J. Phys. A: Math. Gen. 20 (1987) 4789–4798. Printed in the UK

Stochastic quantisation of the Fermi oscillator: non-Z_2 route

Piotr Garbaczewski†

Research Center Bielefeld-Bochum-Stochastics, Bielefeld University, D-4800 Bielefeld, West Germany and Dipartimento di Fisica, Università di Salerno, I-84100 Salerno, Italy

Received 8 September 1986, in final form 9 February 1987

Abstracts. Bose stochastic mechanics is sufficient for the probabilistic description of the quantised two-level system in terms of Gaussian Markov processes.

1. Motivation

Nelson's stochastic quantisation idea [1, 2] as developed by Guerra [3–5] allows us to interpret the Bose quantised harmonic oscillator in terms of a Gaussian Markov process associated with its ground-state wavefunction. The situation appeared much less transparent when applying the stochastic mechanics strategy to Fermi systems. The simplest case of the Fermi model, the quantised two-level problem (Fermi oscillator), has been interpreted in terms of a classical statistical mechanics problem [6, 7] but the underlying Markov process is then Z_2 valued. Random processes with values on a discrete configuration space and a discrete stochastic mechanics were developed for the description of simple Fermi systems [7, 8].

This particular line of research (see, e.g., [9, 10]) is an obvious result of the choice of one specific candidate for the classical relative of the Fermi system. It seems to be rooted in the well known statistical mechanics observation that Z_2 valued classical systems on a lattice (Ising model for example) in two Euclidean dimensions admit a description in terms of lattice Fermi systems in one space dimension. However, it is also known that pseudoclassical systems with values in the Grassmann algebra provide an equivalent description (giving rise to correct partition and correlation functions) when fermions enter the game. Would this also be a reasonable classical interpretation?

On the other hand, the series of papers on quantisation of spinor fields [11–13] (see also [14, 15]) results in the conclusion that genuine c-number (commuting function ring) classical field theory has a quantum meaning. This statement has been demonstrated for two Fermi models in $(1+1)$ spacetime dimensions: the massive Thirring model and the chiral invariant Gross–Neveu model. As a byproduct, a relation of the Fermi oscillator to the standard classical harmonic oscillator problem was established. However such a choice of the classical relative to the fermion would automatically suggest another non-Z_2 stochastic quantisation programme.

As we shall show, the Bose stochastic quantisation [1–5] in fact suffices to recover the probabilistic description of the quantised two-level (Fermi oscillator) system in terms of the Gaussian Markov processes associated with the coherent states of the quantum oscillator. The discrete stochastic mechanics is thus not necessary for the understanding of the two-level problem, albeit it remains a useful alternative.

† Permanent address: Institute of Theoretical Physics, University of Wroclaw, PL-50-205 Wroclaw, Poland.

0305-4470/87/144789+10$02.50

2. Stochastic mechanics of the harmonic oscillator

The departure point for our considerations is § 6 of [4] where the stochastic quantisation of the harmonic oscillator is accomplished and then compared with the outcomes of the standard (Euclidean) quantum formalism.

The harmonic oscillator in one (time) dimension:

$$H_{\mathrm{cl}}=\frac{p^2}{2m}+\tfrac{1}{2}m\omega^2q^2 \tag{2.1}$$

upon canonical quantisation:

$$\{q,p\}=1\to[\hat{q},\hat{p}]_-\subseteq -\mathrm{i}\hbar \tag{2.2}$$

gives rise to the Schrödinger problem:

$$\mathrm{i}\hbar\,\partial_t\psi=-\frac{\hbar^2}{2m}\partial_x^2\psi+\tfrac{1}{2}m\omega^2x^2\psi \tag{2.3}$$

$$\hat{q}\equiv x\qquad\qquad \hat{p}\equiv\mathrm{i}\hbar\partial_x.$$

The ground state of (2.3) [1-4]:

$$\psi_0(x,t)=(2\pi\sigma)^{-1/4}\exp\left(-\frac{x^2}{4\sigma}-\mathrm{i}\frac{\omega}{2}t\right) \tag{2.4}$$

$$\sigma=\hbar/2m\omega$$

is associated with the Gaussian Markov process $q_0(t)$ with the expectations

$$\begin{aligned}&E(q_0(t))=0\\&E(q_0(t)q_0(t'))=\sigma\exp[-\omega(t-t')]\qquad t\geqslant t'\end{aligned} \tag{2.5}$$

while the higher-order non-vanishing correlations are given by the recursive formula

$$E(q_0(t_1)\ldots q_0(t_{2n}))=\sum_{j=2}^{2n}E(q_0(t_1)q_0(t_j))E(q_0(\not{t}_1)\ldots q_0(\not{t}_j)\ldots). \tag{2.6}$$

