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CANONICAL ACTION-ANGLE FORMALISM FOR QUANTIZED NONLINEAR FIELDS

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The canonical quantizations of field and action-angle coordinates which (locally) parametrize the phase manifold for the same nonlinear field theory model (e.g. sine-Gordon and nonlinear Schrödinger with the attractive coupling) are reconciled on the common for both cases state space. The classical-quantum relationship is maintained in the mean: coherent state expectation values of operators give rise to classical objects.

1. Motivation

In the modern field theory literature, it is rather common to consider classical and quantum problems separately, and the old tradition that a physically relevant quantum theory should (must?) contain a prescription for going over to the classical partner is often neglected as an unnecessary complication. The reverse problem of the (canonical) quantization of the once given classical theory becomes reduced to the formal replacements of phase space variables by operators, so that the quantum meaning of classical field theory[1,2,3] effectively evaporates from the formalism.

There are many ambiguities inherent in passing from the classical to quantum description[4] and reversly.[5] We shall pay a particular attention to the following points:

(i) When passing to the quantum picture a specification of the Hilbert space domain for quantum operators is necessary and it is usually not contained in the quantization prescription. It is our aim to maintain the classical-quantum relationship (in the mean) which imposes the domain limitation to guarantee a proper classical image of the quantum model.

(ii) The quantization procedures are sensitive to the initial choice of coordinates on the symplectic manifold, and usually are not invariant under general canonical transformations. The passage from field variables to the action-angle coordinates may strongly affect the arising quantum structure.

A particularly convenient playground for studying these issues is provided by completely integrable models in $1 + 1$ dimensions, and we shall confine our discussion to the two classic examples: the nonlinear Schrödinger field with the attractive coupling and the sine-Gordon field. Albeit specialized, the underlying analysis pertains to the general theoretical foundations of the quantization of nonlinear fields.

As is well-known the action-angle version of the classically integrable model in addition to soliton coordinates includes the so-called radiation contribution. The problem of quantizing the field, so that the action-angle problem gets quantized in the state space of the quantum field simultaneously, has not received adequate attention in the literature. Especially since we must reconcile the canonical quantization of the field coordinates (ψ, ψ^*), (ϕ, Π) and this of the action-angle coordinates $(q, p, \tau, m, \varphi, \varphi^*)$, $(q, p, Q, P, \rho, \theta, \varphi, \pi)$ for the nonlinear Schrödinger and sine-Gordon field respectively. A more detailed discussion of the problem is postponed to Secs. 3 and 5 of the present paper.

2. Canonical Formalism for Classical Fields

2.1. *Nonlinear Schrödinger field: Galilei invariance in terms of the scattering data*

Our classical model is defined as follows

$$H = \int_{R^1} \left(\frac{1}{2\mu} |\psi_x|^2 - \frac{g}{2} |\psi|^4 \right) dx,$$
$$\partial_t \psi = \psi_t = \{\psi(x), H\}, \tag{2.1}$$
$$\{A, B\} = i \int \left(\frac{\delta A}{\delta \psi^*(x)} \frac{\delta B}{\delta \psi(x)} - \frac{\delta A}{\delta \psi(x)} \frac{\delta B}{\delta \psi^*(x)} \right) dx.$$

We assume $g > 0$ i.e., the attractive case. The system is invariant under the action of the extended Galilei group with generators[6]

$$M = \mu \int |\psi|^2 \, dx,$$
$$P = \int \psi^*(-i\psi_x) \, dx, \quad K = -\mu \int x\psi^*\psi \, dx, \tag{2.2}$$
$$\{M, H\} = \{M, P\} = \{M, K\} = \{H, P\} = 0,$$
$$\{H, K\} = P, \quad \{P, K\} = M.$$

M and $2MH - P^2$ are the Casimir operators. The inverse scattering approach to this system results in the following change of the local coordinates on the symplectic manifold

$$(\psi, \psi^*) \to (q_l, p_l, \tau_l, m_l, q(k), p(k)), \quad 1 \le l \le N, \tag{2.3}$$

where

$$\{q_l, p_{l'}\} = \{\tau_l, m_{l'}\} = \varepsilon_{ll'}, \quad \{q(k), p(k')\} = \delta(k - k'),$$

$$\varphi(x) = (2\pi)^{-1/2} \int \exp(ikx)[-p(k)^{1/2}] \exp[-iq(k)]\, dk$$

$$\Rightarrow \{\varphi^*(x), \varphi(y)\} = i\delta(x-y),$$

$$\{\varphi(x), \varphi(y)\} = 0 = \{\varphi^*(x), \varphi^*(y)\},$$

$$i\varphi_t = -\frac{1}{2\mu}\varphi_{xx} \tag{2.4}$$

and

$$\{A, B\} = \sum_l \left[\left(\frac{\partial A}{\partial q_l}\frac{\partial B}{\partial p_l} - \frac{\partial A}{\partial p_l}\frac{\partial B}{\partial q_l}\right) + \left(\frac{\partial A}{\partial \tau_l}\frac{\partial B}{\partial m_l} - \frac{\partial A}{\partial m_l}\frac{\partial B}{\partial \tau_l}\right)\right]$$

$$+ i\int\left(\frac{\delta A}{\delta\varphi^*(x)}\frac{\delta B}{\delta\varphi(x)} - \frac{\delta A}{\delta\varphi(x)}\frac{\delta B}{\delta\varphi^*(x)}\right)dx,$$

$$M = \sum_l m_l + \mu\int|\varphi|^2\, dx, \quad P = \sum_l p_l + \int\varphi^*(-i\varphi_x)\, dx, \tag{2.5}$$

$$H = \sum_l\left(\frac{p_l^2}{2m_l} - \frac{g^2}{24\mu^2}m_l^3\right) + \frac{1}{2\mu}\int|\varphi_x|^2\, dx,$$

$$K = -\sum_l m_l q_l - \mu\int x|\varphi|^2\, dx.$$

Under the action of the extended Galilei group the respective coordinates transform as follows

$$\begin{aligned} q_l'(t') &= q_l(t) + vt + a, \\ p_l'(t') &= p_l(t) + m_l v, \\ \tau_l'(t') &= \tau_l(t) - vq_l(t) - \tfrac{1}{2}v^2 t - \theta, \\ m_l'(t') &= m_l(t), \end{aligned} \tag{2.6}$$

hence each (q, p, τ, m) set of data gives rise to the free classical particle with a well-defined Galilean position and momentum. Here m does not appear as the mass parameter but plays the role of the action variable with τ being the conjugate angle. The continuous segments in (2.6) correspond to the radiation, i.e., the free field contribution to the interacting problem. The time development of the soliton action-angle data reads

$$\dot{q}_l = \frac{p_l}{m_l}, \quad \dot{p}_l = 0, \quad \dot{\tau}_l = -\left(\frac{p_l^2}{2m_l^2} + \frac{g^2}{8\mu^2} m_l^2\right),$$
$$\dot{m}_l = 0, \tag{2.7}$$

while the radiation component obeys

$$i\varphi_t = -\frac{1}{2\mu}\varphi_{xx}. \tag{2.8}$$

It is useful to know that one can re-construct the soliton field $\psi(x)$ from the (q, p, τ, m) data[6]

$$\psi(x) = -\frac{1}{2}\left(\frac{g}{\mu}\right)^{1/2} m \exp\left[i\frac{\mu}{m}(px - m\tau - qp)\right]$$
$$\times \operatorname{sech}\left[\frac{1}{2} mg(x - q)\right], \tag{2.9}$$

where (2.7) implies

$$\psi(x, t) = -\frac{1}{2}\left(\frac{g}{\mu}\right)^{1/2} m \exp\left\{i\frac{\mu}{m}\left[px - \left(\frac{p^2}{2m} - \frac{g^2}{8\mu^2}m^3\right)t\right]\right\}$$
$$\times \operatorname{sech}\left\{\frac{1}{2} mg[x - q(t)]\right\}. \tag{2.10}$$

Here $q(t) = q(0) + \frac{p}{m}t$ and the notation $\dot{k} = \frac{\mu}{m}p$, $\omega = \frac{\mu}{m}\left[\frac{p^2}{2m} - \frac{g^2}{8\mu^2}m^3\right]$ is commonly in use. Observe that (2.10) equals (2.9) provided we replace q, τ by $q(t), \tau(t)$ respectively.

