the forward and backward diffusive propagation by means of the kernel $h(y,0,x,t)$:

$$\partial_t\theta_* = D\triangle\theta_* \tag{38}$$

$$\partial_t\theta = -D\triangle\theta$$

$$\theta_*(x,0) = \theta_{*0}(x)\,,\ \theta(x,T) = \theta_T(x)\,,\ t \in [0,T]$$

where

$$\theta_*(x,t) = \int h(y,0,x,t)\,\theta_{*0}(y)\,dy \tag{39}$$

$$\theta(x,t) = \int h(x,t,y,T)\,\theta_T(y)\,dy$$

$$0 \le t \le T$$

The local conservation laws (33), and (35) are satisfied by:

$$\rho(x,t) = (\theta\theta_*)(x,t) \tag{40}$$

$$v(x,t) = D\nabla\, ln\,\frac{\theta}{\theta_*}(x,t)$$

$$x \in R^3\,,\ t \in [0,T]$$

A complete statistical information about the *most likely* way the individual particles propagate, is provided by the transition density $p(y,s,x,t) = h(y,s,x,t)\frac{\theta(x,t)}{\theta(y,s)}$ which solves the Kolmogorov (Fokker-Planck) equation associated with the (individual particle motion recipe) stochastic differential equation

$$dX(t) = b(X(t),t)dt + \sqrt{2D}dW(t) \tag{41}$$

$$b(x,t) = (u+v)(x,t)$$

$$u(x,t) = D\frac{\nabla\rho}{\rho}$$

Notice that the standard Brownian motion comes here in a trivial way by substituting $\theta_* = \rho(x,t), \theta = 1$ for all times $t \in [0,T]$.

We can still have a more detailed insight into the standard Brownian dynamics. Let us consider the initial probability distribution of the random variable $X(0)$ of the Wiener (Brownian in the high friction regime) process in the form

$$\rho_0(x) = (\pi\alpha^2)^{-1/2}\exp[-\frac{x^2}{\alpha^2}] \tag{42}$$

Then its statistical evolution is given by the familiar heat kernel

$$p(y,s,x,t) = [4\pi D(t-s)]^{-1/2}\exp[-\frac{x^2}{4D(t-s)}] \tag{43}$$

$$\rho(x,t) = [\pi(\alpha^2+4Dt)]^{-1/2}\exp[-\frac{x^2}{\alpha^2+4Dt}]$$

where $s \le t$.

Let us notice that since the density distribution is now defined for all times $t > s$ we can introduce a convenient device allowing to reproduce a statistical past of the (irreversible on physical grounds, but admitting this specific inversion mathematically)

$$p_*(y,s,x,t) = p(y,s,x,t)\frac{\rho(y,s)}{\rho(x,t)} \tag{44}$$

with the properties (set $s = t - \Delta t$)

$$\int p_*(y,s,x,t)\rho(x,t)dx = \rho(y,s) \quad s \leq t \tag{45}$$

$$\int y\, p_*(y,s,x,t)dy = x\,\frac{\alpha^2+4Ds}{\alpha^2+4Dt} = x\, -\, \frac{4Dx}{\alpha^2+4Dt}\Delta t = x\, -\, b_*(x,t)\Delta t$$

where $b_*(x,t) = -2D\nabla\rho(x,t)/\rho(x,t)$ and quite trivially $b(x,t) = 0$. Notice furthermore that by defining $v(x,t) = \frac{1}{2}b_*(x,t)$, as a consequence of the heat equation we have satisfied $(\rho v)(x,t) = \int p(y,s,x,t)\rho_0(y)v_0(y)dy$ and equations (38).

Our previous discussion was entirely devoted to the free evolution, and it is quite natural to address the issue of the effects of external force fields on the random propagation. If to accept the high friction regime, like in case of (33), we should consider the Brownian motion in a field of force, in the Smoluchowski approximation.