Since, by virtue of [1, 2] each wavefunction solving (2.3) specifies its own stochastic process, it is useful to know that to each coherent state of (2.1)

$$\begin{aligned}&\psi(x_1,t)=(2\pi\sigma)^{-1/4}\exp\left(-\frac{1}{4\sigma}(x-q_{\mathrm{cl}}(t))^2+\frac{\mathrm{i}}{\hbar}xp_{\mathrm{cl}}(t)-\frac{\mathrm{i}}{2\hbar}p_{\mathrm{cl}}(t)q_{\mathrm{cl}}(t)-\mathrm{i}\frac{\omega}{2}t\right)\\&\langle\psi,\hat{q}\psi\rangle=q_{\mathrm{cl}}(t)\qquad\langle\psi,(\hat{q}-\langle\psi,\hat{q}\psi\rangle)^2\psi\rangle=\sigma\end{aligned} \tag{2.7}$$

we have associated a corresponding stochastic process:

$$\mathrm{d}q(t)=[(1/m)p_{\mathrm{cl}}(t)-\omega(q(t)-q_{\mathrm{cl}}(t))]\,\mathrm{d}t+\mathrm{d}w(t) \tag{2.8}$$

characterised by the following density and phase data:

$$\begin{aligned}&\rho(x,t)=(2\pi\sigma)^{-1/2}\exp[-(1/2\sigma)(x-q_{\mathrm{cl}}(t))^2]\\&S(x,t)=xp_{\mathrm{cl}}(t)-\tfrac{1}{2}p_{\mathrm{cl}}(t)q_{\mathrm{cl}}(t)-\tfrac{1}{2}\hbar\omega t.\end{aligned} \tag{2.9}$$

Obviously, if we define

$$q(t)=q_{\mathrm{cl}}(t)+q_0(t) \tag{2.10}$$

then (2.8) reduces to

$$\mathrm{d}q_0(t) = -\omega q_0(t)\,\mathrm{d}t + \mathrm{d}w \tag{2.11}$$

corresponding to the wavefunction (2.4) with

$$\begin{aligned} \rho_0(x, t) &= (2\pi\sigma)^{-1/2}\exp(-x^2/2\sigma) \\ S_0(x, t) &= -\tfrac{1}{2}\hbar\omega t \\ \langle\psi_0, \hat{q}\psi_0\rangle &= 0 = \int x\rho_0(x)\,\mathrm{d}x = E(q_0(t)). \end{aligned} \tag{2.12}$$

The ground-state wavefunction and the associated Markov process are especially important because of their relation to the correlation functions of the quantum oscillator problem:

$$\begin{aligned} W(\mathrm{i}t_1, \ldots, \mathrm{i}t_n) &= \langle\psi_0, \hat{q}(\mathrm{i}t_1)\ldots\hat{q}(\mathrm{i}t_n)\psi_0\rangle \\ &= \sum_{j=2}^{n=2k} W(\mathrm{i}t_1, \mathrm{i}t_j)\,W(\mathrm{i}\not{t}_1, \ldots, \mathrm{i}\not{t}_j, \ldots) \\ W(\mathrm{i}t, \mathrm{i}t') &= \sigma\exp[-\omega(t'-t)] \end{aligned} \tag{2.13}$$

which are the formal analytic continuation ($t \to \mathrm{i}t$) of the standard time correlation functions. Obviously in the above

$$\begin{aligned} \psi_0 &\equiv \psi_0(x, 0) = \exp(-\mathrm{i}\hat{H}t)\psi_0(x, t) \\ \hat{q}(t) &= \exp(\mathrm{i}\hat{H}t)\hat{q}\exp(-\mathrm{i}\hat{H}t) \\ \hat{H} &= \hat{p}^2/2m + \tfrac{1}{2}m\omega^2\hat{q}^2. \end{aligned} \tag{2.14}$$

Let us notice that the stochastic processes (2.8)-(2.10) fall into the category of the controlled ones of [5], e.g. we have

$$\mathrm{d}q(t) = v_+(q(t), t)\,\mathrm{d}t + \mathrm{d}w(t) \tag{2.15}$$

where (see [4]) the control field $v_\pm$ in the oscillator case is

$$\begin{aligned} v_\pm(x, t) &= (1/m)p_{\mathrm{cl}}(t) \mp \omega(x - q_{\mathrm{cl}}(t)) \\ v &= \tfrac{1}{2}(v_+ + v_-) = (1/m)\nabla S = (1/m)p_{\mathrm{cl}}(t) \\ u &= \tfrac{1}{2}(v_+ - v_-) = (\hbar/2m)\nabla\rho/\rho \end{aligned} \tag{2.16}$$

and ρ and S are specified by (2.9).

