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Bosons, fermions and spins $\frac{1}{2}$: the interplay on lattices of arbitrary dimension

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Abstract. We establish relationships between Bose, Fermi and spin-$\frac{1}{2}$ systems on lattices of arbitrary dimension and arbitrary but finite size which are defined to live in the (common) state space of the boson system. The underlying systems display an essential affinity by being described by precisely the same form of Hamiltonians, which are the polynomial functions of canonical operator variables: boson, fermion or spin-$\frac{1}{2}$ variables, respectively.

For large enough, densely populated finite volume lattices it entails the computation of partition and correlation functions for lattice fermions in terms of bosons only. Albeit an approximation, its accuracy increases as the continuum limit is approached.

1. Motivation

The main motive for our investigation is the recent paper [1] which together with the preceding one [2] attempts to develope the mapping of free fermion systems on arbitrary lattices into systems of spins. The idea behind it was that of mapping fermion (anticommuting) degrees of freedom into the boson (commuting) ones, which by itself has a long history and quite a variety of realisations (see, e.g., [3, 4]). In [1, 2] it has been limited to the replacement of fermions by spins which are not bosons in the strict sense of the word. Moreover, as usual when the generalised Jordan-Wigner map is in use, the resulting spin Hamiltonians display a complicated non-local structure, which is due to the lack of the natural site ordering in higher dimensions. Monte Carlo computations with fermions on a lattice, despite being developed into the self-contained domain of searching for less computer time-consuming procedures [5] have not as yet reached the stage of efficiency with respect to its primary goal, i.e. that of computing the energy spectrum of the given system for not too small lattices.

The main difficulty arises here from the use of anticommuting variables (Grassmann algebra elements) which traditionally enter the game when Fermi systems are studied via Monte Carlo methods. Since the commuting variables are associated with bosons, it is not a purely academic exercise to study mappings from fermionic to bosonic models which automatically involve a passage from the Grassmann Monte Carlo to the standard Bose Monte Carlo techniques, this was our goal both in [3, 4] and in more recent papers [6, 7]. The underlying fermions-to-bosons passage according to

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the recipes of [4, 6, 7] always induces a spin-$\frac{1}{2}$ system which is related to the Fermi system under study. Hence it pertains to the analysis of [1, 2].

Our main goal in the present paper is to reveal connections between the formally distinct boson, fermion and spin-$\frac{1}{2}$ versions of the same polynomial function, a good example being provided by the quadratic Hamiltonians $\Sigma_{jk} b_j^* W_{jk} b_k$, $\Sigma_{jk} a_j^* W_{jk} a_k$, $\Sigma_{jk} \sigma_j^+ W_{jk} \sigma_k^-$ where (W_{jk}) is the $N \times N$ Hermitian matrix, $b_j^\#$ stands for bosons and $a_j^\#$ for fermions, while $\sigma_j^\#$ represents Pauli level raising/lowering operators at each site. Extensions to the simplest (density-density) interacting models are straightforward. The particular choice of the lattice dimension and size is irrelevant.

The underlying three model Hamiltonians, denoted by $H_{\rm B}$, $H_{\rm F}$ and $H_{1/2}$ respectively, can be defined to live in a common Hilbert space $\mathscr{H}_{\rm B}$. We demonstrate that on a proper subspace of $\mathscr{H}_{\rm B}$ selected by the projection P: $\mathscr{H}_{\rm F} = P\mathscr{H}_{\rm B}$ the following operator identities hold true:

$$PH_{\rm B}P = H_{1/2} \qquad P\tilde{H}_{\rm B}P = H_F$$

$$\tilde{H}_{\rm B} = \Gamma H_{\rm B} \Gamma^{-1} \qquad [\tilde{H}_{\rm B}, P]_- = 0$$

where Γ is a unitary transformation in $\mathscr{H}_{\rm B}$.

By using the argument of the boson and fermion Fock space unification which is valid in the continuum [8–10] we arrive at the conjecture that in a finite volume, with the growth of the site density, i.e. lowering the lattice spacing, it makes sense to disregard differences between the observable features of the Hamiltonian systems $H_{\rm B}$, $H_{\rm F}$, $H_{1/2}$. Consequently we may freely pass from the boson to fermion or spin-$\frac{1}{2}$ degrees of freedom and the reverse, since they can be viewed as different manifestations of (almost) the same physical situation.

This means that for densely populated finite lattices, properties of the fermion systems should satisfactorily reproduce (approximately) those of their pure boson relatives and on the reverse.

In connection with the above reversibility claim a warning is necessary that, in general, the reverse route does not apply, as explained in reference [10], namely, all fermion field theory models (not only in the case of Fock representations of the canonical algebra) admit boson equivalents, while the reverse statement is invalid: not all boson models allow for a pure fermion reconstruction. This feature seems to offer an explanation of the issue raised by a referee of this paper, which is worth mentioning to avoid interpretational problems or erroneous results while applying our discussion in concrete calculations.