The Fokker-Planck equation governing the time developement of the spatial probability distribution in case of the phase space noise with high friction, in the Smoluchowski form reads

$$\partial_t\rho \;=\; D\,\triangle\rho\; -\; \nabla(b\rho) \tag{46}$$

$$b(x,t) \;=\; \frac{1}{\beta}\,F(x), \quad , \quad \rho_0(x) = \rho(x,0)$$

where β is the friction constant and the external force we assume to be conservative

$$F(x) \;=\; -\,\nabla\Phi(x) \tag{47}$$

It is well known that the substitution

$$\rho(x,t) \;=\; \theta_*(x,t)\,\exp[-\frac{\Phi(x)}{2D\beta}] \tag{48}$$

converts the Fokker-Planck equation into the generalised diffusion equation for $\theta_*(x,t)$

$$\partial_t\theta_* \;=\; D\,\triangle\theta_*\; -\; \frac{V(x)}{2mD}\theta_* \tag{49}$$

where (the mass m was here introduced per force, but with a very concrete purpose of embedding our discussion in the formalism of the "Euclidean quantum mechanics", the name coined by J. C. Zambrini for a natural extension of the standard nonequilibrium statistical physics)

$$V(x) = \frac{m}{\beta}(\frac{F^2}{2\beta} + D\nabla F) \tag{50}$$

Since F^2, D, β are positive, a sufficient condition for the auxiliary potential $V(x)$ to be bounded from below (its continuity is taken for granted) is that the source term $g(x)$ in the familiar Poisson equation

$$\nabla F = -\triangle\Phi = g \tag{51}$$

is bounded from below: $g(x) > -c, c > 0, c$ is finite. Under this boundedness condition, we know that the equation (49) defines the fundamental *semigroup transition mechanism* underlying the Smoluchowski diffusion. Indeed, by (49) we have in hands the well defined semigroup operator $\exp[-t(-D\triangle + V/2mD)]$, whose integral kernel is a strictly positive solution of (49) with the initial condition $\lim_{t\to 0} h(y,0,x,t) = \delta(y-x)$.

The kernel is defined by the Feynman-Kac formula (in terms of the conditional Wiener measure, which sets an obvious link with the Brownian propagation). It is immediate that

$$\theta_{*0}(x) = \rho_0(x)\exp[\frac{\Phi(x)}{2D\beta}] \longrightarrow \tag{52}$$

$$\theta_*(x,t) = \int h(y,0,x,t)\theta_*(y,0)dy$$

while, apparently

$$\theta(x,t) = \exp[-\frac{\Phi(x)}{2D\beta}] = \int h(x,t,y,T)\theta_T(y)dy = \theta_T(x) \tag{53}$$

for all $t \in [0,T]$. Indeed $\theta(x,t)$, (53) solves

$$\partial_t\theta = -D\triangle\theta + \frac{V}{2mD}\theta \tag{54}$$

where $\partial_t\theta = 0$ and

$$D\triangle\theta = [\frac{(\nabla\Phi)^2}{4D\beta^2} - \frac{\triangle\Phi}{2\beta}]\theta = \frac{V}{2mD}\theta \tag{55}$$

as should be. Since the deterministic evolution governed by the Smoluchowski equation gives rise to a definite terminal (in the interval $[0,T]$) outcome $\rho_T(x)$ given $\rho_0(x)$, a straightforward inspection demonstrates that the Schrödinger system is solved by $\theta_{*0}(x)$ and $\theta_T(x)$ with the kernel $h(V;y,s,x,t)$. As a consequence, we have completely specified the unique Markov-Bernstein diffusion interpolating between $\rho_0(x)$ and $\rho_T(x)$, which is identical with the Smoluchowski diffusion itself. We know here the transition probability density (e.g. the law of random displacements modified by the presence of external force fields)

$$p(y,s,x,t) \;=\; h(y,s,x,t)\,\frac{\theta(x,t)}{\theta(y,s)} \tag{56}$$

which is responsible for the *most likely* particle propagation scenario. We have also automatically satisfied the local conservation laws

