January 1985 ITP-UWr-85/630

BOSONS VERSUS FERMIONS: IS THERE A FUNDAMENTALITY PROBLEM?

Piotr Garbaczewski

Indian Statistical Institute, Calcutta, 700 035 India

and

Institute of Theoretical Physics, University of Wroclaw
50-205 Wroclaw, Poland *)

"Whereof we cannot speak,
thereof we must be silent"

(L.Wittgenstein, "Tractatus
Logico-Philosophicus")

*) Permanent address

Talk delivered at the International Conference on Quantum Statistics and Foundational Problems of Quantum Mechanics, Jan.28-Feb.1, 1985, Calcutta

1. The quest for elementary quantum statistics: a synopsis

1924	Bose statistics
1925	Pauli exclusion principle
	Fermi statistics
	Spin hypothesis
1927	Spin in Quantum mechanics: Pauli equation
1928	Dirac equation
1932	de Broglie, fusion (neutrino) theory of light(1)
1935	Jordan, neutrino theory of light, fermionization of bosons and its reverse i.e. bosonization of fermions(2)
1938	Pryce, no-go statement for the neutrino theory of light(3)
1940	Holstein and Primakoff, boson expansions of (finite spin) Lie algebras(4)
1956	Dyson, spin wave theory by means of the finite bosons expansions(5)
1958	Burgoyne, Lüders, Zumino, spin-statistics connection in relativistic local field theory(6)
1958	Heisenberg, nonlinear unified field theory of the spinorized universe(7)
1958	Skyrme, fermions from bosons in 1+1 (8)
1960	Marumori, boson expansions for the (nuclear physics) Lie algebras(9)
1964	neutrino theory of light, a renaissance(10-14), also(15,16)
1965	Schwinger, bosonization of Lie algebras(17)
1965	Penney: bosonization of Lie algebras(17)
1965	Penney: fermionization impossible if the system is finite(18)
1970	Streater and Wilde, fermion states of a boson field in 1+1(19)
1970	Kademova and Kalnay, bosonization of finite Fermi systems(20-22)
1970	Freundlich, bosonization and fermionization of massless fields in 1+1(23)

1974 Okubo, bosonization of Lie algebras[24]

1974 Garbaczewski and Rzewuski, strict bosonization of Fock representation of the CAR algebra, irrespective of the space-time dimension[25,26]

1974 Luther and Peschel, Jordan's construction further developed

1975 Coleman, the concept of fermion-boson equivalence (duality, reciprocity) in 1+1[28-32]

1976 Fermions as boson composites in 1+3, dyonization of fermions[33,34]

1977 Nakanishi, bosonization of Thirring and Schwinger models[35,36]

1978 Garbaczewski, quantization of c-number (non-Grassmann) spinor fields: fermions can be achieved via bosonization in 1+3[37]

1979 Zhelnorovich, tensorized universe: tensorial description of spinor fields[38]

1979 Luther, bosonization in 1+3 for Tomonaga fermions[39]

1981 Frenkel, affine Lie algebras and bosonization in 1+1[40,41]

1981 Dobaczewski, bosonization of Lie algebras, unification[42]

1982 Zhelnorovich, Takahashi, spinorization versus tensorization[43-45]

1983 Aratyn, Bose representation for the massless Dirac field in 1+3[46]

1983 Sorkin, particle statistics in three space dimensions, dyonization developed[47,48]

1983 Apostol, improvements of Jordan-Luther-Haldane bosonization--fermionization theories[49]

1983 Garbaczewski, mechanisms of the fermion-boson reciprocity[50] the quantum meaning of classical (field) theory for Fermi systems, via bosonization[51,52]

1984 Garbaczewski, joint Bose-Fermi spectral problems, or fermion-boson unduality in 1+1 and 1+3,[37,55-57]

1984 Luther ans Schotte, boson-fermion duality in 1+3: neutrinos from photons and vice versa[53]

1984 Witten, non-abelian bosonization in 1+1[54]

1984 Rajeev, fermions from bosons in 1+3 through anomalous commutators[58]

... ...

2. Fermion-boson relationships: duality or unduality.

For identical quantum particles, the respective multi-point wave functions, according to the folk lore recipe, are either symmetric or antisymmetric. Indeed, the symmetrization postulate (apart from problems with the experimental meaning of the concept of identical particles[59]) can be justified and even proved[60,61] on the level of non-relativistic quantum theory. The two possibilities, corresponding to symmetric and antisymmetric wave functions, appear in a natural way in three- or higher dimensional (Euclidean) spaces. In one or two dimensions there is allowed a continuum of intermediate possibilities which connects the extremal boson and fermion cases.