2.2. *The sine-Gordon field: Poincare invariance in terms of the scattering data*

The field equation

$$\partial_t^2\phi - \partial_x^2\phi + \frac{m^3}{g^{1/2}}\sin\left(\frac{g^{1/2}}{m}\phi\right) = 0, \quad m, g > 0, \tag{2.11}$$

exhibits[7] a manifest Poincare invariance, since the transformations

$$(t, a, \sigma) : \phi(t, x) \to \phi'(t', x') = \phi(t, x),$$

$$t' = \gamma(t + vx) + b,$$

$$x' = \gamma(x + vt) + a,$$

$$\gamma = \cosh\sigma, \quad v = \tanh\sigma,$$

$$\hbar = c = 1 \tag{2.12}$$

leave it intact. Here b is the parameter of time translations, σ is of pure Lorentz transformations.

In the phase space V of the system, the Poincare group Lie algebra consists of functionals

$$\begin{aligned} H &= \frac{1}{2}\int\left\{\Pi^2 + \phi_x^2 + \frac{2m^4}{g}\left[1 - \cos\left(\frac{g^{1/2}}{m}\phi\right)\right]\right\}dx, \\ P &= -\int \phi_x \Pi\, dx, \\ K &= -\frac{1}{2}\int x\left\{\Pi^2 + \phi_x^2 + \frac{2m^4}{g}\left[1 - \cos\left(\frac{g^{1/2}}{m}\phi\right)\right]\right\}dx, \end{aligned} \tag{2.13}$$

which with respect to the Poisson bracket

$$\{A, B\} = \int\left[\frac{\delta A}{\delta\phi(x)}\frac{\delta B}{\delta\Pi(x)} - \frac{\delta A}{\delta\Pi(x)}\frac{\delta B}{\delta\phi(x)}\right]dx \tag{2.14}$$

satisfy

$$\begin{aligned} &\{H, P\} = 0, \quad \{K, H\} = -P, \quad \{K, P\} = -H, \\ &\dot\phi(x,t) = \{\phi(x,t), H\}, \quad \dot\Pi(x,t) = \{\Pi(x,t), H\}. \end{aligned} \tag{2.15}$$

The main result of the inverse scattering approach is that a mapping from the initial data $\phi(x)$, $\Pi(x)$ to the scattering ones implies the following change of the local coordinate system

$$\begin{aligned} &(\phi(x), \Pi(x)) \to (q_i, p_i, Q_l, P_l, \rho_l, \theta_l, \varphi(x), \pi(x)), \\ &\{q_i, p_j\} = \delta_{ij} = \{Q_i, P_j\} = \{\rho_i, \theta_j\}, \\ &\{\varphi(x), \pi(y)\} = \delta(x - y), \end{aligned} \tag{2.16}$$

all the remaining Poisson brackets are vanishing. Hence

$$\{A,B\} = \sum_j \left(\frac{\partial A}{\partial q_j}\frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_j}\frac{\partial B}{\partial q_j}\right) + \sum_l \left[\left(\frac{\partial A}{\partial Q_l}\frac{\partial B}{\partial P_l} - \frac{\partial B}{\partial Q_l}\frac{\partial A}{\partial P_l}\right)\right.$$

$$\left. + \left(\frac{\partial A}{\partial \rho_l}\frac{\partial B}{\partial \theta_l} - \frac{\partial B}{\partial \rho_l}\frac{\partial A}{\partial \theta_l}\right)\right] + \int \left(\frac{\delta A}{\delta\varphi(x)}\frac{\delta B}{\delta\pi(x)} - \frac{\delta B}{\delta\varphi(x)}\frac{\delta A}{\delta\pi(x)}\right)dx, \quad (2.17)$$

while

$$H = \sum_j (P_j^2 + M^2)^{1/2} + \sum_l (P_l^2 + M^2(\theta_l))^{1/2} + \frac{1}{2}\int (\pi^2 + \varphi_x^2 + m^2\varphi^2)\,dx$$

$$= H_s + H_b + H_0\,, \quad (2.18)$$

$$P = \sum_j p_j + \sum_l P_l - \int \pi\varphi_x\,dx\,,$$

$$K = -\sum_j (p_j^2 + M^2)^{1/2} q_j - \sum_l (P_l^2 + M^2(\theta_l))^{1/2} Q_l - \frac{1}{2}\int x(\pi^2 + \varphi_x^2 + m^2\varphi^2)\,dx\,,$$

$$M(\theta_l) = 2M\sin\theta_l\,,$$

$$M = 8m^3/g\,.$$

Let us emphasize the kinematic independence of the three sets of action-angle variables in (2.18). The canonically conjugate pairs (p, q) describe a system of free (noninteracting) elementary relativistic particles of mass M which are just sine-Gordon solitons in the action-angle representation. The pair $(\varphi(x), \pi(x))$ gives rise to the infinite dimensional realization of the Poincare group, which coincides with this for the mass m Klein-Gordon field (the radiation). The coordinates (P, Q, ρ, θ) correspond to the so-called breathing solutions, which with respect to

$$Q(t) \to Q'(t') = \gamma[Q(t) + vt] + a\,,$$

$$t' = \gamma[t + vQ(t)] + b \quad (2.19)$$

$$\rho'(t') = \rho(t),$$

$$\theta'(t') = \theta(t),$$

behave like classical relativistic particles, but with too rich (four dimensional while dimensionality two is required) phase space.

Let us mention that in terms of the scattering data (2.16) the one soliton field reads

$$\phi_s(x,t) = \frac{4m}{g^{1/2}}\tan^{-1}\exp\left[\varepsilon\frac{m}{M}H(x - q(t))\right],$$

$$H = (p^2 + M^2)^{1/2}, \quad \varepsilon = \pm 1, \quad M = \frac{8m^3}{g}, \quad q(t) = q + \frac{p}{M}t, \tag{2.20}$$

while the breather has the form

$$\phi_6(x,t) = \frac{4m}{g^{1/2}} \tan^{-1}\left[(\tan\theta)\sin\left[\frac{m}{2M}\left(\frac{m^2}{g}\phi(t) - \frac{Px}{\tan\theta}\right)\right.\right.$$
$$\times \left\{\cosh\left[\frac{m}{2M}H_b(x - Q(t))\right]\right\}^{-1}, \tag{2.21}$$

where $\rho = \frac{m^2}{g}\phi - \frac{QP}{\tan\theta}$, $\{\phi,\theta\} = \frac{g}{m^2}$. If we insert (2.20) or (2.21) to (2.13) the result would be (2.18) but reduced to the one soliton or breather contribution. The radiation piece would not arise at all.

Note a particularly simple time evolution of the soliton action-angle variables in the above

$$\dot{q} = pH_s^{-1},$$
$$H_s = (p^2 + M^2)^{1/2}, \tag{2.22}$$
$$\dot{p} = 0.$$

breather coordinates obey

$$\dot{Q} = PH_b^{-1}, \quad \dot{P} = \dot{\theta} = 0,$$
$$\dot{\phi} = \frac{g}{m^2}H_b(\tan\theta)^{-1}, \quad H_b = (P^2 + M^2(\theta))^{1/2}, \tag{2.23}$$

which, in virtue of $\phi \in R$ (mod 32π), $0 < \theta < \frac{\pi}{2}$ implies the periodic motion in the center-of-mass frame ($P = 0$) with a period $T = \frac{2\pi}{m\cos\theta}$. The respective dynamics results from (2.23) upon equating P to zero in all the formulas, so that $\{\phi,\theta\} = \frac{g}{m^2}$ implies

$$\dot{\theta} = 0, \quad \dot{\phi} = 16m\cos\theta,$$
$$H_b = 2M\sin\theta, \quad M = \frac{8m^3}{g}. \tag{2.24}$$

Because of

$$I = \frac{m^2}{g}\theta,$$
$$\{\phi, I\} = 1, \tag{2.25}$$

it is rather natural to pass from θ to I in (2.24)

$$\dot{\theta} = 0 = \dot{I}, \quad \dot{\phi} = 16m\cos\left(\frac{g}{m^2}I\right),$$
$$H_b = 2M\sin\left(\frac{g}{m^2}I\right) = \frac{16m^3}{g}\sin\left(\frac{g}{m^2}I\right), \tag{2.26}$$

where I is the new action coordinate.