$$\partial_t v \;=\; -\nabla(\rho v) \tag{57}$$

$$\partial_t v \,+\, (v\nabla)v \;=\; \frac{1}{m}\nabla(V \,-\, Q)$$

$$\rho(x,0) = \rho_0(x)\,,\; v(x,0) = v_0(x)$$

where $\rho(x,t), v(x,t)$ are defined by the formula (40). Notice that in the detailed derivation, the above momentum balance equation does not appear directly, but in the indirect way by taking the gradient of the much weaker (Hamilton-Jacobi) identity

$$V \,-\, Q \;=\; 2mD[\partial_t S + D(\nabla S)^2] \tag{58}$$

$$S(x,t) = \frac{1}{2}\,ln\,\frac{\theta}{\theta_*}$$

In our case, apparently

$$v(x,t) = D\nabla(-\frac{\Phi}{D\beta} - ln\rho) = -\frac{1}{\beta}\nabla\Phi - D\frac{\nabla\rho}{\rho} \longrightarrow \tag{59}$$

$$\partial_t\rho = \nabla[\frac{1}{\beta}(\nabla\Phi)\rho] + D\Delta\rho$$

to be compared with the Smoluchowski equation.

The above discussion admits various generalisations. As example, by choosing a definite (reference) Smoluchowski force potential and then the auxiliary (induced) one V, we have fixed the strictly positive kernel $h(V;y,s,x,t)$. By playing with different choices of the boundary data ρ_0, ρ_T (unrelated to the initially considered) and seeking solution of the Schrödinger system, we can generate a rich class of the (conditional) random motions, all of which are governed by the local conservation laws with the potential V. However, their forward drifts $b(x,t)$ would have the functional form completely divorced from the simple Smoluchowski expression.

We can as well start from the general Cauchy problem with the completely arbitrary V (except for being continuous and bounded from below). Then, the corresponding Smoluchowski diffusion can be reproduced only if the potential allows to decouple from the defining identity, the force field F.

4 Quantum mechanical games : schrödinger wave mechanics as the theory of markov diffusion processes

Now, we shall analyze some consequences of the primordial for quantum theory, albeit frequently underestimated statistical postulate due to Max Born: the identification of the squared modulus of the Schrödinger wave function with the probability density ("of something if anything", but undoubtedly of a certain probability measure) is what makes quantum mechanics a part of the theory of stochastic processes, and in particular of Markov diffusions.

4.1 A specific example of the invariant probability measure: measure preserving stochastic dynamics

We indicate at this point certain amusing features of the harmonic attraction. Let us consider the Sturm-Liouville problem on $L^2(R^1)$

$$-D\triangle\psi + \frac{\omega^2 x^2}{4D}\psi = \epsilon\psi \tag{60}$$

The substitutions:$\alpha^4 = \omega^2/4D^2, \lambda = \epsilon/\omega, x = \xi/\alpha$ give rise to the equivalent eigenvalue problem

$$(-\frac{1}{2}\triangle_\xi + \frac{\xi^2}{2})\phi = -\lambda\phi \tag{61}$$

$$\phi(\xi) = \psi(\frac{\xi}{\alpha}) = \psi(x)$$

with the well known solution (normalised relative to x)

$$\lambda_n = n + \frac{1}{2} \leftrightarrow \epsilon_n = (n + \frac{1}{2})\omega\ , \ n = 0, 1, 2, ...$$

$$\psi_n(x) = \phi_n(\xi) = (\frac{\alpha}{2^n n!\sqrt{\pi}})^{1/2} \exp[-\frac{\xi^2}{2}]\, H_n(\xi) \tag{62}$$

$$H_0 = 1\, ,\ H_1 = 2\xi\, ,\ H_2 = 2(2\xi^2 - 1)\, ,\ H_3 = 4\xi(2\xi^2 - 3)\, , ...$$

