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Remark on Kàlnay Theory of Fermions Constructed from Bosons

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Abstract

Theories of Bose description for fermions developed by A. J. Kàlnay (1974) and the present author (Garbaczewski, 1975) are compared. It is proved that the underlying constructions can be in principle summarized as follows:

$$\text{CAR and CCR} \Rightarrow \text{new CAR} \tag{1}$$

$$\text{CCR} \Rightarrow \text{CAR} \tag{2}$$

where CCR and CAR are abbreviations for representations of the canonical commutation (and anticommutation, respectively) relations algebra. According to this result (1), though independent of (2), can appear as a secondary step only in the quantum theory of fermions constructed from bosons.

Section 1

In the period 1970-1975 there appeared a few papers concerned with the formulation of quantum field theory for fermions constructed from bosons (Streater et al., 1970; Kàlnay et al., 1973; Kàlnay, 1974a, b; Garbaczewski, 1975a, b, c). These theories are mainly based on investigations of boson expansions for Fermi operators, and such an idea is extensively used also in the nuclear many-body problems (Okubo, 1974). The approaches mentioned were developed independently and no connection between them has up to now been proved. In the present paper, we take into account the theory proposed by Kalnay (1974a, b) and attempt to establish its relation to the theory proposed by Garbaczewski (1975a, b, c), considering, however, the free field case only (Fock representations of the canonical algebras).

The simplest insight into the Kàlnay method, according to Kàlnay (1974a, b) goes as follows: One constructs quantum fermions $f_\zeta(z)$ (where ζ is bispinor index and $z \in \mathbb{R}^3$) from pure quantum bosons $b_\xi(x)$ (where ξ is an appropriate boson index, $x \in \mathbb{R}^3$) with the additional use of *c-number* coefficients $F_{\zeta\xi\xi'}(z, x, x')$ which provide a "matrix" representation of the canonial anti-commutation relations. These last coefficients are even interpreted as classical fields. Quoting the original (Kàlnay, 1974a, b) sentence:

> The quantum Fermi field $f_\zeta(z)$ is constructed from a quantum Bose field $b_\xi(x)$ and a trilinear *classical* (in the sense of *c-number*) field $F_{\zeta\xi\xi'}(z, x, x') \cdots$ the only quantum fundamental entities are the Bose field $b_\xi(x)$ and its state vector space B.

We attempt to prove that, in fact, that is not always so and "the trilinear *classical* field $F_{\zeta\xi\xi'}(z, x, x')$" can appear as an *operator* field revealing thus representation of the CAR algebra as one more fundamental quantum entity in the theory. We present here an alternative method for construction of such field without the use of binary arithmetic originally employed in Kàlnay et al. (1973).

Compared with the results of Garbaczewski (1975a, b, c) this version of Kàlnay theory appears as a secondary step only in the theory of fermions constructed from bosons.

Section 2

To prove correctly the above conclusion, let us begin from a simple construction proposed recently by Kàlnay (1974b). One begins here from a momentum space description (which is the proper language for any algebraic formulation) and then translates results into the field-theoretic framework.

Given an enumerably infinite set $\{F_i\}_{i=1,2,\ldots}$ of infinite-dimensional matrices providing a matrix representation of the CAR (see in this connection, for example Guichardet, 1966; Powers, 1967; Rzewuski, 1969; Emch, 1972), we have

$$\sum_s (F_i)_{rs}(F_j^*)_{st} + \sum_s (F_j^*)_{rs}(F_i)_{st} = (F_iF_j^*)_{rt} + (F_j^*F_i)_{rt} = \delta_{ij}\delta_{rt} \tag{2.1}$$

$$[F_i, F_j^*]_+ = \delta_{ij}\,\mathbb{1}, \qquad [F_i, F_j]_+ = 0$$

This matrix algebra acts in a certain infinite-dimensional linear vector space V.