3. Some Aspects of the Canonical Quantization for Soliton Bearing Fields

Because of the well-defined symplectic structure, both field theory models of Sec. 2 admit a canonical quantization either with respect to the field or action-angle variables. Leaving aside the problem of Hilbert space domains for operators and accounting for the fact that the standard (Dirac) quantization prescription

$$\{f, g\} = k \to [\hat{f}, \hat{g}]_- = i\hbar\hat{k}, \tag{3.1}$$

cannot be extended to all classical observables,[8] we use (3.1) to quantize the respective (symmetry group) Lie algebra generators.

There is however one subtlety which must be kept in mind while comparing the field or action-angle quantizations of the same classical system. Namely the change of the local parameterization on the symplectic manifold of the classical model (e.g., (2.3) and (2.15)) is not quite innocent from the quantal point of view. The problem can be best exemplified by discussing the nonlinear Schrödinger model, whose quantal features are better understood[9,10,11] than those of the sine-Gordon model (see Refs. 12–18, and Refs. 3, 19 and 20). The reason for this is that the Fock representation of the canonical commutation relations (CCR) algebra suffices for the quantization of (2.1 and 2.2)

$$\{\psi^*(x), \psi(y)\} = i\delta(x-y) \xrightarrow[\hbar=c=1]{} [\hat{\psi}(x), \hat{\psi}^*(y)]_- = \delta(x-y),$$
$$\hat{\psi}(x)|0\rangle = 0, \quad \forall x \in R^1, \tag{3.2}$$

and for the construction of the (Bethe Ansatz) eigenvectors of the quantum Hamiltonian for the model.

If the state space vector $|0\rangle$ is the unique cyclic vector (e.g., Fock vacuum) the representation (3.2) is irreducible, and its domain is the Hilbert space closure $\mathscr{F}$ of the set of all vectors obtained from $|0\rangle$ by acting upon it with polynomial functions of fields

$$\mathscr{F} = \overline{\{W(\hat{\psi}^*, \hat{\psi})|0\rangle\}}\,. \tag{3.3}$$

Each nonzero operator with a domain in $\mathscr{F}$ can be expressed as a function of $\hat{\psi}^*$, $\hat{\psi}$ variables, and by virtue of the irreducibility of the representation it does not commute with the whole of the field (CCR) algebra unless it is a multiple of the operator unit in $\mathscr{F}$ (see Ref. 21).

On the other hand, if to depart from (2.3) and (2.4), the action-angle quantization of the nonlinear Schrödinger field amounts to representing three sets (q_l, p_l), (τ_l, m_l), $(\varphi(x), \varphi^*(x))$ of kinematically independent canonical variables by (kinematically independent again) operators.

However then (we disregard for a while the subtleties connected with the quantization of the action-angle coordinates[22]) the Fock space quantization of the radiation piece of the problem

$$\begin{aligned}
&[\hat{\varphi}(x), \hat{\varphi}^*(y)]_- = \delta(x-y), \quad [\hat{\varphi}(x), \hat{\varphi}(y)]_- = 0,\\
&\hat{\varphi}(x)|0\rangle = 0, \quad \forall x \in R^1 \Rightarrow \mathscr{F} = \overline{\{W(\hat{\varphi}^*, \hat{\varphi})|0\rangle\}},\\
&\hat{M}_0 = \mu \int \hat{\varphi}^*(x)\hat{\varphi}(x)\,dx = \mu \int \hat{n}(p)\,dp,\\
&\hat{H}_0 = \frac{1}{2\mu}\int \hat{\varphi}_x^*\hat{\varphi}_x\,dx = \int \frac{p^2}{2\mu}\hat{n}(p)\,dp,\\
&\hat{P}_0 = \int \hat{\varphi}^*(-i\hat{\varphi}_x)\,dx = \int p\hat{n}(p)\,dp,\\
&\hat{K}_0 = -\mu \int x\hat{\varphi}^*(x)\hat{\varphi}(x)\,dx,
\end{aligned} \tag{3.4}$$

precludes the existence of any (except for multiples of identity) operators which would commute with the field algebra.

Hence as long as the $(\hat{\varphi}^*, \hat{\varphi})$ algebra is irreducible, there is no room in $\mathscr{F}$ for quantum images of (q_l, p_l), (τ_l, m_l), $1 \le l \le N$ which would be kinematically independent of the radiation.

Since in $\mathscr{F}$ the $(\hat{\psi}^*, \hat{\psi})$ and $(\hat{\varphi}^*, \hat{\varphi})$ field algebras are related by a unitary transformation, we realize that the only way to allow for a consistent quantization of nonradiation coordinates (consistent means: representing operators in the state space of the fundamental $\hat{\psi}^*$, $\hat{\psi}$ field) is to abandon the irreducibility requirement.

It is well-known that the reducible representations of the canonical algebras allow for the existence of operators which cannot be solely expressed in terms of field generators. Let us mention that the need for reducible representations is suggested by the quantum analysis[23] and has been raised in the more general context[24] of the existence problem for nontrivial quantum field theory models.

Remark: A brute force quantization of $(q_l, p_l, \tau_l, m_l, \varphi, \varphi^*)$ would amount to the introduction of the tensor product Hilbert space, each entry corresponding to another canonical pair. Then the relationship to the $(\hat{\psi}, \hat{\psi}^*)$ generated state space would be apparently lost.

Since we wish to maintain the relationship of the (ψ, ψ^*) and $(q_l, p_l, m_l, \tau_l, \varphi, \varphi^*)$ quantizations on the state space level, it makes sense to exploit the most efficient approach to establishing the classical-quantum correspondence, namely of studying the coherent state expectation values of operators in the tree approximation (see examples in Refs. 24, 25 and 26, and also in Refs. 3, 22, 27 and 28).

In connection with soliton fields, we are most strongly motivated by Ref. 25 and the heuristic analysis of soliton states presented in Ref. 29 (see also Refs 19 and 1), all of which can be reconciled only if the field and the action-angle quantizations of the same classical model are compatible.

The direct integral Hilbert space construction to be used in below seems to be a slightly forgotten concept[30,31,32,33] albeit the need for its revival was discussed recently in Ref. 34 (see also Ref. 3). It seems to be a very natural way to encompass families (fields) of Hilbert spaces which arise in many physical contexts[35,36] but also in connection with the (translation mode[19]) vacuum degeneracy of quantized nonlinear fields.[34]

4. Quantum Nonlinear Schrödinger Field with the Attractive Coupling

The Bethe Ansatz (spectral) solution of the model has been found by directly quantizing (2.1) with respect to the fundamental field variables

$$(\psi^*, \psi) \to (\hat{\psi}^*, \hat{\psi}), \quad [\hat{\psi}(x), \hat{\psi}^*(y)]_- = \delta(x - y), \tag{4.1}$$

in Fock space $\mathscr{F} : \hat{\psi}(x)|0\rangle = 0, \quad \forall x \in R^1$. The time instant is chosen equal $t = 0$.