Except for $n = 0$ the solutions $\phi_n(\xi)$ are not positive definite and change sign at nodes. We have

$$n = 0\, ,\ \psi_0(x) > 0\, ,\ x \in (-\infty, +\infty)$$

$$n = 1\, ,\ \psi_1(x) > 0\, ,\ x \in (0, +\infty)$$

$$\psi_1(x) < 0\, ,\ x \in (-\infty, 0)$$

$$n = 2\, ,\ \psi_2(x) > 0\, ,\ x \in (-\infty, -1/\sqrt{2}) \cup (1/\sqrt{2}, +\infty)$$

$$\psi_2(x) < 0\, ,\ x \in (-1/\sqrt{2}, +1/\sqrt{2})$$

and so on. It is convenient to continue further considerations with respect to the rescaled $\xi = \alpha x$ variables, in view of the form $-\frac{1}{2}\triangle_\xi + \frac{\xi^2}{2} = H$ of the Hamiltonian predominantly used in the mathematical physics literature. To proceed in this notational convention it is enough to set $x \to \xi$ and $D \to \frac{1}{2}$ and thus utilize $b = \nabla\Theta/\Theta, \Omega = \frac{1}{2}(b^2 + \nabla b), \nabla\Omega = b\nabla b + \frac{1}{2}\Delta b$.

Although we need Θ, Θ_* of the same sign, and $\rho(x)$ to be strictly positive, we can first make a formal identification $\Theta = \Theta_* = \phi_n, n = 0, 1, 2, ...$ and notice that

$$n = 0\,,\, b_0 = -\xi \to \Omega_0 = \frac{\xi^2}{2} - \frac{1}{2}$$

$$n = 1\,,\, b_1 = \frac{1}{\xi} - \xi \to \Omega_1 = \frac{\xi^2}{2} - \frac{3}{2}$$

$$n = 2\,,\, b_2 = \frac{4\xi}{2\xi^2 - 1} - \xi \to \Omega_2 = \frac{\xi^2}{2} - \frac{5}{2}$$

Obviously $\nabla\Omega_n = \xi$ for all n. Irrespective of the fact that each of $b_n, n > 0$ shows singularities, the auxiliary potentials are well defined for all x, and for different values of n they acquire an additive renormalisation $-\lambda_n = -(n + \frac{1}{2})$.

The case of $n = 0$ is a canonical example of the Feynman-Kac integration, and the classic Mehler formula involves the Cameron-Martin-Girsanov density as well.

Indeed, the integral kernel $[\exp(-Ht)](y, x) = k(y, 0, x, t)$ for $H = -\frac{1}{2}\Delta + (\frac{1}{2}x^2 - \frac{1}{2})$ is known to be given by the formula:

$$k(y, 0, x, t) = \pi^{-1/2}(1 - e^{-2t})^{-1/2} \exp[-\frac{x^2 - y^2}{2} - \frac{(e^{-t}y - x)^2}{2}] \tag{63}$$

$$(e^{-Ht}\Theta)(x) = \int k(y, 0, x, t)\Theta(y)dy$$

where the integrability property

$$\int k(y, 0, x, t) \exp[\frac{x^2 - y^2}{2}]dy = 1 \tag{64}$$

is simply a statement pertaining to the transition density of the homogeneous diffusion, which preserves the Gaussian distribution $\rho(x) = (\Theta\Theta_*)(x) = \frac{\alpha}{\sqrt{\pi}} \exp(-\xi^2)$.

4.2 The imaginary time substitution as a mapping between two families of diffusion processes

Let us invoke the analytic continuation in time concept, which is a notorious technical tool to pass to the so called Euclidean framework whenever any problems with the mathematically rigorous processing appear in the context of quantum theory. In fact it is also well known that the easiest way to generate explicit examples of Markov (actually Markov-Bernstein) diffusions is by analytic continuation of solutions of the Schrödinger equation. For V continuous and bounded from below, the generator $H = -2mD^2\Delta + V$ is essentially selfadjoint, and then the kernel $h(x, s, y, t) = [\exp[-(t - s)H]](x, y)$ of the related dynamical semigroup is strictly positive, so the previous Markov-Bernstein process considerations do follow immediately for time-independent potentials V. On the other hand it is quite traditional to relate this dynamical semigroup evolution to the quantum mechanical unitary evolution operator $\exp(iHt)$ by the imaginary time substitution $t \to it$. In the most pedestrian and naive interpretation of this fact, one might be tempted to invent the concept of "diffusion process in the imaginary time". Actually nothing like that is here allowed, and if taken seriously, becomes self-contradictory.