While boson and spin systems are expected to be similar (spin-$\frac{1}{2}$ systems correspond to hard core bosons), they are usually different from fermion systems. The main difference is that the first two types of systems may show a (non-Fock) ground state with long-range order, such as, for example, the superfluid state with breaking of gauge symmetry for bosons or the magnetically ordered state of the XY model. These ground states occur for infinite systems. For finite systems (and we consider the finite volume which, both on a lattice and in the continuum, implies the use of Fock space methods) there is no such long-range order. However, in the continuum limit, for large enough systems—one should expect, and this is the purpose of this work, to use it for Monte Carlo methods—that the features associated with the long-range order of boson and spin systems begin to show up and consequently fermion systems are allowed to behave differently compared with boson and spin systems in this limit.

2. Quadratic Hamiltonians

Let us consider the finite d-dimensional lattice, where d stands for positive integer. Let $\boldsymbol{n}$ be the site label $\boldsymbol{n} = (n_1, n_2, \dots n_d)$. In principle n_1, n_2, n_3 can be viewed as space coordinates while the remaining labels indicate the internal degrees of freedom. Let us consider the Fock representation of the CAR algebra with generators

$$\begin{aligned} &[a_{\boldsymbol{n}}, a^*_{\boldsymbol{m}}]_+ = \delta_{\boldsymbol{nm}} \\ &[a_{\boldsymbol{n}}, a_{\boldsymbol{m}}]_+ = 0 = [a^*_{\boldsymbol{n}}, a^*_{\boldsymbol{m}}]_+ \\ &a_{\boldsymbol{n}}|0\rangle = 0 \qquad \forall \boldsymbol{n}. \end{aligned} \tag{2.1}$$

Let the Hermitian operator which is quadratic with respect to $\{a^*_{\boldsymbol{n}}, a_{\boldsymbol{n}}\}$ be given by

$$H_{\mathrm{F}} = \sum_{\boldsymbol{n},\boldsymbol{m}} a^*_{\boldsymbol{n}} W_{\boldsymbol{nm}} a_{\boldsymbol{m}} \tag{2.2}$$

where summations extend over all possible labels $\boldsymbol{n}$, $\boldsymbol{m}$. We adopt the following linear ordering convention in the set of site labels $\boldsymbol{n}$:

$$\boldsymbol{n} < \boldsymbol{m} \text{ if } n_1 < m_1 \text{ or } n_1 = m_1, n_2 < m_2, \text{ or} \dots \text{ or } n_1 = m_1, \dots, n_{d-1} = m_{d-1}, n_d < m_d \tag{2.3}$$

which allows us to replace the $\boldsymbol{n}$ labelling by the standard linear one

$$H_{\mathrm{F}} = \sum_{s,t=1}^{N} a^*_s W_{st} a_t. \tag{2.4}$$

N denotes here the overall number of distinct site labels. We shall not impose *any* restrictions on the matrix W except for its Hermiticity: $W^*_{st} = W_{ts}$.

Let $\{b^*_s, b_s, 1 \leq s \leq N\}$ be the generators of the Fock representation of the CCR algebra:

$$\begin{aligned} &[b_s, b^*_t]_- \equiv \delta_{st} \\ &[b_s, b_t]_- = 0 = [b^*_s, b^*_t]_- \\ &b_s|0\rangle = 0 \qquad \forall s \end{aligned} \tag{2.5}$$

where the notation $|0\rangle$ is kept the same for bosons and fermions due to reasons to be clarified below. The boson Fock space constructed about the vacuum $|0\rangle$ we denote $\mathscr{H}_{\mathrm{B}}$.

Let us introduce the boson analogue of the operator (2.4):

$$H_{\mathrm{B}} = \sum_{s,t=1}^{N} b^*_s W_{st} b_t. \tag{2.6}$$

Inspired by our previous investigations [4, 6, 7] we introduce the following projection operator in $\mathscr{H}_{\mathrm{B}}$:

$$\begin{aligned} &P = \prod_{s=1}^{N} P_s \\ &P_s =: \exp(-b^*_s b_s) : + b^*_s : \exp(-b^*_s b_s) : b_s \end{aligned} \tag{2.7}$$

with the obvious properties

$$\begin{aligned} &P: \mathscr{H}_{\mathrm{B}} \to P\mathscr{H}_{\mathrm{B}} = \mathscr{H}_{\mathrm{F}} \subset \mathscr{H}_{\mathrm{B}} \\ &P b^*_s P = \sigma^+_s \\ &P b_s P = \sigma^-_s \\ &[\sigma^*_s, \sigma^*_t]_- = 0 \qquad s \neq t \qquad [\sigma^-_s, \sigma^+_s]_+ = P \\ &P H_{\mathrm{B}} P = \sum_{s,t=1}^{N} \sigma^+_s W_{st} \sigma^-_t = H_{1/2}. \end{aligned} \tag{2.8}$$

By adopting the standard Jordan-Wigner construction we make the following identification of the fermion operators in the boson Fock space:

$$a_s^* = \sigma_s^+ \prod_{j<s} (1-2\sigma_j^+\sigma_j^-)$$
$$a_s = \sigma_s^- \prod_{j<s} (1-2\sigma_j^+\sigma_j^-). \qquad (2.9)$$

It allows us to represent the fermion Hamiltonian H_F in $\mathscr{H}_B$ and in particular in its proper subspace $\mathscr{H}_F$, together with $H_{1/2}=PH_BP$.