In the relativistic quantum field theory, the celebrated spin-statistics theorem[6] infers from the locality postulate that (identical) particles with integral spin are bosons, while these with half-odd-integral spin are fermions. The possibility of para-statistics we leave aside.[62]

For the fermions-from-bosons constructions which involve magnetically charged particles, the respective statistical properties must be formulated as a certain kind of relationship among non-identical particles[47,48]. Then, the conclusion is that bosons can combine to form fermions without violation of the normal connection of spin with statistics (dyonization in 1+3).

It is obvious that on the level of conventional (identical particles) quantum theory, the only realistic objection against the universality of the fermions- from- bosons route may come from the spin statistics theorem: the spinorial description is then related to anticommutativity, while the tensorial one to commutativity. Nevertheless it does not yet mean that "Whereof we cannot speak, thereof we must be silent" (L.Wittgenstein) since before, one should ignore that:

(1) not only tensors can be completely given in terms of spinors (spinorization), the reverse - tensorization procedure works as well[43-45]

(2) for a consistent particle interpretation Fock representations are necessary, and each Fock representation of the CAR can be embedded (bosonization) in the bicommutant of this of the CCR[25,26], see e.g. also at the non-Fock extension in[37,55].

Mathematically nothing forbids one from viewing the bosonic and tensorial description as primary (elementary) against the fermionic-spinorial one. Let us also mention that not all Bose models can be fermionized, while there is no Fermi model which would not admit any form of bosonization. Hence we do not find the search for elementary (primary) quantum statistics unfounded. But once such a problem is stated, we should realize that irrespective of the fact that the known (less or more elementary) particles are identified either with fermions or bosons, we thus admit that the deeper elementary levels may in principle display the bosonic rather than fermionic features. This quite unusual hypothesis of the statistical asymmetry in favour of bosons, in different forms persists in the literature(8,19,22,26,37). One may obviously insist on the fermion-boson duality(53) that: "since the photon can be constructed from neutrinos, and inversely neutrinos from the massless bosons, neither bosons nor fermions are truly fundamental", "an obvious issue raised by the duality relations concerns the meaning of a fundamental particle. If a boson Hamiltonian with purely boson eigenstates, can nonetheless produce a fermion, these bosons could be termed fundamental. But the inverse construction obviously denies this role. Neither can be accepted as fundamental". But this is merely the manifestation of the "Coleman-route-slavery", since one insists on the boson-fermion correspondence:

$$H_B = H_F \qquad (1)$$

on the level of Hamiltonians, completely ignoring the rigorous proof (25,26) of the algebraic superiority of bosons with respect to fermions. It is ironic that the authors of(53) after demonstrating a remarkable relationship of internal energies of the system of massless bosons and fermions:

$$\sum_{\{\Omega\}}\sum_{n=1}^{\infty} \frac{k_n}{\exp(\beta k_n)+1} = \sum_{\{\Omega\}}\sum_{n=1}^{\infty}{}' \frac{k_n}{\exp(\beta k_n)-1} \qquad (2)$$

where Σ' indicates a summation over odd integers only (thus a part of the available bosonic data suffices to get the Fermi system reconstructed), could have only mentioned that "there is an alternative that views the boson degrees of freedom as, in some sense,more fundamental"... which is indeed the case, provided one does not ignore the existing literature on the boson-fermion relationships. For example the boson-fermion unduality summarized in the joint Bose-Fermi spectral problem of[37,55-57]:

$$H_B = H_F + (1 - P)H_B(1 - P) \tag{3}$$

$$H_F = PH_BP \qquad P = P^* , \qquad P^2 = P$$

shows that (1) is an exception rather than a rule. Moreover the more sophisticated version of (3) may arise: in principle one can admit that for some Bose models there exists a countable family of projections:

$$\sum_k P_k = 1, \qquad P_kP_l = \delta_{kl}P_k , \qquad [P_k,H_B]_- = 0 \tag{4}$$

such that (the thermal it → ∞ interpretation is here particularly appealing):

$$\begin{aligned} &\mathrm{tr}\,\exp(-itH_B) = \sum_{k=1}^{\infty} \mathrm{tr}\,\exp(-iH_F^k\, t) \\ &H_F^k = P_k\, H_B\, P_k \end{aligned} \tag{5}$$

and the respective Bose model can be viewed as a (infinitely) reducible Fermi model, or even as a "tower" of possibly distinct Fermi models, each one with its own Hamiltonian H_F^k.