The routine illustration for the imaginary time transformation is provided by considering the force-free propagation, where apparently the formal recipe gives rise to (one should be aware that to execute a mapping for concrete solutions, the proper adjustment of the time interval boundaries is indispensable):

$$
\begin{aligned}
i\partial_t \psi = -D\triangle\psi &\longrightarrow \partial_t \overline{\theta_*} = D\triangle\overline{\theta_*} \\
i\partial_t \overline{\psi} = D\triangle\overline{\psi} &\longrightarrow \partial_t \overline{\theta} = -D\triangle\overline{\theta} \\
it &\longrightarrow t
\end{aligned}
\tag{65}
$$

Then

$$
\psi(x,t) = [\rho^{1/2}\exp(iS)](x,t) = \int dx' G(x-x',t)\psi(x',0)
$$

$$
G(x-x',t) = (4\pi iDt)^{-1/2}\exp[-\frac{(x-x')^2}{4iDt}] \tag{66}
$$

$$
\overline{\theta_*}(x,t) = \int dx' h(x-x',t)\overline{\theta_*}(x',0)
$$

$$
h(x-x',t) = (4\pi Dt)^{1/2}\exp[-\frac{(x-x')^2}{4Dt}]
$$

where the imaginary time substitution recipe

$$
h(x-x',it) = G(x-x',t)\ ,\ \ h(x-x',t) = G(x-x',-it) \tag{67}
$$

seems to persuasively suggest the previously mentioned "evolution in imaginary time" notion, except that one *must decide in advance*, which of the two considered evolutions:the heat or Schrödinger transport, would deserve the status of the "real time diffusion".

At this point let us recall that given the initial data

$$
\psi(x,0) = (\pi\alpha^2)^{-1/4}\exp(-\frac{x^2}{2\alpha^2}) \tag{68}
$$

the free Schrödinger evolution $\partial_t\psi = -D\triangle\psi$ implies

$$
\psi(x,t) = (\frac{\alpha^2}{\pi})(\alpha^2 + 2iDt)^{-1/2}\exp[-\frac{x^2}{2(\alpha^2+2iDt)}] \tag{69}
$$

with $(\frac{m}{2}(D_+D_- + D_-D_+)X(t) = 0$ applies):

$$
\rho(x,t) = |\psi(x,t)|^2 = \frac{\alpha}{[\pi(\alpha^4 + 4D^2t^2)]^{1/2}}\exp(-\frac{x^2\alpha^2}{\alpha^4+4D^2t^2})
$$

$$= \int p(y,0,x,t)\rho(y,0)dy \tag{70}$$

$$p(y,0,x,t) = (4\pi Dt)^{-1/2} \exp[-\frac{(x-y-\frac{2D}{\alpha^2}yt)^2}{4Dt}]$$

where $p(y,0,x,t)$ is the (distorted Brownian) transition probability density for Nelson's diffusion derivable from $\psi(x,t)$. On the other hand we can straightforwardly pass to

$$\psi(x,-it) = \overline{\theta_*}(x,t) = (\frac{\alpha^2}{\pi})^{1/4}(\alpha^2+2Dt)^{-1/2}\exp[-\frac{x^2}{2(\alpha^2+2Dt)}] \tag{71}$$