By inverting (2.9) we can always rewrite the spin-$\frac{1}{2}$ lattice Hamiltonian in terms of fermion variables, but the resulting formula (see [1, 2]) will in general not preserve the functional form of $H_{1/2}$. The notation H_F is exclusively reserved for the fermion version (2.4) of the boson or spin-$\frac{1}{2}$ Hamiltonians under study.

The Fock representation of the CAR algebra generated by (2.9) is defined in the whole of $\mathscr{H}_B$ but has a non-trivial irreducible component on $\mathscr{H}_F=P\mathscr{H}_B$ only. Vectors which are not elements of $\mathscr{H}_F$ are annihilated both by a_s^* and a_s.

By virtue of (2.7)-(2.9) we can make the Hamiltonians H_B, H_F, $H_{1/2}$ operate in $\mathscr{H}_B$, which includes the Fermi states in $\mathscr{H}_F=P\mathscr{H}_B$. It is easy to check that the basis vectors in $\mathscr{H}_F$ are

$$\begin{aligned} a_{j_1}^* \ldots a_{j_s}^*|0\rangle &= \varepsilon_{j_1\ldots j_s}\sigma_{j_1}^+ \ldots \sigma_{j_s}^+|0\rangle \\ &= \varepsilon_{j_1\ldots j_s} b_{j_1}^* \ldots b_{j_s}^*|0\rangle \end{aligned} \qquad (2.10)$$

where $\varepsilon_{j_1\ldots j_1}=\pm 1$ or 0 is the completely antisymmetric Levi-Civita tensor.

Hermitian operators (2.4) and (2.6) become diagonal once the matrix W is diagonalised. It can be achieved by means of the $N\times N$ unitary matrix U which gives rise to the appropriate linear transformation of canonical generators

$$\begin{array}{ll} b_s = \sum_{j=1}^{N} U_{sj}B_j & a_s = \sum_{j=1}^{N} U_{sj}A_j \\ [B_j, B_k^*]_- = \delta_{jk} & [A_j, A_k^*]_+ = \delta_{jk}P \\ [B_i, B_j]_- = 0 & [A_i, A_j]_+ = 0 \\ B_j|0\rangle = 0 \quad \forall j & A_j|0\rangle = 0 \quad \forall j. \end{array} \qquad (2.11)$$

Notice that since fermions operate in $\mathscr{H}_F$ the corresponding operator unit is P: $P\mathscr{H}_B=\mathscr{H}_F$.

Transformation U is supposed to diagonalise H_B and H_F. Hence

$$\begin{aligned} H_B &= \sum_{ik} b_i^* W_{ik} b_k = \sum_{ik}\sum_{st} \bar{U}_{is} B_s^* W_{ik} U_{kt} B_t \\ &= \sum_k \varepsilon_k B_k^* B_k \end{aligned} \qquad (2.12)$$

and

$$H_F = \sum_{ik} a_i^* W_{ik} a_k = \sum_k \varepsilon_k A_k^* A_k. \qquad (2.13)$$

It is obvious that the eigenvectors of H_F do belong to the previously introduced $\mathscr{H}_F$:

$$A_1^* \ldots A_s^*|0\rangle = \sum_{j_1 \ldots j_s} \bar{U}_{1j_1} \ldots \bar{U}_{sj_s} a_{j_1}^* \ldots a_{j_s}^*|0\rangle$$

$$= \sum_{j_1 \ldots j_s} \bar{U}_{1j_1} \ldots \bar{U}_{sj_s} \varepsilon_{j_1 \ldots j_s} b_{j_1}^* \ldots b_{j_s}^*|0\rangle. \tag{2.14}$$

However, in general it is not so in the case of boson eigenvectors:

$$B_1^* \ldots B_s^*|0\rangle = \sum_{j_1 \ldots j_s} \bar{U}_{1j_1} \ldots \bar{U}_{sj_s} b_{j_1}^* \ldots b_{j_s}^*|0\rangle \tag{2.15}$$

because the coinciding pairs of indices are not eliminated from the summations.

A comparison of (2.14) and (2.15) with (2.11) shows that, in contrast to $\{b^*, b\}$ and $\{a^*, a\}$, the generators $\{B^*, B\}$ are not related to $\{A^*, A\}$ by means of the procedure (2.7)-(2.9).