As is well-known, if $f(x)$ is a square integrable on R^1 complex function, then the operator

$$U_f = \exp\left\{\int\left[\hat{\psi}^*(x)f(x) - \bar{f}(x)\hat{\psi}(x)\right]dx\right\} \tag{4.2}$$

maps $|0\rangle$ into a normalized vector (coherent state) in $\mathscr{F}$

$$\begin{gathered} |f\rangle = U_f|0\rangle, \\ \hat{\psi}(x)|f\rangle = f(x)|f\rangle, \\ U_f^{-1}\hat{\psi}(x)U_f = \hat{\psi}(x) + f(x). \end{gathered} \tag{4.3}$$

Since there is no operator ordering problems, the representation of the extended Galilei group in $\mathscr{F}$ is generated by operators $\hat{H}$, $\hat{M}$, $\hat{P}$, $\hat{K}$ of exactly the same functional form as (2.1) and (2.2).

$$\hat{H} = \int_{R^1} \left(\frac{1}{2\mu} \hat{\psi}_x^* \hat{\psi}_x - \frac{g}{2} \hat{\psi}^{*2} \hat{\psi}^2 \right) dx ,$$

$$\hat{M} = \mu \int \hat{\psi}^*(x) \hat{\psi}(x) \, dx , \quad \hat{K} = -\mu \int x \hat{\psi}^*(x) \hat{\psi}(x) \, dx , \tag{4.4}$$

$$\hat{P} = \int \hat{\psi}^*(x)(-i\hat{\psi}_x) \, dx ,$$

so that by taking the coherent state expectation value of any generator (4.4) we map it into a c-number expression, like for example,

$$\langle f | {:}W(\hat{\psi}^*, \hat{\psi}){:} | f \rangle = W(\bar{f}, f) \tag{4.5}$$

and in particular

$$\begin{aligned} \langle f | \hat{H} | f \rangle &= \int \left(\frac{1}{2\mu} \bar{f}_x f_x - \frac{g}{2} |f|^4 \right) dx , \\ \langle f | \hat{M} | f \rangle &= \mu \int |f|^2 \, dx , \\ \langle f | \hat{P} | f \rangle &= \int \bar{f}(-if_x) \, dx , \\ \langle f | \hat{K} | f \rangle &= -\mu \int x |f|^2 \, dx . \end{aligned} \tag{4.6}$$

The proper classical-quantum relationship arises upon identifying $f = f(x)$ with a (square integrable on R^1) solution of the field equation (2.1) and supplementing the c-number objects (4.5) and (4.6) with the Poisson bracket operation

$$\{A, B\} = i \int dx \left(\frac{\delta A}{\delta \bar{f}(x)} \frac{\delta B}{\delta f(x)} - \frac{\delta A}{\delta f(x)} \frac{\delta B}{\delta \bar{f}(x)} \right) . \tag{4.7}$$

Though we are inspired by the variational idea of Ref. 25, the static solutions are inappropriate for our purposes, while some inconveniences appear if to admit the time dependent solutions.

Remark: A well-known property of the familiar harmonic oscillator coherent states is that

$$|\alpha, t\rangle = \exp(-i\hat{H}_0 t)|\alpha\rangle = \exp(-i\omega t/2)|\alpha(t)\rangle, \tag{4.8}$$

where $|\alpha\rangle = \exp(\alpha a^* - \bar{\alpha}a)|0\rangle$ and $\alpha(t) = \alpha \exp(-i\omega t)$ solves the classical equation of motion $\ddot{\alpha} + \omega^2 \alpha = 0$.

Consequently

$$\langle\alpha|\hat{A}(t)|\alpha\rangle = \langle\alpha, t|\hat{A}|\alpha, t\rangle,$$
$$\hat{A}(t) = \exp(i\hat{H}_0 t)\hat{A}\exp(-i\hat{H}_0 t), \tag{4.9}$$

and the classical motion rule is compatible with the quantum one, since $|\alpha(t)\rangle$ is the Schrödinger picture vector labeled by the time-dependent classical variable. All that is not the case for the nonlinear Schrödinger model: $|\alpha(t)\rangle$ is not a Schrödinger picture vector if $\alpha(t)$ is supposed to solve the field equation.

To simplify our further arguments, we shall consider the 1 soliton solution of (2.1) (the N-soliton generalization is immediate) which is determined by the scattering data (p, q, τ, m) at time t

$$\psi(x, t) = -\frac{1}{2}\left(\frac{g}{\mu}\right)^{1/2} m \exp\left\{i\frac{\mu}{m}[px - m\tau(t) - q(t)p]\right\}$$
$$\times \operatorname{sech}\left[\frac{1}{2}mg(x - q(t))\right]. \tag{4.10}$$

If to identify $f = f(x)$ of (4.6) with $\psi(x, t)$ the coherent state expectation value of the operator in $\mathscr{F}$ becomes a function of the soliton data (p, q, τ, m)

$$\langle\psi|:A(\hat{\psi}^*, \hat{\psi}):|\psi\rangle = A(\bar{\psi}, \psi) = \mathscr{A}(p, q, \tau, m). \tag{4.11}$$

It is enough to compute (4.11) at time $t = 0$ to have determined the classically driven time development of $\langle\psi(t)|:A(\hat{\psi}^*, \hat{\psi}):|\psi(t)\rangle$. It automatically follows from the knowledge of the classical Hamiltonian (4.6), which upon inserting (4.10) equals

$$\langle\psi|\hat{H}|\psi\rangle = \frac{p^2}{2m} - \frac{g^2}{24\mu^2}m^3. \tag{4.12}$$

The symplectic structure (2.5) implies the time evolution (2.7) of the action-angle variables, and hence of the coherent state expectation values. Since (p, q, τ, m) can be varied and for each (p, q, τ, m) set a corresponding soliton field is generated, we deal with a four-parameter family of coherent soliton states $|\psi\rangle$ and $\forall(p, q, \tau, m)$ we have

$$\overline{\{W(\hat{\psi}^*, \hat{\psi})|\psi\rangle\}} = \mathscr{F} = \overline{\{W(\hat{\phi}^*, \hat{\phi})|\psi\rangle\}}. \tag{4.13}$$

As long as we do not insist on the time dependence of the q, τ labels ($\dot{p} = 0 = \dot{m}$) neither coherent state $|\psi\rangle = |p, q, \tau, m\rangle$ can be particularly distinguished among the others. The situation drastically changes if we would admit $q = q(t)$, $\tau(t)$, since then we encounter the problem mentioned in the *Remark* accompanying (4.9). Namely, the time evolution $|p, q, \tau, m\rangle \to |p, q(t), \tau(t), m\rangle$ is of the purely classical origin and is not implemented by the Hamiltonian of the quantum model. For clarity of further discussion, let us consider the 1 soliton solution of (2.1) (see (4.10)) in the form

$$\psi(x, t) = -\frac{1}{2}\left(\frac{g}{\mu}\right)^{1/2} m \exp\left\{i\frac{\mu}{m}p(x - \alpha)\right\}$$

$$\times \operatorname{sech}\left\{\frac{1}{2}mg(x - \beta)\right\}, \tag{4.14}$$

where we have incorporated the time dependence into the freedom of choice of the two (independent!) translation parameters $\alpha, \beta \in R^1$. Notice that apart from having allowed α to run through the whole of R^1, the effective α-dependence of $\psi(x, t)$ is periodic with the period $2\pi m/p\mu$. The expectation value $\langle\psi|\hat{H}|\psi\rangle$ is supposed to have something in common with the tree approximation value of the rest mass of the extended soliton-type particle.[25] At this point the (α, β)-degeneracy of states which give rise to the same classical energy seems to be slightly inconvenient. The standard approach to such (translation mode) problem is to get rid of the redundant states by any means[19] so that the "classical soliton plus quantum perturbations" problem is isolated.

Since a priori all $|\alpha, \beta\rangle$ states can be placed in the Fock space $\mathscr{F}$, the simplest procedure would be for example, to take an appropriately weighted superposition

$$|F\rangle = \int d\alpha \int d\beta\, F(\alpha, \beta)|\alpha, \beta\rangle \in \mathscr{F}. \tag{4.15}$$

Then however the CCR algebra representation $(\hat{\psi}^*, \hat{\psi})$ would still remain irreducible and the problem of reconciling (4.1)–(4.4) with the action-angle quantization of the system would not be solved. There is however another possibility of dealing with the translation freedom[30–33,3] which offers a resolution of these problems. We should work with a reducible representation of the CCR algebra generated by $(\hat{\psi}^*, \hat{\psi})$ instead of the irreducible (unique Fock vacuum) one. Let us consider a two parameter family of Hilbert spaces $\mathscr{F}_{\alpha\beta}$ each one being a copy of the initial Fock space $\mathscr{F}$

$$\mathscr{F}_{\alpha\beta} = \{\overline{W(\hat{\psi}^*, \hat{\psi})|\alpha, \beta\rangle}\}. \tag{4.16}$$

Instead of picking up the coherent states $|\alpha, \beta\rangle$ from the same Hilbert space $\mathscr{F}$, we take each $|\alpha, \beta\rangle$ from its own Fock space $\mathscr{F}_{\alpha\beta}$.