By virtue of [5] the symplectic structure can be related to (2.1) with a local (ρ, S) parametrisation and the Poisson bracket:

$$\{\mathscr{A}, \mathscr{B}\} = \int \mathrm{d}x\left(\frac{\delta\mathscr{A}}{\delta\rho(x)}\frac{\delta\mathscr{B}}{\delta S(x)} - \frac{\delta\mathscr{B}}{\delta\rho(x)}\frac{\delta\mathscr{A}}{\delta S(x)}\right) \tag{2.17}$$

for any two functions $\mathscr{A} = \mathscr{A}(\rho, S)$, $\mathscr{B} = \mathscr{B}(\rho, S)$ on phase space. The following phase space function:

$$\mathscr{H} = \mathscr{H}(\rho, S) = \int \tfrac{1}{2}(mv^2 + mu^2 + m\omega^2x^2)\rho(x)\,\mathrm{d}x \tag{2.18}$$

can be used to generate the time development of ρ and S, since [5]

$$\partial_t \rho(x,t) = -\nabla(\rho v) = \{\rho, \mathscr{H}\} = \frac{\delta \mathscr{H}}{\delta S(x,t)}$$

$$\partial_t S(x,t) = \frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} - \frac{(\nabla S)^2}{2m} + \tfrac{1}{2}m\omega^2 x^2 \tag{2.19}$$

$$= \{S, \mathscr{H}\} = -\frac{\delta \mathscr{H}}{\delta \rho(x,t)}.$$

In terms of another canonical parametrisation:

$$\psi = \sqrt{\rho}\exp(\mathrm{i}S/\hbar)$$
$$\{\mathscr{A}, \mathscr{B}\} = \frac{1}{\mathrm{i}\hbar}\int \mathrm{d}x\left(\frac{\delta\mathscr{A}}{\delta\psi(x)}\frac{\delta\mathscr{B}}{\delta\psi^*(x)} - \frac{\delta\mathscr{B}}{\delta\psi(x)}\frac{\delta\mathscr{A}}{\delta\psi^*(x)}\right) \tag{2.20}$$

we immediately realise that

$$\{\psi(x), \psi^*(x')\} = \delta(x-x')/\mathrm{i}\hbar$$
$$\{\psi(x), \psi(x')\} = 0 = \{\psi^*(x), \psi^*(x')\} \tag{2.21}$$

and moreover

$$\mathscr{H} = \mathscr{H}(\psi, \psi^*) = \langle\psi, \hat{H}\psi\rangle = \int \psi^*(x)(\hat{H}\psi)(x)\,\mathrm{d}x \tag{2.22}$$

which furthermore implies

$$\partial_t \psi = \{\psi, \mathscr{H}\} = \frac{1}{\mathrm{i}\hbar}\frac{\delta\mathscr{H}}{\delta\psi^*(x)} = \frac{1}{\mathrm{i}\hbar}(\hat{H}\psi)(x). \tag{2.23}$$

Since the ρ, S dependence enters the object $\mathscr{A} \in \mathbb{C}$:

$$\mathscr{A} = \mathscr{A}(\psi^*, \psi) = \langle\psi, \hat{A}\psi\rangle = \int \psi^*(x)(\hat{A}\psi)(x)\,\mathrm{d}x \tag{2.24}$$

through wavefunctions ψ^*, ψ only, while $\hat{A}$ is some (bounded if we wish to avoid domain problems) operator acting in the Hilbert space $h = L^2(R^1)$, we can view (2.24) as a general prescription relating the operator-valued objects in h, with functions of phase space data $\mathscr{A} = \mathscr{A}(\rho, S)$. What is more important here is that for any two phase space functions $\mathscr{A}$, $\mathscr{B}$ given by (2.24), the formula (2.20) implies that

$$C = \{\mathscr{A}, \mathscr{B}\} = \frac{1}{\mathrm{i}\hbar}\int \mathrm{d}x[\psi^*(x)(\hat{A}\hat{B}\psi)(x) - \psi^*(x)(\hat{B}\hat{A}\psi)(x)]$$

$$= \frac{1}{\mathrm{i}\hbar}\langle\psi, [\hat{A}, \hat{B}]_-\psi\rangle = \langle\psi, \hat{C}\psi\rangle. \tag{2.25}$$