Let us confine t to the time interval $[-T/2, T/2]$ with $DT < \alpha^2$. Then we arrive at

$$\begin{gathered} \partial_t \overline{\theta_*} = D\triangle\overline{\theta_*} \\ \partial_t \overline{\theta} = -D\triangle\overline{\theta} \\ -\frac{T}{2} \le t \le \frac{T}{2} \end{gathered} \tag{72}$$

$$\overline{\theta} = (\frac{\alpha^2}{\pi})^{1/4}(\alpha^2-2Dt)^{-1/2}\exp[-\frac{x^2}{2(\alpha^2-2Dt)}]$$

where

$$\overline{\rho}(x,t) = (\overline{\theta\theta_*})(x,t) = [\frac{\alpha^2}{\pi(\alpha^4-4D^2t^2)}]^{1/2}\exp[-\frac{\alpha^2x^2}{\alpha^4-4D^2t^2}] \tag{73}$$

with the interesting, and certainly unpredictable if to follow the traditional Brownian intuitions, outcome:

$$\overline{\rho}(x,-T/2) = \overline{\rho}(x,T/2) \tag{74}$$

However strange the probabilistic evolution appropriate for (74) would seem, it does not need an imagination effort, to realize that it refers to a conditional Brownian motion (in fact the Brownian bridge with smooth ends) for which the acceleration formula $D_+^2X = D_-^2X = 0$ holds true. Here the intermediate probability density (73) can be represented as the conditional transition probability density formula (identifiable as the Bernstein transition density)

$$\overline{\rho}(x,t) = P(x_1,t_1;x,t;x_2,t_2) = \frac{h(0,-\alpha,x,t)h(x,t,0,\alpha)}{h(0,-\alpha,0,\alpha)} \tag{75}$$

$$h(0,-\alpha,x,t) = [4\pi D(t+\alpha)]^{-1/2}\exp(-\frac{x^2}{4D(t+\alpha)})$$

Clearly nothing like the "imaginary time diffusion" is here involved. We have rather executed a mapping from one real time diffusion to another, with the *incompatible* dynamical principles (previously introduced microscopic conservation laws) at work. Since the Schrödinger equation plays here the role of the linear problem associated (linearisation) with the nonlinear diffusion equations, there are *not* the diffusions themselves which are related directly by the Wick rotation. The link can be established on the auxiliary (for the Nelson diffusion) level of description:the corresponding linear problem (Schrödinger equation which itself generates nonlinear diffusions) can be mapped into the linear diffusion problem, with all the reservations concerning the proper choice of the time interval boundaries.

4.3 Free Schrödinger dynamics as the diffusion process

By defining

$$p(y,0,x,t) = (4\pi Dt)^{-1/2} \exp[-\frac{(x-y+2Dty/\alpha^2)^2}{4Dt}] \tag{76}$$

we realise that

$$\int p(y,0,x,t)(\pi\alpha^2)^{-1/2}\exp(-y^2/\alpha^2)dy =$$

$$\frac{\alpha}{[\pi(\alpha^4+4D^2t^2)]^{1/2}} \exp\,[-\frac{x^2\alpha^2}{\alpha^4+4D^2t^2}] = \rho(x,t) \tag{77}$$

and

$$\int p(y,0,x,t)[\frac{2Dy}{\alpha^2}(\pi\alpha^2)^{-1/2}]\exp[-\frac{y^2}{\alpha^2}]dy =$$

$$\frac{2D(\alpha^2-2Dt)x}{\alpha^4+4D^2t^2}\rho(x,t) = -b(x,t)\rho(x,t) \tag{78}$$

where evidently

$$v(x,t) = b(x,t) - D\nabla\rho(x,t)/\rho(x,t) \tag{79}$$

solves local conservation identities (laws) with $V = 0$ and via the familiar Madelung transcription of the free Schrödinger dynamics $i\partial_t\psi(x,t) = -D\triangle\psi(x,t)$ with $\psi = \exp(R+iS), \rho = \exp(2R), v = 2D\nabla S$ the link between the Brownian type diffusion and the quantum mechanical evolution is established.