Nevertheless, since the representations of the algebras we deal with are always determined up to a unitary transformation, we have a guarantee that for the just introduced Fermi generators $\{A^*, A\}$ it is possible to select a unitary transformation Γ in $\mathscr{H}_B$

$$\Gamma B_k^* \Gamma^{-1} = \tilde{B}_k^* \qquad \Gamma|0\rangle = |0\rangle \tag{2.16}$$

such that the procedure (2.7)-(2.9) can be adopted to pass from $\{\tilde{B}^*, \tilde{B}\}$ to $\{A^*, A\}$. For this purpose replace $\{b^*, b\}$ by $\{\tilde{B}^*, \tilde{B}\}$, P then should be replaced by $P' = P(b^* \to \tilde{B}^*, b \to \tilde{B})$ and finally $\{a^*, a\}$ by $\{A^*, A\}$. Since the arena for our investigation remains unchanged, it is $\mathscr{H}_B$, we realise that

$$\tilde{B}_1^* \ldots \tilde{B}_s^*|0\rangle = A_1^* \ldots A_s^*|0\rangle \tag{2.17}$$

where

$$\mathscr{H}_F = P\mathscr{H}_B = P'\mathscr{H}_B \Rightarrow P' = P. \tag{2.18}$$

Let us emphasise that $P = P(b^*, b)$ while $P' = P(\tilde{B}^*, \tilde{B})$. Since the transformation Γ is unitary in $\mathscr{H}_B$ we can write

$$\Gamma b_k^* \Gamma^{-1} = \tilde{b}_k^*. \tag{2.19}$$

Because of (2.15) and (2.17)

$$\tilde{B}_1^* \ldots \tilde{B}_s^*|0\rangle = \sum_{j_1 \ldots j_s} \bar{U}_{1j_1} \ldots \bar{U}_{sj_s} \tilde{b}_{j_1}^* \ldots \tilde{b}_{j_s}^*|0\rangle$$

$$= \Gamma^{-1} \sum_{j_1 \ldots j_s} \bar{U}_{1j_1} \ldots \bar{U}_{sj_s} b_{j_1}^* \ldots b_{j_s}^*|0\rangle$$

$$= \sum_{j_1 \ldots j_s} \bar{U}_{1j_1} \ldots \bar{U}_{sj_s} \varepsilon_{j_1 \ldots j_s} b_{j_1}^* \ldots b_{j_s}^*|0\rangle \tag{2.20}$$

which determines the action of Γ in $\mathscr{H}_B$.

After accounting for (2.19) and (2.17) we realise that

$$\tilde{H}_B = \Gamma H_B \Gamma^{-1} = \sum_{s,t} \tilde{b}_s^* W_{st} \tilde{b}_t = \sum_k \varepsilon_k \tilde{B}_k^* \tilde{B}_k \tag{2.21}$$

conserves the projection $P' = P$ in $\mathscr{H}_B$:

$$[\tilde{H}_B, P]_- = 0 \tag{2.22}$$

and thus

$$\tilde{H}_{\mathrm{B}} = P\tilde{H}_{\mathrm{B}}P + (1-P)\tilde{H}_{\mathrm{B}}(1-P)$$
$$P\tilde{H}_{\mathrm{B}}P = H_{\mathrm{F}}. \tag{2.23}$$

The boson Hamiltonian $\tilde{H}_{\mathrm{B}}$ has Fermi states, which are eigenvectors common with H_{F}.

Let us recall that due to (2.8) we have

$$PH_{\mathrm{B}}P = H_{1/2} \tag{2.24}$$

but P does not necessarily commute with H_{B}, while it does with $\tilde{H}_{\mathrm{B}}$. Consequently the two Hamiltonians $H_{1/2}$ and H_{F}, despite arising in $\mathscr{H}_{\mathrm{F}}$ from the 'same' boson Hamiltonian (H_{B} and $\tilde{H}_{\mathrm{B}}$, respectively) do not seem to have very much in common.

Let us also observe that by departing from $P = P(b^*, b)$ we can introduce in $\mathscr{H}_{\mathrm{B}}$ another projection

$$\tilde{P} = \Gamma P \Gamma^{-1} = P(\tilde{b}^*, \tilde{b}) \tag{2.25}$$

selecting another (different from $\mathscr{H}_{\mathrm{F}}$) proper subspace $\tilde{\mathscr{H}}_{\mathrm{F}} = \tilde{P}\mathscr{H}_{\mathrm{B}}$ of $\mathscr{H}_{\mathrm{B}}$.