According to the recipes of[30–33,3] we shall form the (double) direct integral Hilbert space with respect to the appropriate measure

$$\mathscr{H} = \int^{\oplus} \mathscr{F}_{\alpha\beta}\, d\mu(\alpha,\beta) \ni \xi = \int^{\oplus} f(\alpha,\beta)\,|\xi,\alpha,\beta\rangle\, d\mu(\alpha,\beta),$$

$$|\xi,\alpha,\beta\rangle \in \mathscr{F}_{\alpha\beta}, \tag{4.17}$$

$$(\xi,\xi') = \int d\mu(\alpha,\beta)\,|f(\alpha,\beta)|^2 \langle \xi,\alpha,\beta|\xi',\alpha,\beta\rangle,$$

where $(\xi,\alpha,\beta|\xi',\alpha,\beta)$ stands for the scalar product of two Hilbert space vectors in $\mathscr{F}_{\alpha\beta}$, $\{|\xi,\alpha,\beta\rangle\}$ is a two parameter family of vectors with the same Fock space coordinates.

For any polynomial function $W(\hat{\psi}^*,\hat{\psi})$ of the initial field operators (whose domain was primarily in $\mathscr{F}$) we define the following extension to the whole of $\mathscr{H}$

$$\mathscr{W}(\hat{\psi}^*,\hat{\psi})\xi = \int^{\oplus} f(\alpha,\beta)\, W(\hat{\psi}^*,\hat{\psi})\,|\xi,\alpha,\beta\rangle\, d\mu(\alpha,\beta),$$

$$\mathscr{W}(\hat{\psi}^*,\hat{\psi}) = \int^{\oplus} [W(\hat{\psi}^*,\hat{\psi})]_{\alpha\beta}\,. \tag{4.17'}$$

So introduced operators do not intertwine between different (α,β) sectors, which is however the case for operators

$$V_{\underline{s}} = \int^{\oplus} \sum_n |n,\underline{a}\rangle \exp(\underline{s}\nabla_{\underline{a}})\langle n,\underline{a}|,$$

$$V_{\underline{t}} = \int^{\oplus} \exp(i\underline{t}\underline{a}) \sum_n |n,\underline{a}\rangle\langle n,\underline{a}|, \tag{4.18}$$

$$\underline{a} = (\alpha,\beta), \quad \underline{s}\nabla_{\underline{a}} = s_1\partial_\alpha + s_2\partial_\beta,$$

$$\underline{s} = (s_1,s_2), \quad \underline{t}\underline{a} = t_1\alpha + t_2\beta.$$

Here $\{|n,\underline{a}\rangle,\ n = 0,1,\ldots\}$ is an orthonormal (Fock space) basis system in $\mathscr{F}_{\underline{a}}$, $\underline{a} = (\alpha,\beta)$ and upon adopting the action rule

$$V_{\underline{s}}\xi = \int^{\oplus} f(\alpha,\beta) \sum_n |n,\alpha,\beta\rangle \exp(s_1\partial_\alpha + s_2\partial_\beta)\langle n,\alpha,\beta)|\xi;\alpha,\beta\rangle\, d\mu(\alpha,\beta), \tag{4.19}$$

and assuming the (translation) invariance of the integration measure, we realize that

$$V_{\underline{t}}V_{\underline{s}}\xi = \exp(-i\underline{t}\underline{s})\, V_{\underline{s}}V_{\underline{t}}\xi, \quad \forall \xi \in \mathscr{H}, \tag{4.20}$$

while for all vectors ξ from the domain $\mathscr{D} \subset \mathscr{H}$ for $\mathscr{W}(\hat{\psi}^*,\hat{\psi})$ there holds

$$[V_{\underline{t}}, \mathscr{W}(\hat{\psi}^*, \hat{\psi})]_- \xi = 0 = [V_{\underline{s}}, \mathscr{W}(\hat{\psi}^*, \hat{\psi})]_- \xi . \tag{4.21}$$

The infinitesimal generators associated with (4.18) read

$$\begin{aligned}
\hat{q} &= \int^{\oplus} \alpha \sum_n |n, \alpha, \beta\rangle \langle n, \alpha, \beta| , \\
\hat{Q} &= \int^{\oplus} \beta \sum_n |n, \alpha, \beta\rangle \langle n, \alpha, \beta| , \\
\hat{p} &= \int^{\oplus} \sum_n |n, \alpha, \beta\rangle \left(i \frac{\partial}{\partial \alpha} \right) \langle n, \alpha, \beta| , \\
\hat{P} &= \int^{\oplus} \sum_n |n, \alpha, \beta\rangle \left(i \frac{\partial}{\partial \beta} \right) \langle n, \alpha, \beta| , \\
[\hat{q}, \hat{p}]_- \xi &= -i\xi = [\hat{Q}, \hat{P}]_- \xi , \quad \xi \in \mathscr{D} \subset \mathscr{H} , \\
[\hat{q}, \hat{Q}]_- &= [\hat{q}, \hat{P}]_- = 0 = [\hat{p}, \hat{Q}]_- = [\hat{p}, \hat{P}]_- ,
\end{aligned} \tag{4.22}$$

where $\hat{q}$, $\hat{p}$ are related to the periodic dependence of $\psi(x, t)$ on α while $\hat{Q}$, $\hat{P}$ correspond to translations of the soliton energy center. Obviously the generators $\hat{q}$, $\hat{p}$, $\hat{Q}$, $\hat{P}$ commute with all elements of the $(\hat{\psi}^*, \hat{\psi})$ field algebra.

The just constructed operators in $\mathscr{H}$ suffice for the quantization of the action-angle version of the nonlinear Schrödinger system, since what we have been in need were the three pairs of mutually independent canonical variables.

Let us restrict considerations to the 1 soliton plus radiation segment of (2.1)

$$\begin{aligned}
H &= \left(\frac{p^2}{2m} - \frac{g^2}{24\mu^2} m^3 \right) + \frac{1}{2\mu} \int |\varphi_x|^2 \, dx , \\
\{q, p\} &= \{\tau, m\} = 1 , \\
\{\varphi^*(x), \varphi(y)\} &= i\delta(x - y) .
\end{aligned} \tag{4.23}$$

Upon recalling that $(\hat{\psi}^*, \hat{\psi})$ and $(\hat{\varphi}^*, \hat{\varphi})$ algebras are unitarily equivalent in the Fock space, the quantization of $(\hat{\varphi}^*, \hat{\varphi})$ is immediate. The $(\hat{P}, \hat{Q})$ pair of variables can be chosen as the quantization of the pair (p, q) of soliton coordinates. The other pair should thus be exploited for the quantization of the (m, τ) coordinates, where however some care is necessary:[22] the angle τ itself should not be quantized, but rather its functions $\sin \tau \to \hat{S}$, $\cos \tau \to \hat{C}$.

We adopt the number-phase quantization pattern of Ref. 22, which is corrected to avoid the noninvertibility of the operator $\hat{M}$

$$m \to \hat{M} = \hat{N} + 1, \quad \hat{N} = a^* a,$$

$$a = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}), \tag{4.24}$$

$$a^* = \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p}).$$

It implies

$$\hat{H} = \frac{\hat{P}^2}{2\hat{M}} - \frac{g^2}{24\mu^2}\hat{M}^3 + \int \frac{p^2}{2\mu}\hat{n}(p)\,dp,$$
$$\hat{n}(p) = \hat{\phi}^*(p)\hat{\phi}(p). \tag{4.25}$$

Since the eigenvalues of $\hat{N}$ equal 0, 1, 2, ... we may follow the tradition[9] and renormalize $\hat{H}$ by adding to it the term $\hat{M}g^2(24\mu^2)^{-1}$ so that the standard soliton dispersion formula follows (originally it comes out due to the definite choice of the operator ordering, or due to the demand that the $m = 1$ excitation is structureless i.e., without any internal energy)

$$E_r(n) = \frac{p^2}{2n} - \frac{q^2}{24\mu^2}(n^3 - n), \quad n \geq 1. \tag{4.26}$$

In the form

$$\hat{H}_r = \frac{\hat{P}^2}{2\hat{M}} - \frac{g^2}{24\mu^2}(\hat{M}^3 - \hat{M}) + \int \frac{p^2}{2\mu}\hat{n}(p)\,dp, \tag{4.27}$$

we have canonically quantized both the 1 soliton and radiation pieces of the classical action-angle expression (4.18) in the direct integral Hilbert space $\mathscr{H}$.