However, it seems instructive to have a detailed demonstration that the pertinent dynamics is a well defined solution of the Schrödinger problem as formulated in Section 1. To simplify considerations we shall rescale the variables so that effectively $D = 1$ appears everywhere. Certainly, we deal with the evolution associated with the continuous mapping:

$$\rho_0(x) = (2\pi)^{-1/2}\exp[-\frac{x^2}{2}] \longrightarrow \rho(x,t) = [2\pi(1+t^2)]^{-1/2}\exp[-\frac{x^2}{2(1+t^2)}] \tag{80}$$

We have defined the transition probability density effecting the "quantum job" from the initial time instant 0 till any finite time t. However, our diffusion process is definitely *not* homogeneous in time, hence the fundamental trasport mechanism for arbitrary times is very different from what the previously utilized formula might suggest. Indeed, let us consider :

$$p(y,s,x,t) = [4\pi(t-s)]^{-1/2}\exp[-\frac{(x-cy)^2}{4(t-s)}] \tag{81}$$

$$c = c(s,t) = [\frac{(1-t)^2 + 2s}{1+s^2}]^{1/2}$$

which by setting $c(0,t) = 1-t$ reduces to the previously considered $p(y,0,x,t) = (4\pi t)^{-1/2} \exp[-\frac{(x-y-yt)^2}{4t}]$. One can easily calculate the drift $b(x,t) = -\frac{(1-t)}{1+t^2}x$ following the standard stochastic methods and check the validity of both the continuity and momentum balance equations. It is however more interesting to realize that by taking $\psi(x,t) = \exp(R+iS)$ where$R(x,t), S(x,t)$ are real functions, we can as well introduce the new real functions $\theta = \exp(R+S), \theta_* = \exp(R-S)$ such that :

$$R(x,t) = -\frac{1}{4} ln 2\pi(1+t^2) - \frac{x^2}{4(1+t^2)} \tag{82}$$

$$S(x,t) = \frac{x^2}{4}\frac{t}{1+t^2} - \frac{1}{2} arctan\ t$$

implies

$$\theta(x,t) = [2\pi(1+t^2)]^{-1/4} \exp(-\frac{x^2}{4}\frac{1-t}{1+t^2}) \exp(-\frac{1}{2} arctan\ t) \tag{83}$$

$$\theta_*(x,t) = [2\pi(1+t^2)]^{-1/4} \exp(-\frac{x^2}{4}\frac{1+t}{1+t^2}) \exp(\frac{1}{2} arctan\ t)$$

where, strikingly there holds:

$$\partial\theta = -\triangle\theta + \Omega\theta \tag{84}$$

$$\partial\theta_* = \triangle\theta_* - \Omega\theta_*$$

$$\Omega(x,t) = \frac{x^2}{2(1+t^2)^2} - \frac{1}{1+t^2} = 2\frac{\triangle\rho^{1/2}}{\rho^{1/2}}$$

Moreover, the function $h(y,s,x,t) = p(y,s,x,t)\frac{\theta(y,s)}{\theta(x,t)}$ is a fundamental solution of the above equations:

$$\theta_*(x,t) = \int h(y,s,x,t)\theta_*(y,s)dy \tag{85}$$

$$h(y,s,x,t) = [4\pi(t-s)]^{-1/2}(\frac{1+t^2}{1+s^2})^{1/4} \exp\frac{1}{2}(arctan\ t - arctan\ s)$$

$$\exp[-\frac{(x-cy)^2}{4(t-s)} - \frac{y^2}{4}\frac{1-s}{1+s^2} + \frac{x^2}{4}\frac{1-t}{1+t^2}]$$

Although the form of the strictly positive semigroup kernel $h(y,s,x,t)$ does not look that promising,it is possible to check through a direct (albeit a little bit involved) computation that the dynamical semigroup implemented identity

$$\lim_{\triangle s\to 0}\frac{1}{\triangle s}[1 - \int h(y,s,x,s+\triangle s)dx] = \Omega(y,s) \tag{86}$$

is valid, as expected from the fundamental solution of the generalized diffusion equation. All probabilistic features characteristic for solution of the Schrödinger problem were there-by recovered.

Ackonwledgement

The talk was delivered by the first named author with support coming from the KBN research grant No 2 P302 057 07

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