Obviously then

$$\tilde{P}\tilde{H}_{\mathrm{B}}\tilde{P} = \tilde{H}_{1/2} = \sum_{s,t} W_{st}\tilde{\sigma}_s^+\tilde{\sigma}_t^- \tag{2.26}$$

to be compared with the previous formula

$$P\tilde{H}_{\mathrm{B}}P = H_{\mathrm{F}}. \tag{2.27}$$

The domain for H_{F} is $P\mathscr{H}_{\mathrm{B}} = \mathscr{H}_{\mathrm{F}}$ while the domain for $\tilde{H}_{1/2}$ is $\tilde{P}\mathscr{H}_{\mathrm{B}} = \tilde{\mathscr{H}}_{\mathrm{F}}$ where $\tilde{\mathscr{H}}_{\mathrm{F}} = \Gamma\mathscr{H}_{\mathrm{F}}$. Since $\tilde{H}_{1/2}$ admits its corresponding fermion realisation, we thus have two distinct fermion systems which can be related with $\tilde{H}_{\mathrm{B}}$: one is H_{F} which preserves the functional form of the Hamiltonian. The other is the one replacing $\tilde{H}_{1/2}$ after passing to fermion variables.

Remark. The difference between H_{B}, H_{F}, $H_{1/2}$ once we attempt at the diagonalisation of H_{B}, H_{F} is a bit annoying in the context of our previous investigations [6] where the one-dimensional cyclic problem was studied:

$$H_{\mathrm{B}} = -J \sum_{s=1}^{N} (b_s^* b_{s+1} + b_{s+1}^* b_s). \tag{2.28}$$

The procedure (2.7)-(2.9) entails

$$PH_{\mathrm{B}}P = -J \sum_{s=1}^{N} (\sigma_s^+\sigma_{s+1}^- + \sigma_{s+1}^+\sigma_s^-) = H_{1/2}$$
$$= -J \sum_{s=1}^{N} (a_s^* a_{s+1} + a_{s+1}^* a_s) \tag{2.29}$$

i.e. precisely

$$H_{\mathrm{F}} = PH_{\mathrm{B}}P = H_{1/2} \tag{2.30}$$

to be contrasted with the previous analysis.

Nevertheless the same problem as before arises in connection with the eigenvectors of H_B which do not necessarily belong to $\mathcal{H}_F = P\mathcal{H}_B$. It is due to the diagonalisation [6]:

$$H_B = \sum_k \left(-2J \cos \frac{2\pi k}{N}\right) B_k^* B_k$$
$$H_F = \sum_k \left(-2J \cos \frac{2\pi k}{N}\right) A_k^* A_k. \tag{2.31}$$

After passing to unitarily transformed bosons $\{\tilde{B}_k^*, \tilde{B}_k\}$ we can recover the projection $P' = P(b^* \to \tilde{B}^*, b \to \tilde{B})$ which gives rise to

$$P'\mathcal{H}_B = \mathcal{H}_F = P\mathcal{H}_B \tag{2.32}$$

and hence consistently to

$$P\tilde{H}_B P = H_F = H_{1/2} = PH_B P. \tag{2.33}$$

It means that while computing the matrix elements of PH_BP in $\mathcal{H}_F$ we can diagonalise the respective matrix by passing to $P\tilde{H}_BP$. This is just the case in the boson approximation problem considered in reference [6].

3. Boson approximation of Fermi systems on a lattice: the boson–fermion Fock space unification argument

The analysis of § 2 indicates that, though a spin-$\frac{1}{2}$ lattice system can be related to H_B by adopting the spin-$\frac{1}{2}$ approximation concept $PH_BP = H_{1/2}$, it may not be of much use with respect to the fermion problem arising in $\mathcal{H}_F$ via $P\tilde{H}_BP = H_F$, $\Gamma H_B \Gamma^{-1} = \tilde{H}_B$. Consequently there is no apparent way to extend the boson approximation arguments of reference [6] to describe basic features of the fermion model H_F in terms of the boson one H_B.

However, the situation is not that hopeless. Let us admit that our lattice systems are in fact the lattice approximations of some continuum field theory models (such problems as the fermion species doubling we leave aside since, if arising, they would be shared by bosons as well). In such a case we must account for an explicit dependence of all quantities on the lattice spacing Δ. To simplify our task we shall work in a finite volume so that passing with Δ to 0 means to increase the site number from N to infinity. It guarantees that we shall remain in the Fock space after taking the limit $\Delta \to 0$.

Our problem is now how the previously discussed lattice systems would behave under such a limiting operation. Unfortunately we are not as yet able to keep things under control from the strictly mathematical point of view (with respect to the smoothness of the limiting procedure). Consequently we shall use rather intuitive and partly formal arguments, though a piece of strict mathematics will be used below. For further simplification, let us assume that the initial discrete labels of (2.1) refer to the space lattice only. Then let us admit

$$b_s^* = \frac{1}{\sqrt{\Delta^3}} \int d^3x\, \chi_s(x) \phi^*(x)$$
$$[\phi(x), \phi^*(y)]_- = \delta^3(\bar{x} - \bar{y}) \tag{3.1}$$
$$[\phi(x), \phi(y)]_- = 0$$

where $\chi_s(x)$ is a characteristic function of the set Δ_s in R^3, while Δ_s is a cube of the volume Δ^3 centred about the point x_s.