It means that the commutator algebra is consistently mapped into the Poisson bracket algebra for functions on stochastic phase space. In particular

$$\dot{\mathscr{A}} = \{\mathscr{A}, \mathscr{H}\} = \frac{\mathrm{i}}{\hbar}\langle\psi, [\hat{A}, \hat{H}]_-\psi\rangle = \langle\psi, \dot{\hat{A}}\psi\rangle \tag{2.26}$$

which provides us with a correct image of the quantum dynamics. Here $\mathcal{H}$ of (2.18) arises via (2.24) as follows:

$$\mathcal{H} = \mathcal{H}(\rho, S) = \int\int \mathrm{d}x\, \mathrm{d}y\, \psi^*(x)\left(-\frac{\hbar^2}{2m}\partial_y^2 + \tfrac{1}{2}m\omega^2 y^2\right)\delta(x-y)\psi(y). \tag{2.27}$$

Everywhere in the above the expectation values of quantum operators display an explicit dependence on the defining functions ρ and S of the random (Markov) process associated with the wavefunction in use. Moreover the classically patterned (stochastic) mechanics reflects all the necessary quantum features, and specifically the Schrödinger equation comes out as the Hamilton equation of motion on the ρ, S symplectic manifold. In fact we manipulate with the probabilistic expectations of the (related) random variables and functions of them. Let us mention that extensions of this harmonic oscillator strategy are possible for less trivial examples of quantum systems like, e.g., the anharmonic oscillator [16] or the hydrogen atom [17], see also [18, 19] where the stochastic reinterpretation of the Bopp–Haag quantum rotator model is presented.

3. The Fermi oscillator

We are now ready to demonstrate that the Fermi oscillator can be completely embedded in the above stochastic framework, hence without any reference to discrete stochastic processes.

For a complete explanation of this conjecture one should solve the Schrödinger equation for the Fermi oscillator (arising via its embedding in the harmonic oscillator Hilbert space) and explicitly specify the (x, t) dependence of the probability density $\rho(x, t)$ and phase $S(x, t)$ determining the solution $\psi(x, t) = \rho^{1/2}(x, t)\exp(\mathrm{i}S(x, t)/\hbar)$. Then the class of controlled stochastic Markov processes would arise with the drift velocity completely defined in terms of ρ, S.

Let the harmonic oscillator problem be given in the form

$$\hat{H}_{\mathrm{B}} = \hbar\omega(a^*a + \tfrac{1}{2}) \qquad [a, a^*]_- \subseteq 1,\ a|0) = 0. \tag{3.1}$$

The two-level projection p in $L^2(R^1)$ is defined as follows:

$$\begin{aligned} &p = {:}\exp(-a^*a){:} + a^*{:}\exp(-a^*a){:}a \\ &[\hat{H}_{\mathrm{B}}, p]_- = 0 \\ &\hat{H}_{\mathrm{B}} = p\hat{H}_{\mathrm{B}}p + (1-p)\hat{H}_{\mathrm{B}}(1-p) \\ &\hat{H}_{\mathrm{F}} = p\hat{H}_{\mathrm{B}}p \end{aligned} \tag{3.2}$$

so that H_{F} is the Hamiltonian of the two-level system:

$$\begin{aligned} \hat{H}_{\mathrm{F}} &= \hbar\omega(pa^*ap + \tfrac{1}{2}p) \\ &= \tfrac{1}{2}\hbar\omega[3a^*{:}\exp(-a^*a){:}a + {:}\exp(-a^*a){:}]. \end{aligned} \tag{3.3}$$

For any state vector $|\psi) \in ph = h_{\mathrm{F}}$, $h = L^2(R^1)$, we have

$$p|\psi) = |\psi) \Rightarrow H_{\mathrm{B}}|\psi) = H_{\mathrm{F}}|\psi) \qquad (\psi|H_{\mathrm{B}}|\psi) = (\psi|H_{\mathrm{F}}|\psi). \tag{3.4}$$

Let us consider the coherent state in h and its two-level projection:

$$\begin{aligned} &|\alpha) = \exp(\alpha a^* - \bar{\alpha}a)|0) = \exp(-\tfrac{1}{2}|\alpha|^2)\exp(\alpha a^*)|0) \\ &p|\alpha) = \exp(-\tfrac{1}{2}|\alpha|^2)(1 + \alpha a^*)|0). \end{aligned} \tag{3.5}$$