Given

$$K = \bigoplus_1^N k, k = \mathscr{L}^2(\mathbb{R}^3), K \supset \{g_r^m\}_{r=1,2,\ldots}^{m=1,\ldots,N}$$

where $\{g_r^m\}$ is an orthonormal complete set in K:

$$\begin{aligned} &\sum_m (\bar{g}_r^m, g_s^m) = \delta_{rs} \text{ orthonormality} \\ &\sum_s \bar{g}_s^m \otimes g_s^n = \delta_{mn} \text{ completeness} \end{aligned} \tag{2.2}$$

where (,) denotes a bilinear form in K, and the overbar denotes an involution in K. Let us now introduce;

$$F_i^{mn}(k,p) = \sum_{rs} (F_i)_{rs} \bar{g}_r^{\,m}(k) g_s^{\,n}(p) = (\bar{g}^m, F_i g^n)(k,p) \tag{2.3}$$

and similarly

$$F_i^{*mn}(k,p) = \sum_{rs} (F_i^*)_{rs} \bar{g}_r^{\,m}(k) g_s^{\,n}(p) = F_i^{nm}(p,k)^* \tag{2.4}$$

One finds easily that the pair $\{F_i^{nm}, F_i^{*n'm'}\}$ provides again certain "matrix" (continuously indexed) representations of the CAR algebra:

$$\sum_n \int dq\, \{F_i^{mn}(k,q) F_j^{*np}(q,k') + F_j^{*mn}(k,q) F_i^{np}(q,k')\} = \delta_{ij}\delta(k-k')\delta_{mp} \tag{2.5}$$

$$\sum_n \int dq\, \{F_i^{mn}(k,q) F_j^{np}(q,k') + F_j^{mn}(k,q) F_i^{np}(q,k')\} = 0$$

which follows from (2.1) and (2.2).

This representation acts in a new representation space V':

$$V \ni v = \{v_r\}_{r=1,2,\ldots} \Rightarrow V' \ni v'$$
$$v'^m(k) = \sum_r \bar{g}_r^m(k) v_r = (\bar{g}^m, v)(k) \tag{2.6}$$

Here V' is isomorphic to

$$\bigoplus_1^N \mathscr{L}^2(\mathbb{R}^3)$$

Given now a Fock representation of the CCR algebra over

$$K = \bigoplus_1^N k$$

generated by the triple $\{b, b^*, |0\rangle\}_K$, $k = \mathscr{L}^2(\mathbb{R}^3)$

$$[b_m(p), b_n^*(q)]_+ = \delta(p-q)\delta_{nm}\,\mathbb{1}, \qquad [b_m(p), b_n(q)]_- = 0 \tag{2.7}$$

The domain we denote $\mathscr{D} \subset B$.

Then (for example), operators

$$\begin{aligned} f_i &= \iint F_i^{mn}(p,q) b_m^*(p) b_n(q)\, dp\, dq \\ f_i^* &= \iint F_i^{*mn}(p,q) b_m^*(p) b_n(q)\, dp\, dq \end{aligned} \tag{2.8}$$

fulfill

$$\begin{aligned} [f_i, f_j^*]_+ &= \delta_{ij} \\ [f_i, f_j]_+ &= 0 \end{aligned} \tag{2.9}$$

providing in B^1, the one-particle boson subspace of B, a Fock representation of the CAR algebra [on how to introduce a Fock vacuum for the new representatioı see Kàlnay et al. (1973) and Kàlnay (1974b)].

Section 3

Now we will give a constructive proof for the following:

> *Statement.* Kàlnay theory of fermions constructed from bosons (i) can be improved in the foundations to admit also operator realizations of trilinear fields, and henceforth (ii) can be added as a secondary one to the theory proposed by Garbaczewski (1975).

Proof. Given the triple $\{f^*, f, |0\rangle\}_K$, generating a Fock representation of the CAR algebra over K, acting in a certain Hilbert space H, for $K = \mathscr{L}^2(\mathbb{R}^3) \ni g, g'$; $p \in \mathbb{R}^3$, we have

$$\begin{gathered} f(g) = (f, \bar{g}) = \int dp \cdot f(p)\bar{g}(p) \\ [f(g), f(g')^*]_+ = (\bar{g}, g')\mathbb{1} \\ [f(g), f(g')]_+ = 0 \end{gathered} \tag{3.1}$$