To introduce the Hilbert space $\mathscr{H}_F = P\mathscr{H}_B$ we need a projection P which is product of single site projections P_s. No assumptions about any specific site ordering are necessary as long as we do not explicitly pass to fermions. Hence

$$P = \prod_{s=1}^{N} P_s =: \exp\left(-\sum_s b_s^* b_s\right): + \sum_s b_s^* :\exp\left(-\sum_t b_t^* b_t\right): b_s$$
$$+ \frac{1}{2!}\sum_{s\neq t} b_s^* b_t^* :\exp\left(-\sum_r b_r^* b_r\right): b_t b_s + \ldots \tag{3.2}$$

If now we pass formally with Δ to 0 we observe that

$$P = P_\Delta \to :\exp\left(-\int d^3x\, \phi^*(x)\phi(x)\right): + \int d^3x\, \phi^*(x)$$
$$\times :\exp\left(-\int d^3y\, \phi^*(y)\phi(y)\right): \phi(x) + \ldots$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \ldots \int dx_n\, \phi^*(x_1)\ldots\phi^*(x_n) :\exp\left\{-\int d_2^3 \phi^*(z)\phi(z)\right\}:$$
$$\times \phi(x_1)\ldots\phi(x_n)[\sigma(x_1,\ldots,x_n)]^2 \tag{3.3}$$

where 1_F is a continuum field theory operator known from our earlier studies of relationships between fermions and bosons [4, 8]. In the above $[\sigma(x_1, \ldots, x_n)]^2$ takes the value 1 except when any two x indices in the sequence coincide, when the squared Friedrichs-Klauder symbol vanishes. Consequently in the boson Fock space $\mathscr{H}_B$ spanned by vectors

$$|f\rangle = \sum_n \frac{1}{\sqrt{n!}} \int d^3x_1 \ldots \int d^3x_n f(x_1, \ldots, x_n)\phi^*(x_1)\ldots\phi^*(x_n)|0\rangle \tag{3.4}$$

we deal with a continuum relative of P which has all the features of the projection as well: $1_F^2 = 1_F$, $1_F^* = 1_F$. However, as long as the standard Lebesgue-Riemann measure is used to give a meaning to the expression (3.4) we encounter the rather striking phenomenon of boson and fermion Fock space unification on the contiuum level [9, 10].

Indeed

$$|f\rangle = \sum_n \frac{1}{\sqrt{n!}} \int d^3x_1 \ldots \int d^3x_n\, [\sigma^2 + (1-\sigma^2)] f(x_1, \ldots, x_n)\phi^*(x_1)\ldots\phi^*(x_n)|0\rangle$$
$$= \sum_n \frac{1}{\sqrt{n!}} \int d^3x_1 \ldots \int d^3x_n f(x_1, \ldots, x_n)[\sigma(x_1, \ldots, x_n)]^2 \phi^*(x_1)\ldots\phi^*(x_n)|0\rangle$$
$$= \sum_n \frac{1}{\sqrt{n!}} \int d^3x_1 \ldots \int d^3x_n \{f(x_1, \ldots, x_n)\sigma(x_1, \ldots, x_n)\}\psi^*(x_1)\ldots\psi^*(x_n)|0\rangle \tag{3.5}$$

provided we observe that

(i) $(1-\sigma^2)f$ is non-zero on the set of Lebesgue measure zero in R^n; hence its contribution can be omitted in (3.5), see [9, 10];

(ii) the Fock representation of the CCR algebra induces the Fock representation of the CAR algebra in $\mathscr{H}_B$ such that the corresponding fermion and boson generators are related [8, 10, 4] precisely as in (3.5): $\psi^*(x) = \psi^*(\phi^*, \phi, x)$.

Since by departing from (3.4) we would have obtained

$$1_F|f\rangle = \sum_n \frac{1}{\sqrt{n!}} \int d^3x_1 \ldots \int d^3x_n f(x_1, \ldots, x_n)[\sigma(x_1, \ldots, x_n)]^2 \phi^*(x_1) \ldots \phi^*(x_n)|0\rangle \tag{3.6}$$

accounting for (3.5) leads to the conclusion that 1_F effectively coincides with the operator unit 1_B of the boson algebra: $1_F|f\rangle = |f\rangle$. This observation is rather striking because it implies that while approaching the contunuum limit we are in fact extending the non-trivial proper subspace $P\mathscr{H}_B = \mathscr{H}_F$ of $\mathscr{H}_B$ (on a lattice) to the whole of $\mathscr{H}_B$ (in continuum).