We normalise $p|\alpha)$ to unity so that

$$|\psi) = \frac{p|\alpha)}{(\psi|\psi)^{1/2}} = (1+|\alpha|^2)^{-1/2}(1+\alpha a^*)|0)$$
$$= (1+|\alpha|^2)^{-1/2}\exp(|\alpha|^2/2)p|\alpha). \tag{3.6}$$

The following property holds true:

$$(\psi|\hat{H}_B|\psi) = \frac{\exp|\alpha|^2}{1+|\alpha|^2}(\alpha|p\hat{H}_B p|\alpha)$$
$$= (\alpha|\hat{H}_F|\alpha)\frac{\exp|\alpha|^2}{1+|\alpha|^2} = (\psi|\hat{H}_F|\psi)$$
$$= \frac{\hbar\omega}{2}\left(\frac{2|\alpha|^2}{1+|\alpha|^2}+1\right) = \mathscr{H}_F \tag{3.7}$$

to be compared with

$$(\alpha|\hat{H}_B|\alpha) = \tfrac{1}{2}\hbar\omega(2|\alpha|^2+1) = \mathscr{H}_B. \tag{3.8}$$

Notice that (3.7) goes over to (3.8) in the small $|\alpha|$ regime, compare, e.g., the discussion of [13] at this point.

Since $|\psi)$ is defined in the proper subspace of the harmonic oscillator Hilbert space, its configuration space realisation can be obtained by passing to the Schrödinger representation of the CCR algebra. By inserting the well known expressions for the Weber-Hermite functions, we arrive at the following wavefunction

$$|\psi) \to (x|\psi) = \psi(x) = (1+|\alpha|^2)^{-1/2}$$
$$\times\left(\frac{1}{2\pi\sigma}\right)^{1/4}\left(1+\frac{\alpha x}{\sigma^{1/2}}\right)\exp(-x^2/4\sigma) \qquad \sigma = \frac{\hbar}{2m\omega}. \tag{3.9}$$

Since $\alpha = J^{1/2}\exp(+i\Delta) = J^{1/2}(\cos\Delta + i\sin\Delta)$ we have furthermore ($\psi(x)$ is normalised)

$$\psi(x) = \rho^{1/2}(x)\exp(i/\hbar)S(x) \tag{3.10}$$

where

$$\begin{aligned}\rho(x) &= (1+J)^{-1}\left(\frac{\omega m}{\pi\hbar}\right)^{1/2}\exp\left(-x^2\frac{m\omega}{\hbar}\right)\\ &\quad\times\left[1+Jx^2\frac{m\omega}{\hbar}+x\left(J\frac{m\omega}{\hbar}\right)^{1/2}\cos\Delta\right]\\ S(x) &= \hbar\cos^{-1}\left(\frac{1+x(Jm\omega/\hbar)^{1/2}\cos\Delta}{[1+J(m\omega/\hbar)x^2+x(Jm\omega/\hbar)^{1/2}\cos\Delta]^{1/2}}\right)\\ \hat{H}_B &= (1/2m)(p^2+m^2\omega^2q^2) \qquad [q,p]_- \subseteq i\hbar.\end{aligned} \tag{3.11}$$

By virtue of (3.10) we can adopt the canonical formalism of § 2, which implies that for time-dependent (Schrödinger picture) wavepackets, the Schrödinger equation arises from the classical Hamiltonian formalism in the framework of stochastic mechanics:

$$\partial_t\psi(x,t) = \{\psi(x,t),\mathscr{H}\}_{\psi,\bar{\psi}}$$

$$\{A, B\}_{\psi,\bar{\psi}} = \frac{1}{\mathrm{i}\hbar}\int \mathrm{d}x \left(\frac{\delta A}{\delta\psi(x)}\frac{\delta B}{\delta\bar{\psi}(x)} - \frac{\delta A}{\delta\bar{\psi}(x)}\frac{\delta B}{\delta\psi(x)}\right)$$
$$= \int \mathrm{d}x\left(\frac{\delta A}{\delta\rho(x)}\frac{\delta B}{\delta S(x)} - \frac{\delta A}{\delta S(x)}\frac{\delta B}{\delta\rho(x)}\right) = \{A, B\}_{\rho,S} \tag{3.12}$$

provided we have

$$(\psi|\hat{H}_{\mathrm{B}}|\psi) = \mathscr{H} = \mathscr{H}(\rho, S)$$
$$= \int (\tfrac{1}{2}v^2 + \tfrac{1}{2}u^2 + V)(x)\rho(x)\,\mathrm{d}x \tag{3.13}$$
$$V(x) = m^2\omega^2x^2$$

where

$$v(x,t) = v = \nabla S \qquad u(x,t) = u = \tfrac{1}{2}\nabla\rho/\rho. \tag{3.14}$$

The particular functional form of ρ and S relies on the choice of the state vector $|\psi)$ which solves the Schrödinger equation.