Let k, k' be separable complex Hilbert spaces with an involution (–) and a bilinear form $(\cdot\cdot\cdot)$ implementing a sesquilinear form $(\bar{\cdot}, .)$. Assume $k = \mathscr{L}^2(\mathbb{R}^1)$, $k' = \mathscr{L}^2(\mathbb{R}^2)$ and take into account $K = k \otimes k'$. Elements from K are of the form $a(p) = a(q, q')$ with $q \in \mathbb{R}^1$, $q' \in \mathbb{R}^2$ and $p = (q, q') \in \mathbb{R}^3$. Hence we can consider $\{f, f^*, |0\rangle\}_{\mathscr{L}^2(\mathbb{R}^3)}$ as the restriction of $\{f, f^*, |0\rangle\}_{k \otimes k'}$ to $\mathscr{L}^2(\mathbb{R}^3)$.

Let us now introduce

$$K = \left(\bigoplus_1^N k\right) \otimes \left(\bigoplus_1^M k'\right)$$

and choose an orthonormal set $\{a_{ik}\}_{k=1,\ldots,M}^{i=1,\ldots,N}$

$$\int dp\, \bar{a}_{ik}(p) a_{ji}(p) = (\bar{a}_{ik}, a_{ji}) = \delta_{ij}\delta_{ki} \tag{3.2}$$

with $p = (q, q') \in \mathbb{R}^3$
Now we can define for $\{f, f^*, |0\rangle\}_{\mathscr{L}^2(\mathbb{R}^3)}$:

$$\begin{aligned} f_{ik} &= (f, \bar{a}_{ik}) = f(a_{ik}) = \int dp f(p)\bar{a}_{ik}(p) \\ f_{jl}^* &= (f^*, a_{jl}) \end{aligned} \tag{3.3}$$

which for $N = M = 1$ becomes simply $f(a)$. As an immediate consequence of the CAR (3.1), we have

$$[f_{ik}, f_{jl}^*]_+ = (\bar{a}_{ik}, a_{jl})\mathbb{1} = \delta_{ij}\delta_{kl}\mathbb{1} \qquad (3.4)$$

$$[f_{ik}, f_{jl}]_+ = 0$$

Choosing in

$$\bigoplus_1^N k$$

a complete orthonormal set $\{g_r^i\}_{r=1,2,\ldots}^{i=1,\ldots,N}$ we are able to construct a "matrix" representation of the CAR algebra, where matrix elements are not *c-valued* but *operator-valued* entities: $k = \mathscr{L}^2(\mathbb{R}^3)$

$$(F_k)_{rs} = \sum_i \int dp\, g_r^i(p) f_{ik} \bar{g}_s^i(p) \qquad (3.5)$$

Owing to (3.4) we have

$$[F_k, F_l^*]_+ = \delta_{kl}\mathbb{1}$$
$$[F_k, F_l]_+ = 0 \qquad (3.6)$$

which provides the underlying "matrix" representation of the CAR algebra; (3.6) follows from (3.4) and orthogonality-completeness relations (2.2):

$$([F_k, F_l^*]_+)_{rt} = \sum_i \sum_j \sum_s \int dp \int dq\, \{g_r^i(p) f_{ik} \bar{g}_s^i(p)$$
$$\times g_s^j(q) f_{jl}^* \bar{g}_t^j(q) + g_r^j(p) f_{jl}^* \bar{g}_s^j(q) g_s^i(p) f_{ik} \qquad (3.7)$$
$$\times \bar{g}_t^i(p)\} = \sum_i \int dp\, g_r^i(p)\, [f_{ik}, f_{il}^*]_+ g_t^i(p) = \delta_{rt}\delta_{kl}\mathbb{1}$$

$\mathbb{1}$ is the operator unit in the representation of the CAR algebra generated by $\{f, f^*, |0\rangle\}_K$.

Turning back to Section 2, we see that after the introduction of "matrices" F_k, F_l^*, Statement (i) is proved. To assert (ii) it is enough to remark that by Garbaczewski (1975b) there is established that there exist Fock representations of the CAR algebra constructed in Fock representations of the CCR algebra, and hence Bose constructed representations of the CCR algebra can be used (mentioned secondary step) to introduce matrices F_k, F_l^* with *operator-valued* matrix elements, implementing further *operator-velued* trilinear fields. The Statement is proved.

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