Let us now recall that on a lattice we have dealt with the identities

$$P\tilde{H}_B P = H_F \qquad PH_B P = H_{1/2} \tag{3.7}$$

where $\tilde{H}_B = \Gamma H_B \Gamma^{-1}$ and Γ is a unitary transformation in $\mathscr{H}_B$. If we adopt formally our previous argument while passing to the continuum we obtain $P \to 1_F = 1_B$, then (3.7) should be replaced by the following identities in $\mathscr{H}_B$:

$$\tilde{H}_B = H_F \qquad H_B = H_{1/2} \tag{3.8}$$

where continuum versions of previous lattice operators arise in (3.8), and $\tilde{H}_B = \Gamma H_B \Gamma^{-1}$ where Γ is a corresponding unitary transformation on the continuous level.

By virtue of (3.8) the continuum limits of H_F and $H_{1/2}$ are related by the unitary in $\mathscr{H}_B = \mathscr{H}_F$ transformation Γ, i.e. if we would have redefined $H_{1/2}$ in terms of fermion variables $\{\psi^*, \psi\}$ we can write

$$H_B(\phi^*, \phi) = H_F(\tilde{\psi}^*, \tilde{\psi}) = H_{1/2}(\psi^*, \psi). \tag{3.9}$$

Hence, as long as we intend to develop the working approximation schemes for fermions in terms of bosons, we arrive at the conclusion that; once in a finite volume but with a high site density, differences between the lattice problems H_B, H_F, $H_{1/2}$ can be disregarded. They describe almost the same (while on a lattice) phenomena.

At this point let us raise the challenging problem of investigating the properties of the lattice partition $\mathrm{Tr}\exp(-\beta H_B)$, $\mathrm{Tr}\exp(-\beta H_{1/2})$ and correlation functions while making explicit their dependence on the lattice spacing. In the absence of analytic tools the Monte Carlo specialists are the proper addresses of this challenge. In particular, in contrast to the Fermi case, the boson computations allow for studying the large site numbers. The challenge can be pursued further since the convergence of correlation functions for corresponding boson and spin-$\frac{1}{2}$ lattice systems once we vary Δ is worth investigation. (Compare, e.g., also [6] where another procedure for approximating $H_{1/2}$ in terms of H_B was proposed.)

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Appendix. Uses of the two-level approximation: an illustrative example in (1+1) dimensions

In the course of our investigations extensive use was made of the spin-$\frac{1}{2}$ approximation concept, which amounts to replacing each boson degree of freedom by the respective two-level (spin-$\frac{1}{2}$) degree. Technically it was realised by means of single-site two-level projections p_s, $P = \Pi_s p_s$ so that $PH_BP = H_{1/2}$ could have been related to H_B. Since the Jordan–Wigner transformation replaces $H_{1/2}$ by the pure fermion problem it amounts to associating the Fermi problem to H_B. Generally it is not at all apparent how to extract the $H_{1/2}$ contribution from the partition function $\mathrm{Tr}\exp(-\beta H_B)$, unless some additional prescriptions (like that of § 3) are adopted. The situation would greatly simplify in the case of

$$[P, H_B]_- = 0 \Rightarrow \mathrm{Tr}\exp(-\beta H_B) = \mathrm{Tr}[P\exp(-\beta H_{1/2})] + \mathrm{Tr}\{(1-P)\exp[-\beta(1-P)H_B(1-P)]\}. \tag{A1}$$

It may however also happen that the non-$H_{1/2}$ contribution is not relevant at all and that the following identity holds true:

$$\mathrm{Tr}\exp(-\beta H_B) = \mathrm{Tr}\, P\exp(-\beta PH_BP) = \mathrm{Tr}\exp(-\beta H_{1/2}) \tag{A2}$$

for lattice systems themselves and hence without any reference to the previous boson–fermion Fock space unification argument. To demonstrate that (A1) is possible, we shall follow reference [11], mainly its § 4. The one-dimensional spinless fermion lattice model

$$\begin{aligned} &H = H_1 + H_2 \\ &H_1 = -t\sum_s (c_s^* c_{s+1} + c_{s+1}^* c_s) \\ &H_2 = v\sum_s c_s^* c_s c_{s+1}^* c_{s+1} = v\sum_s n_s n_{s+1} \\ &[c_s, c_t^*]_+ = \delta_{st} \qquad 1 \leqslant s \leqslant M \end{aligned} \tag{A3}$$

upon denoting

$$\sigma_s^+ = \tfrac{1}{2}(\sigma_s^x + \mathrm{i}\sigma_s^y) \qquad \sigma_s^- = \tfrac{1}{2}(\sigma_s^x - \mathrm{i}\sigma_s^y) \tag{A4}$$

and using the Jordan–Wigner transformation:

$$\begin{aligned} c_s^* &= \sigma_s^+ \exp\left(\frac{\mathrm{i}\pi}{2}\sum_{p=1}^{s-1}(1+\sigma_p^z)\right) \\ c_s &= \sigma_s^- \exp\left(-\frac{\mathrm{i}\pi}{2}\sum_{p=1}^{s-1}(1+\sigma_p^z)\right) \end{aligned} \tag{A5}$$

admits a reformulation in terms of (free boundary conditions are adopted)

$$H_{1/2} = -\frac{t}{2}\sum_s\left(\sigma_s^x\sigma_{s+1}^x + \sigma_s^y\sigma_{s+1}^y + \frac{v}{2t}\sigma_s^z\sigma_{s+1}^z - \frac{v}{t}\sigma_s^z - \frac{v}{2t}\right). \tag{A6}$$