For the harmonic oscillator coherent state (2.7), we apparently have

$$\psi(x,0) = \rho'(x)\exp(\mathrm{i}S'(x)/\hbar)$$
$$\mathscr{H}' = \mathscr{H}(\rho', S') = \int (\tfrac{1}{2}v'^2 + \tfrac{1}{2}u'^2 + V)\rho'(x)\,\mathrm{d}x \tag{3.15}$$

and by virtue of (3.4), the hydrodynamic Hamiltonian $\mathscr{H}$ allows for both the choice of ρ, S and ρ', S', the difference being coded in the particular (x, t) dependence of $|\psi)$ and $|\alpha) \to (x|\alpha) = \psi(x, 0)$. At this point we observe that

$$\exp(-\mathrm{i}\hat{H}_{\mathrm{B}}t/\hbar)|\alpha) = \exp(-\tfrac{1}{2}|\alpha|^2)\sum_n \frac{(\alpha a^*)^n}{n!}\exp[-\omega(n+\tfrac{1}{2})t]|0)$$
$$= \exp(-\omega t/2)|\alpha(t)) = |\alpha, t) \tag{3.16}$$

which, by virtue of

$$\hat{H}_{\mathrm{B}}|\alpha, t) = \mathrm{i}\hbar\partial_t|\alpha, t) \tag{3.17}$$

and

$$p\hat{H}_{\mathrm{B}}|\alpha, t) = \hat{H}_{\mathrm{B}}p|\alpha, t) \tag{3.18}$$

implies that the time development of the normalised wavepacket (3.6) is

$$|\psi, t) = (1+|\alpha|^2)^{-1/2}\exp(|\alpha|^2/2)\exp(-\mathrm{i}\hat{H}_{\mathrm{B}}t)p|\alpha) \tag{3.19}$$

with

$$\exp(-\mathrm{i}\hat{H}_{\mathrm{B}}t)p|\alpha) = \exp(-\omega t/2)p|\alpha(t))$$
$$= p|\alpha, t). \tag{3.20}$$

Because

$$\alpha(t) = J^{1/2}\exp[\mathrm{i}(\Delta - \omega t)] \tag{3.21}$$

we finally arrive at the following motion formulae for $\rho(x)$ and $S(x)$ as given by (3.11):

$$\rho(x,t) = \rho(x, \Delta \to \Delta - \omega t)$$
$$S(x,t) = S(x, \Delta\to\Delta-\omega t) - \hbar\omega t/2 \tag{3.22}$$

where, obviously,

$$\hat{H}_F|\psi, t) = \hat{H}_B|\psi, t) = i\hbar\partial_t|\psi, t). \tag{3.23}$$

Because we have $\rho(x, t)$ and $S(x, t)$, the corresponding stochastic process is given by the Ito stochastic differential equation with the drift velocity $v_+ = (1/m)\nabla S + (\hbar/2m)\nabla\rho/\rho$. Since any solution of $\hat{H}_F\psi = i\hbar\partial_t\psi$ is automatically a solution of $\hat{H}_B\psi = i\hbar\partial_t\psi$, we deal in fact with a subset of processes which can be related to the quantum harmonic oscillator.

Remark. By virtue of (3.2) the following properties hold true:

$$\hat{H}_B = p\hat{H}_B p + (1-p)\hat{H}_B(1-p)$$

$$p\hat{H}_B p = \hbar\omega(\sigma^+\sigma^- + \tfrac{1}{2}p) = \hat{H}_F$$

$$\sigma^+ = pa^*p \qquad \sigma^- = pap$$

$$[\sigma^-, \sigma^+]_+ = p.$$

Notice that on $h_F = ph$, p acts as an identity. All operators p, H_F, σ^+, σ^- are bounded in h. Moreover, $\{\sigma^+, \sigma^-, \psi_0\}$ provides us with an irreducible representation of the CAR algebra in h_F, so that any operator acting invariantly in h_F can be given as a function of σ^+, σ^-, and hence is bounded in h. We introduce