The passage to the multi soliton case amounts to rather straightforward generalizations of the previous procedure with the result

$$\hat{H}_r = \sum_{l=1}^{k} l\left\{\frac{\hat{P}_l^2}{2\hat{M}_l^2} - \frac{g^2}{24\mu^2}(\hat{M}_l^3 - \hat{M}_l)\right\} + \int \frac{p^2}{2\mu}\hat{n}(p)\,dp,$$
$$\hat{M}_l = \hat{N}_l + 1, \quad \hat{M} = \sum_{l=1}^{k}\hat{M}_l. \tag{4.28}$$

Our goal has thus been accomplished: the quantized action-angle version of the model is shown to co-exist (in the same state space) with the reducible representation of the CCR algebra used to realize the field quantization. Moreover the radiation modes were incorporated.

Remark: We use the coherent state expectation values of relate the quantum and classical level of the same model. For completeness let us mention that there exists a procedure of passing from the quantum to classical soliton fields in the nonlinear Schrödinger model, which amounts to studying matrix elements of the field operator among n-particle (Bethe Ansatz) bound states in the $n \to \infty$ limit.[37,38,39]

5. Quantum sine-Gordon Field

The situation here is more involved than previously. First of all there is no Bethe Ansatz solution for the model. There are mathematically rigorous arguments for the existence of the canonically quantized sine-Gordon field, and it is rather well cofirmed that the non-Fock representations of the CCR algebra enter the game.

On the other hand, although the inverse spectral transform technique allows to recover the quantized action-angle spectrum of (soliton) excitations, no information about the soliton states and no information about the quantized radiation is obtained. Moreover the above-mentioned spectral solution refers to the even soliton numbers.

It seems that the WKB quantization[19] of the model accounts for the quantized radiative perturbation of classical soliton fields, but it does not give any insight into the state space structure of the system.

The rigorous approach of Ref. 13, albeit involving the notion of coherent soliton states, does not refer to the problem of the action-angle quantization of the classical model, and is confined to the analysis of topologically arising state space sectors. The attempt of Refs. 17 and 18 seems to be incompatible with the correspondence principle, being incapable of reproducing the classical action-angle Hamiltonian for the sine-Gordon field. The quantization of the sine-Gordon system with respect to the field variables, amounts to replacing the Poisson bracket $\{\phi(x), \pi(y)\} = \delta(x - y)$ by the commutator $[\hat{\phi}(x), \hat{\Pi}(y)] = i\delta(x - y)$ so that

$$\hat{H} = \frac{1}{2}\int :\left\{\hat{\Pi}^2 + \hat{\phi}_x^2 + \frac{2m^4}{g}\left[1 - \cos\left(\frac{g^{1/2}}{m}\hat{\phi}\right)\right]\right\}: dx. \tag{5.1}$$

If to remain (for a while) on the Fock space level, then the standard coherent state analysis would enter the game

$$\begin{aligned} |f, g\rangle &= \exp i \int dx(g\hat{\phi} - f\hat{\Pi})|0\rangle = U(f, g)|0\rangle, \\ \langle f, g|\hat{\phi}(x)|f, g\rangle &= f(x), \\ \langle f, g|\hat{\Pi}(x)|f, g\rangle &= g(x). \end{aligned} \tag{5.2}$$

However as mentioned above, the non-Fock representations of the CCR algebra are necessary for the description of the classical soliton fields in terms of coherent state expectation values of the quantized sine-Gordon field. In that case the notation

$U(f,g)|0\rangle$ would be incorrect since $U(f,g)$ does no longer exist as a well defined operator in the Fock space. Nevertheless we can give the meaning to the generalized coherent states (see example in Refs. 3 and 26 and references therein). Once non-Fock, each state $|f,g\rangle$ obeying $\langle f,g|\hat{\phi}(x)|f,g\rangle = \phi(x)$, $\langle f,g|\hat{\Pi}(x)|f,g\rangle = \Pi(x)$ can be used to generate a corresponding Hilbert space via $\{W(\hat{\phi},\hat{\Pi})|f,g\rangle\} = \mathscr{H}(|f,g\rangle)$ in which the (non-Fock) representation of the CCR algebra is irreducible.

At this point we encounter the very same problem as before in connection with the nonlinear Schrödinger equation. Namely, in each Hilbert space $\mathscr{H}(|f,g\rangle)$ in which the fundamental (sine-Gordon) field is defined, we can pick up a unitary transformation to the irreducible (non-Fock again) representation of the CCR algebra, which is generated by the radiation field

$$[\hat{\phi}(x), \hat{\pi}(y)]_- = i\delta(x-y). \tag{5.3}$$

The associated (free field) representation of the Poincare group is given by

$$\begin{aligned} \hat{H}_0 &= \frac{1}{2}\int :(\hat{\pi}^2 + \hat{\phi}_x^2 + m^2\hat{\phi}^2): dx, \\ \hat{P}_0 &= -\int :\hat{\pi}\hat{\phi}_x: dx, \\ \hat{K}_0 &= -\int x\hat{H}_0(x)\,dx, \end{aligned} \tag{5.4}$$

and acts invariantly in each Hilbert space $\mathscr{H}(|f,g\rangle)$. As in the case of the nonlinear Schrödinger model, there is no room for quantized soliton coordinates, unless we abandon the irreducibility requirement. The construction of the Hilbert space, in which the quantized action-angle variables coexist together with the quantized sine-Gordon field, follows the previous direct integral pattern.[34] The form of the 1 soliton field clearly indicates that the time variability can be absorbed in the freedom of choice of the translation parameters q in

$$\begin{aligned} &\varphi = \varphi(x) = \exp m\gamma(x+q), \\ &q = v(t-t_0), \quad v = \frac{|a|^2-1}{|a|^2+1}, \quad a \in R^1, \quad |a| \in [0,\infty), \\ &\gamma = (\operatorname{sgn} a)(1-v^2)^{-1/2}. \end{aligned} \tag{5.5}$$

Consistently we shall deal with a one-parameter (a determines the momentum variable for which $\dot{p} = 0$) family of coherent states corresponding to the ath classical soliton. States $\{|a,q\rangle\}$ induce the respective Hilbert spaces $\{\mathscr{H}(|a,q\rangle) = h_q\}$. By using any standard measure we arrive at the direct integral Hilbert space

$$h = \int^{\oplus} h_q \, d\mu(q),$$

$$h \ni |\xi\rangle = \int^{\oplus} f(q)|\xi, q\rangle \, d\mu(q), \quad f \in L^2(R^1), \tag{5.6}$$

$$\langle \xi | \xi' \rangle = \int d\mu(q) |f(q)|^2 \langle \xi, q | \xi', q \rangle, \quad |\xi, q\rangle \in h_q .$$