The evaluation of the partition function proceeds by studying its Trotter approximations

$$Z_m = \sum_{\{i_{1,j}<\ldots<i_{N,j}\}} \sum_{\{P_j\}} \prod_{j=1}^{m} \prod_{\mu,\nu=1}^{N} \operatorname{sgn}(P_j) I\left(\frac{2\beta t}{m}, i_{\mu,j} - i_{P_j\mu,j+1}\right) \exp\left(-\frac{\beta v}{m}\delta_{|i_{\mu,j}-i_{\nu,j}| \bmod M,1}\right)$$

$$N = \sum_{i=1}^{M} n_i \tag{A7}$$

$$I(z,l) = \frac{1}{M}\sum_{n=1}^{M} \cos\frac{2\pi l n}{M} \exp\left(z\cos\frac{2\pi n}{M}\right).$$

Let us get rid of the minus sign problem and replace Z_m by the subsidiary formula

$$Z_m^{1/2} = \sum_{\{i_{1,j}<\ldots<i_{N,j}\}} \sum_{\{P_j\}} \prod_{j=1}^{m} \prod_{\mu,\nu=1}^{N} I\left(\frac{2\beta}{m}, i_{\mu,j} - i_{P_j\mu,j+1}\right) \times \exp\left(-\frac{\beta v}{m}\delta_{|i_{\mu,j}-i_{\nu,j}| \bmod M,1}\right). \tag{A8}$$

Let us here observe that, except for the restriction on the sum over possible partitions of the particles, (A8) is identical to the mth approximation of the partition function for the pure Bose system. Namely we have

$$Z_m^B = \operatorname{Tr}[\exp(-\beta\hat{H}_1/m)\exp(-\beta\hat{H}_2/m)]^m \tag{A9}$$

and

$$Z_m^{B,1/2} = \operatorname{Tr}[P\exp(-\beta P\hat{H}_1P/m)P\exp(-\beta P\hat{H}_2P/m)P]^m \tag{A10}$$

where

$$\hat{H}_1 = -t\sum_s (b_s^* b_{s+1} + b_{s+1}^* b_s)$$

and

$$\hat{H}_2 = v\sum_s b_s^* b_s b_{s+1}^* b_{s+1}$$

stand for the boson kinetic energy and potential, respectively. The projection P is precisely the product of two-level projections of ours and introducing it is the same as saying that there is a hard core interaction that prevents two particles from occupying the same site. For the complete projected boson Hamiltonian the corresponding Trotter approximation is

$$Z_m^{B,1/2} = \sum_{\{i_{\mu,j}\}} \sum_{\{P_j\}} \prod_{j=1}^{m} \prod_{\mu,\nu=1}^{N} I\left(\frac{2\beta t}{m}, i_{\mu,j} - i_{P_j\mu,j+1}\right) \exp\left(-\frac{\beta v}{m}\delta_{|i_{\mu,j}-i_{\nu,j}| \bmod M,1}\right). \tag{A11}$$

In general we have

$$Z_m \neq Z_m^{1/2} \neq Z_m^B \neq Z_m^{B,1/2} \tag{A12}$$

but due to the fact that in the $m\to\infty$ limit all $i_{\mu,j} = i_{\nu,j}$ contributions die out, we arrive at

$$\lim_{m\to\infty} Z_m = \lim_{m\to\infty} Z_m^{1/2} = \lim_{m\to\infty} Z_m^{B,1/2} = \lim_{m\to\infty} Z_m^B. \tag{A13}$$

Thus in the limit $m\to\infty$ the subsidiary model and the projected one give rise to the same partition function, and as long as the computation of the partition function is of interest for us, we can even make the identification of the models, albeit the operator identities

$$H_{\mathrm{B}} = H_{1/2} \qquad \exp(\mathrm{i}H_{\mathrm{B}}t) = \exp(\mathrm{i}H_{1/2}t)$$
$$H_{\mathrm{B}} = \hat{H} = \hat{H}_1 + \hat{H}_2 \tag{A14}$$

are invalid in general. One must also be aware that in higher dimensions the minus sign trick does not apply and, e.g., the two-dimensional spinless fermion model does not admit the analogous procedure [11].

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