$$\hat{Q} = \sigma^+ + \sigma^- \qquad \hat{P} = -i(\sigma^- - \sigma^+)$$

$$\hat{Q}^2 = p = \hat{P}^2 \qquad [\hat{Q}, \hat{P}]_+ = 0$$

so that the Fermi oscillator equations of motion follow:

$$\begin{aligned} i[\hat{Q}, \hat{H}_F]_- &= -\dot{\hat{Q}} = -\hat{P} \\ i[\hat{P}, \hat{H}_F]_- &= -\dot{\hat{P}} = \hat{Q}. \end{aligned} \tag{3.24}$$

Observe that the harmonic oscillator ground state $\psi_0 \equiv \psi_0(x, 0)$ given by (2.4) is the ground state of H_F as well. It is then apparent that, after introducing

$$\hat{Q}(t) = \exp(i\hat{H}_F t)\hat{Q}\exp(-i\hat{H}_F t) \tag{3.25}$$

the standard [6] Schwinger function formula arises:

$$\begin{aligned} S(t_1, \ldots, t_n) &= \langle\psi_0, \hat{Q}(it_1)\ldots\hat{Q}(it_n)\psi_0\rangle \\ &= \langle\psi_0, \hat{Q}\exp[-(t_2 - t_1)\hat{H}_F]\hat{Q} \\ &\quad \times \exp[-(t_3 - t_2)\hat{H}_F]\hat{Q}\ldots\exp[-(t_n - t_{n-1})\hat{H}_F]\hat{Q}\psi_0\rangle \\ &= S(t_1, t_2)S(t_3, t_4)\ldots S(t_{n-1}, t_n). \end{aligned} \tag{3.26}$$

For n odd, S_n vanishes and

$$S(t, t') = E(q(t)q(t')) = W(it, it').$$

4. Discussion

The main difference, in comparing our approach to that of exploiting the discrete stochastic mechanics, appears to be rather obvious. Namely in the Z_2 approach of references [6-10], the guiding principle was to invent a stochastic description for the Fermi problem such that the Bose formulae for correlations of random variables could have been imitated (by the way, the same principle in the framework of quantum field theory identified Grassmann algebra valued models as correct classical analogues of Fermi ones). In our case the above-mentioned imitation property holds true for the first two correlations

$$\langle\psi_0, \hat{Q}(\mathrm{i}t)\psi_0\rangle = E[q(t)] \qquad \langle\psi_0, \hat{Q}(\mathrm{i}t_1)\hat{Q}(\mathrm{i}t_2)\psi_0\rangle = E[q(t_1)q(t_2)] \tag{4.1}$$

and the higher ones do not fit the boson algorithm since the probabilistic analogue of $Q(\mathrm{i}t)$ is not a random variable undergoing the (harmonic oscillator) stochastic process, but rather a function of this random variable. It appears however [20] that the stochastic construction of ours is not competing with the Z_2 one, but stands rather for the complementary stochastic description of the same model. Indeed, each simple quantum system (e.g. harmonic oscillator) can either be described in terms of stochastic processes in the configuration space (which is the standard Nelson route of relating diffusion processes to stationary states of the Hamiltonian) or in terms of stochastic processes which induce discrete jumps between different energy levels of the system. Our approach does exactly follow the original Nelson route, while the discrete stochastic mechanics in fact departs from the other alternative which corresponds to the momentum space description.

Let us mention that our restriction of the boson reconstructed fermion system to the two-dimensional space in the Hilbert space of the Bose system finds its predecessor in the important stochastic analysis of the spin systems in the framework of the Bopp-Haag rotator model [18, 19].

In our investigation the crucial role is played by the two-level projection p in the harmonic oscillator Hilbert space, and one may argue that its choice is somewhat ambiguous. At this point it is useful to refer to physical systems which in the appropriate scaling regime give rise to two-level systems. One well known example is the double well oscillator [21]. Since we do not have a detailed stochastic mechanics of this problem, to understand the issue we shall simulate the properties of the model by considering the subsidiary quantum Bose system in $h = L^2(R)$:

$$\begin{aligned} H_\lambda &= H + \lambda n(n-1) \\ n &= a^* a. \end{aligned} \tag{4.2}$$

We have here modified the original harmonic problem by adding the quartic interaction term. Our previous stochastic mechanics arguments apply without alteration since H_λ can be diagonalised in h together with $\hat{H}$. The essential feature of the subsidiary model H_λ (which reflects the scaling properties of the ϕ^4 model) is that upon assuming $\lambda \gg 1$ we open a large energy gap between the two lowest and all the other energy levels (this is the reason why in statistical physics the ϕ^4 models are used as approximations of Ising models). Consequently the quantum model H_λ in the imaginary time (Euclidean) formalism allows for an arbitrarily good approximation of the two-level system with the growth of λ.