Since each h_q has its own orthonormal basis system $\{|n, q\rangle\}_{n=1,2,\ldots}$ the unitary in h operators given by

$$V_s = \int^{\oplus} \sum_n |n, q\rangle \exp\left(-s\frac{\partial}{\partial q}\right) \langle n, q| \, d\mu(q),$$

$$U_t = \int^{\oplus} \exp(itq) \sum_n |n, q\rangle \langle n, q| \, d\mu(q), \tag{5.7}$$

$$U_t V_s |\xi\rangle = \exp(its) V_s U_t |\xi\rangle, \quad \forall |\xi\rangle \in h,$$

give rise to the CCR algebra generators in h

$$\hat{q} = \int^{\oplus} q \sum_n |n, q\rangle \langle n, q| \, d\mu(q),$$

$$\hat{p} = \int^{\oplus} \sum_n |n, q\rangle \left(-i\frac{\partial}{\partial q}\right) \langle n, q| \, d\mu(q), \tag{5.8}$$

$$[\hat{q}, \hat{p}]_- |\xi\rangle = i|\xi\rangle, \quad |\xi\rangle \in \mathscr{D} \subset h,$$

which by construction commute with mass m free radiation field generators, and the fundamental sine-Gordon field $\hat{\phi}(x)$, $\hat{\pi}(x)$ algebra as well. The underlying representations of the field CCR algebra are thus reducible. Now it is rather straightforward that in the 1 soliton sector of the sine-Gordon state space the representation of the Poincare group Lie algebra can be introduced as follows

$$\begin{aligned} \hat{H} &= (\hat{p}^2 + M^2)^{1/2} + \hat{H}_0, \\ \hat{P} &= \hat{p} + \hat{P}_0, \\ \hat{K} &= -\tfrac{1}{2}[\hat{H}_s \hat{q} + \hat{q}\hat{H}_s] - \hat{K}_0, \\ \hat{H}_s &= (\hat{p}^2 + M^2)^{1/2}, \end{aligned} \tag{5.9}$$

which is an apparent quantization of the action-angle 1 soliton plus radiation piece of (2.14).

The generalization to the N-soliton case is obvious

$$\begin{aligned} &[\hat{q}_i, \hat{p}_j]_- |\xi\rangle = i\delta_{ij}|\xi\rangle, \\ &\hat{H} = \sum_{i=1}^{N} (\hat{p}_i^2 + M^2)^{1/2} + \hat{H}_0, \\ &\hat{P} = \sum_{i=1}^{N} \hat{p}_i + \hat{P}_0, \\ &\hat{K} = -\frac{1}{2}\sum_{i=1}^{N} (\hat{H}_i\hat{q}_i + \hat{q}_i\hat{H}_i) + \hat{K}_0. \end{aligned} \tag{5.10}$$

Remark: the 1 soliton mass here is the bare form of the (renormalized) mass of Ref. 19. The need for renormalization appears as if to attempt at giving a meaning to the sum of the 1 soliton and radiation Hamiltonian which describes an infinite number of harmonic oscillations over the non-Fock (classical) background field.

The breather generated state space is more complicated, and more complicated is the related action-angle algebra, since the classical phase space is compact.

6. Quantization of Breather Action-angle Coordinates

Let us start from the explicit formula of Ref. 40 for the sine-Gordon 2 solitons

$$\begin{aligned} &\cos\phi(x,t) = 1 - \frac{2}{m^2}(\partial_x^2 - \partial_t^2)\ln f(x,t), \\ &f(x,t) = a_{12}\cosh(\nu_1 + \nu_2) + \cosh(\nu_1 - \nu_2) - (1 - a_{12}), \\ &a_{12} = \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2, \quad a_i \in R^1, \quad i = 1, 2, \\ &\nu_i = m\gamma_i(x - v_i t) + \delta_i, \quad v_i = \frac{1 - a_i^2}{1 + a_i^2}, \\ &\gamma_i = (1 - v_i^2)^{-1/2}\,\mathrm{sgn}\, a_i = \frac{1 + a_i^2}{a_i} \Rightarrow \phi(x,t) \\ &= 4\tan^{-1}\frac{\sinh\frac{1}{2}(\nu_1 - \nu_2)}{\cosh\frac{1}{2}(\nu_1 + \nu_2)}. \end{aligned} \tag{6.1}$$

We shall make the following manipulation to arrive at breathing solutions. Instead of real parameters a_1, a_2 and υ_1, υ_2 the complex ones are admitted

$$a_1 = a, \quad a_2 = a^*, \quad a = a_R + ia_I,$$
$$\upsilon_1 = \upsilon, \quad \upsilon_2 = \upsilon^*, \quad \upsilon = \upsilon_R + i\upsilon_I. \tag{6.2}$$

Then

$$a_{12} = -\frac{a_I}{a_R} = -\frac{1}{r}, \quad f(x,t) = -\frac{1}{r}\cosh 2\upsilon_R + \cos 2\upsilon_I - \left(1 + \frac{1}{r}\right) \tag{6.3}$$

implies

$$\phi(x,t) = 4\tan^{-1}\left[r\frac{\sin \upsilon_I}{\cosh \upsilon_R}\right](x,t),$$

$$\upsilon_R = \frac{a_R}{|a|}m\gamma(x - vt) + \delta_R,$$

$$\upsilon_I = \frac{a_I}{|a|}m\gamma(t - vx) + \delta_I, \tag{6.4}$$

$$\gamma = (1 - v^2)^{-1/2} = \frac{1 + |a|^2}{2|a|}, \quad v = \frac{1 - |a|^2}{1 + |a|^2}, \quad \hbar = c = 1.$$

Observe that $\tan\theta$ of (2.16) turns out to be equal to $r = \frac{a_R}{a_I}$ in the present notation.

So the introduced $\phi(x,t)$ obeys the sine-Gordon equation rescaled to the form of $\partial_t^2\phi - \partial_x^2\phi + m^2\sin\phi = 0$ and is known as its breather solution.

Remark: The standard form of the equation $\phi_{tt} - \phi_{xx} + \frac{m^2}{g}\sin(g\phi) = 0$ comes out from the Hamiltonian $H = \int dx\left[\frac{1}{2}\phi_t^2 + \frac{1}{2}\phi_x^2 + \frac{m^2}{g^2}\{1 - \cos(g\phi)\}\right]$ (see also (2.11)–(2.13)).

Let us introduce the following notation

$$\upsilon_R = \frac{a_R}{|a|}m\gamma(x - \beta), \quad \beta = vt + \frac{\delta_R|a|}{a_R m\gamma},$$

$$\upsilon_I = (\omega + \delta_I) - \frac{a_I}{|a|}m\gamma v x = \alpha - \omega v x \tag{6.5}$$

$$\alpha = \omega t + \delta_I, \quad \omega = \frac{a_I}{|a|}m\gamma,$$

which upon accounting for the fact that the a^*, a labels are time independent, leads to the family of breather states $\{|\alpha,\beta\rangle\}$ and the related family of Hilbert spaces induced by the (a^*,a)th classical breather field. The two-parameter family of Hilbert spaces thus appears, and we can form the direct integral Hilbert space

$$\int^{\oplus} d\mu(\alpha)\mathscr{H}(|\alpha,\beta\rangle) = \mathscr{H}_\beta, \tag{6.6}$$

with which the (internal) quantum variables $\hat{q}$, $\hat{p}$ are automatically associated: $[\hat{q},\hat{p}]_- \subseteq i$.

The translation freedom of the breather motion is finally incorporated in the next direct integral

$$\mathscr{H} = \int^{\oplus} d\mu(\beta)\mathscr{H}_\beta \tag{6.7}$$

which carries the representation of the CCR algebra generated by operators $\hat{Q}$, $\hat{P}$, $[\hat{Q},\hat{P}]_- \subseteq i$ (the external quantum variables).

Remark: The breather field is periodic with respect to α, hence the Hilbert space structure (6.6) is too rich for the quantization purpose. Compare for example the situation in connection with the (m,τ) variables in the nonlinear Schrödinger case.

The (a^*,a) parameterization is implicit in the above though irrelevant for the derivation of the spectrum. Since the related classical phase space variables are Poincare invariant: $\rho = \rho'$, $\theta = \theta'$, instead of considering the general breather problem, we can confine attention to the case at rest: $v = 0 \equiv |a| = 1$, $a_I = \cos\theta$ when the time development of the action-angle coordinates is given by (2.24)–(2.26).