The case $\lambda = 0$ corresponds to the standard harmonic oscillator, while the $\lambda \to \infty$ regime gives rise to the two-level system (see also [15]). Since the stochastic mechanics, albeit referring to processes in the real time, provides the Euclidean description of the corresponding quantum systems, we conjecture that there is a λ interpolation between the stochastic mechanics of the double well oscillator and this for the Fermi oscillator.

Remark 1. An analogue of the above interpolation in the non-Euclidean framework (strong operator limit with respect to $\lambda \to \infty$ was enough) arises in the case of the non-linear Schrödinger model with the repulsive interaction in $(1+1)$ dimensions [11, 14]. The original interacting model in the Fock space, determined by the boson field variables, in the $\lambda \to \infty$ limit goes over to the free fermion model living in the very same boson Hilbert space. The $\lambda = 0$ limit gives rise to the free boson field.

Remark 2. The importance of the two-level projection p used by us becomes clear when passing to the study of lattice systems with the final goal of approaching the continuum limit. It pertains in fact to remark 1. In the continuum limit the product of single site projections p in a natural way accounts for the boson and fermion Fock space unification argument [14, 21–24].

Acknowledgments

I would like to thank Professors S Albeverio, Ph Blanchard and L Streit for an invitation to the BiBoS Research Center, where the first draft of the paper was written. The idea of the paper arose from conversations I had with Professor Edward Nelson. The final version of the paper is a result of illuminating discussions with Professor Diego de Falco, and I want to thank Professor Giuseppe Vitiello for hospitality in Salerno where these discussions took place.

References

[1] Nelson E 1985 *Quantum Fluctuations* (Princeton, NJ: Princeton University Press)
[2] Zambrini J C 1986 *J. Math. Phys.* **27** 2397
[3] Guerra F and Ruggiero P 1973 *Phys. Rev. Lett.* **31** 1023
[4] Guerra F 1981 *Phys. Rep.* **77** 263
[5] Guerra F and Marra R 1983 *Phys. Rev.* D **28** 1916
[6] De Angelis G F, de Falco D and Guerra F 1981 *Phys. Rev.* D **23** 1747
[7] Guerra F and Marra R 1984 *Phys. Rev.* D **30** 2579
[8] de Falco D, De Martino S and De Siena S 1982 *Phys. Lett.* **109B** 34
[9] De Angelis G F, Jona-Lasinio C and Sirugue M 1983 *J. Phys. A: Math. Gen.* **16** 2433
[10] Blanchard Ph, Combe Ph, Sirugue M and Sirugue-Collin M 1985 *Preprint, Probabilistic solution of the Dirac equation* BiBoS Bielefeld
[11] Garbaczewski P 1978 *J. Math. Phys.* **19** 642; 1981 *J. Math. Phys.* **22** 1273; 1983 *J. Math. Phys.* **24** 341; 1984 *J. Math. Phys.* **25** 862
[12] Garbaczewski P 1983 *Ann. Phys., NY* **150** 22
[13] Garbaczewski P 1983 *Nucl. Phys.* B **218** 321
[14] Garbaczewski P 1985 *Classical and Quantum Field Theory of Exactly Soluble Nonlinear Systems* (Singapore: World Scientific)
[15] Garbaczewski P 1985 *J. Math. Phys.* **26** 2039
[16] Garbaczewski P 1986 *Phys. Rev.* D **33** 2916
[17] Garbaczewski P 1987 *Stochastic mechanics for the quantized anharmonic oscillator* in preparation
[18] Dankel T Jr 1970 *Arch. Rat. Mech. Anal.* **37** 192
[19] Dankel T Jr 1977 *J. Math. Phys.* **18** 253
[20] de Falco D private communication
[21] Isaacson D 1977 *Commun. Math. Phys.* **53** 253
[22] Hudson R and Parthasarathy K 1986 *Commun. Math. Phys.* **104** 457
[23] Garbaczewski P 1987 *J. Phys. A: Math. Gen.* **20** 1277
[24] Garbaczewski P 1987 *Int. J. Theor. Phys.* to be published