Let us observe that $(\dot{\theta} = 0)$, $H = 2M\sin\theta$ implies that θ should be restricted to the interval $[0,\pi]$ mod 2π. Since on the other hand $\dot{\phi} = 2M\cos\theta = \omega(\theta)$ has a frequency interpretation, we arrive at the final restriction $\theta \in \left[0,\frac{\pi}{2}\right]$ mod 2π or respectively $I \in [0, m^2\pi/2g]$. We are thus confronted with the problem of matching the previous Hilbert space (and related $\hat{p}$, $\hat{q}$, $\hat{P}$, $\hat{Q}$ CCR algebra) construction with the rest frame quantization of the classical system (2.24). Let us recall that because of

$$\begin{gathered} \dot{I} = 0, \quad \{\phi, I\} = 1, \quad H_b = 2M\sin\frac{gI}{m^2}, \\ \phi(t) = 16m\cos\left(\frac{gI}{m^2}\right)t + \phi, \end{gathered} \tag{6.8}$$

the coordinate I plays the role of the action variable, which is canonically conjugate to the angle ϕ.

If to make one more canonical transformation

$$I = \frac{J}{16}, \quad \phi = 16\varphi, \tag{6.9}$$

we obtain

$$\{\phi, I\} = \{\varphi, J\},$$

$$H_b = \frac{16m^3}{g} \sin \frac{g}{16m^2} J, \tag{6.10}$$

$$\varphi(t) = m \cos\left(\frac{g}{16m^2} J\right) t + \varphi,$$

where $J \in \left[0, \frac{m^2}{g} 8\pi\right]$.

At this point it is particularly instructive to refer to the harmonic oscillator regime, when say $\frac{g}{16m^2} J \leq 10^{-3}$, i.e., $J \leq \frac{16m^2}{g} 10^{-3}$ and

$$H_b \cong mJ, \quad \varphi(t) \cong mt + \varphi. \tag{6.11}$$

Notice that in the weak coupling $g \ll 1$ and (or) large mass regime, the variability range for J can be relatively large, albeit finite.

Since the breather field is periodic both with respect to J and φ, we realize that the effective canonical quantization of (6.10) should result in the direct sum of finite dimensional quantum problems. In fact, by general arguments[4,12] it is known that classical systems with a compact phase space do admit quantization in cases when the total phase space area (number of states) is a multiple of 2π. The formulas (6.10) and (6.11) prove that the (J, φ) quantization is always possible in the weak coupling and (or) large mass regime, when the allowed phase space volume is not too small. If to combine the harmonic oscillator regime with the fact that in the Hilbert space $\mathscr{H}$, the canonical pair $(\hat{p}, \hat{q})$ is in our disposal, we can expect that the quantization of (6.10) may have something in common with the action-angle quantization of the harmonic oscillator.[22] Then (6.11) suggests that the proper quantization of J should be a (multiple of) the occupation number operator $\hat{N} = \hat{a}^* \hat{a}$, for which the restriction $0 \leq J \leq \frac{m^2}{g} 8\pi$ imposes the upper bound on the allowed eigenvalues of $\hat{N}$. It throws up the problem to the finite dimensional space, in which there is no operator realization for the canonical commutator (i.e., a finite dimensional representation of $\{\varphi, J\} = 1$). Let h be a carrier Hilbert space for the Fock representation of the CCR algebra, which is given in the Weyl form

$$\hat{U}_\alpha \hat{V}_\beta = \exp(i\alpha\beta) \hat{V}_\beta \hat{U}_\alpha. \tag{6.12}$$

Its generators obey $[\hat{q}, \hat{p}]_- \subseteq i$. Let $a^* = \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p})$, $\hat{a} = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p})$ and $\hat{N} = \hat{a}^*\hat{a}$ be the occupation number operator. By $\hat{p}_n$ we denote the projection on the finite dimensional subspace of h spanned by all eigenvectors of $\hat{N}$ with the property

$$\hat{N}|k\rangle = k|k\rangle, \quad k \leq n. \tag{6.13}$$

Though there is no finite dimensional realization of the commutator $[\hat{q}, \hat{p}]_- \subseteq i$, it is not widely known that Weyl commutation relations (6.12) have finite dimensional representations, (see example in Refs. 41, 42 and 43), which thus provides a framework for the unambigous quantization of the classical system (6.10). For this purpose it suffices to pass from the φ and J variables to their trigonometric functions (which is also a proper recipe while quantizing the action-angle variables of the harmonic oscillator[22]), especially since such a function explicitly enters the Hamiltonian H_b. Consequently we replace the direct quantization problem for (6.10) by the quantization problem for functions $U = \exp(i\alpha\varphi)$, $V = \exp(i\beta J)$ of canonical variables. There is an example of the finite dimensional representation of (6.12) in the odd dimensional space.[43]

Let $h_n = p_n h$, $\dim h_n = 2J + 1 = n$. Instead of the occupation number operator $\hat{N}$ let us consider its translated version

$$\hat{J} = \hat{N} - J, \tag{6.14}$$

so that in the occupation number basis of h, matrix elements of $\hat{J}$ in h_{2J+1} read

$$\langle j|\hat{J}|j'\rangle = j\delta_{jj'}, \quad -J \leq j \leq J. \tag{6.15}$$

Let us introduce an operator $\hat{\phi}$ whose matrix elements in h_{2J+1} read

$$\langle j|\hat{\phi}|j'\rangle = \frac{1}{(2J+1)} \sum_{s=-J}^{+J} s \exp\left[\frac{i2\pi s(j-j')}{2J+1}\right]. \tag{6.16}$$

Notice that $\langle j|\hat{\phi}|j\rangle = 0$.

Let us now investigate the operators (matrices in h_{2J+1})

$$\begin{aligned} \hat{U}_{n\alpha} &= \exp(in\alpha\hat{\phi}), \quad m, n = 0, 1, 2, \ldots. \\ \hat{V}_{m\beta} &= \exp(im\hat{J}). \end{aligned} \tag{6.17}$$

If $\alpha\beta = \frac{2\pi}{2J+1}$, they satisfy the relation

$$\hat{U}_{n\alpha}\hat{V}_{m\beta} = \exp(in\alpha m\beta)\hat{V}_{m\beta}\hat{U}_{n\alpha}, \tag{6.18}$$

which is clearly an analog of (6.12) provided the shift parameters α, β appear in the discrete form $\alpha \to n\alpha$, $\beta \to m\beta$. Once the shift intervals α and β, $\alpha\beta = \frac{2\pi}{2J+1}$ are fixed, we deal with a finite dimensional realization of the Weyl commutation relations, which is known to provide a unique irreducible representation of (6.18) up to equivalence and constant multiplication factors.

In particular, if we wish to refer to (6.10), the choice of $J \to \hat{N}$ and $\beta = \frac{g}{16m^2}$ would amount to the (finite dimensional) quantization of $H_b = \frac{16m^3}{g} \sin \frac{g}{16m^2} J$, provided we observe that the following matrix identity holds true

$$\exp(im\beta\hat{J}) = \exp(im\beta\hat{N})\exp(-im\beta J), \tag{6.19}$$

so that, effectively (6.17) provides us with the quantization of $\exp(i\beta J)$ as required

$$\exp(in\alpha\hat{\phi})\exp(im\beta\hat{N}) = \exp(in\alpha m\beta)\exp(im\beta\hat{N})\exp(in\alpha\hat{\phi}). \tag{6.20}$$

The choice of $\beta = \frac{g}{16m^2}$ implies $\alpha = \frac{2\pi 16m^2}{(2J+1)g}$, while the value of J follows from the obvious restriction

$$\frac{g}{16m^2}(2J+1) \le \frac{\pi}{2}, \tag{6.21}$$

i.e., $\dim h_n = 2J + 1 \le \frac{8m^2}{g}\pi$.

The energy levels arising from the quantization of H_b read

$$\begin{gathered} E_b(n) = \frac{16m^3}{g} \sin \frac{g}{16m^2} n \\ 0 \le n \le 2J + 1 \le \frac{8m^2\pi}{g}, \end{gathered} \tag{6.22}$$

where J is a positive integer.

A comparison with the classic WKB result for the sine-Gordon breathers[19] shows that we have obtained the "bare" breather spectrum. Accounting for the radiation contribution (quantized sea of harmonic oscillations) needs a renormalization of the coupling constant g (see Ref. 19). It is instructive to mention that the weak coupling regime is of interest in statistical physics. Quantum corrections are then disregarded and the "bare" coupling constant is used.[44]

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