3. Boson-fermion dictionary: Coleman and followers

The popularity of the Coleman's route[28] among the so called pragmatists is unquestionable, hence we shall not embark on this topic. Instead we shall briefly recall the rules of the game in which one replaces fermion bilinears by suitable functions of the boson field. E.g. in 1+1 dimensions one has[54]:

$$
\begin{aligned}
&i\bar{\psi}\gamma^1\gamma_1\psi \leftrightarrow \tfrac{1}{2}(\partial_o\phi)^2 + \tfrac{1}{2}(\partial_1\phi)^2 \\
&i\bar{\psi}\gamma^\mu\partial_\mu\psi \leftrightarrow \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi \\
&\bar{\psi}\psi \leftrightarrow -\frac{1}{\pi\alpha}\cos\sqrt{4\pi}\,\phi \\
&i\bar{\psi}\gamma^5\psi \leftrightarrow \frac{1}{\pi\alpha}\sin\sqrt{4\pi}\,\phi \\
&\bar{\psi}\gamma^\mu\psi \leftrightarrow \frac{1}{\sqrt{\pi}}\,\varepsilon^{\mu\nu}\partial_\nu\,\phi \\
&\bar{\psi}\gamma^5\gamma^\mu\psi \leftrightarrow \frac{1}{\sqrt{\pi}}\,\partial^\mu\,\phi
\end{aligned}
\qquad (6)
$$

Here $1/\alpha$ plays the role of the ultraviolet cutoff, α is taken to be zero at the end of calculations in which bosons are used to replace fermions. The underlying fields are canonical:

$$[\phi(x),\pi(y)]_- = i\delta(x-y)$$
$$[\psi_\rho(x),\psi^+_\sigma(y)]_+ = \delta_{\rho\sigma}\delta(x-y) \qquad \rho,\sigma=1,2 \tag{7}$$

Upon (6) the (fermionic) Thirring model Lagrangian:

$$L_F = i\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{g}{2}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi) \tag{8}$$

allows a transformation to the (bosonic) sine-Gordon model:

$$L_B = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m}{\pi\alpha}\cos\left[\frac{4\pi}{1+\frac{g}{\pi}}\right]^{\frac{1}{2}}\phi \tag{9}$$

which is an essence of the "colemanology". Analogously e.g. the free massless fields satisfy the following equivalence:

$$L_F = \bar{\psi}\, i\, \not{\partial}\psi \equiv \tfrac{1}{2}\,\partial_\mu\phi\partial^\mu\phi = L_B \tag{10}$$

By now a generally accepted folk lore statement is[54] that any Fermi theory in 1+1 dimensions is equivalent to a local Bose theory, which manifestly preserves all the symmetries of the Fermi relative. Obviously it cannot happen in 1+3 dimensions, at least if the locality ansatz is not relaxed. Therefore the construction of Fermi fields as (nonlocal) functions of Bose fields was mostly considered in 1+1. Since in the study of any standard equivalence, the (anti) commutation relations like (7) are the basis of fermions-from-bosons construction, it is not worthwhile to recall the mathematically rigorous construction[25,26] of the canonical Fermi fields (CAR algebra generators) in the Bose field (CCR algebra generators) algebra. The main issue here is that this construction does not rely on the specific choice of the model (Hamiltonian), and space-time dimensionality. Later attempts to get fermions from bosons in 1+3[39,46,53] strictly refer to the particular model (massless Dirac field) and to the duality assumption. Needless to say that except for[46] the universal construction of[25,26,37] is simply ignored.

4. BCS pairons and the fermionization of bosons.

The prescription of mapping fermion bilinears into bosonic expressions has a long story and long before[29] has been coded in the physicists' mentality. Apart from the folk lore the apparent difficulties of this neutrino-theory of -light impact should be mentioned. The best example in this context is the controversy about the so called pairon condensation in the BCS model of superconductivity[63-66]. But both the no-go statement by Penney[18] and the final abandoning of the exact Bose statistics for neutrino composed "photons"[14] fall into the same category.

Let us consider fermions on the three dimensional lattice:

$$[c_{k\sigma}, c^*_{k'\sigma'}]_+ = \delta_{kk'}\,\delta_{\sigma\sigma'} \qquad k,k' \in R^3$$

$$[c_{k\sigma}, c_{k'\sigma'}]_+ = 0 \qquad \sigma,\sigma' = \uparrow,\downarrow \tag{11}$$

An obvious intuition about the ground state of the BCS system is that of the condensate of electron pairs. The respective pairon creation and annihilation operators are introduced as follows:

$$b_k = c_{-k\downarrow}\, c_{k\uparrow} \quad , \quad b^*_k = c^*_{k\uparrow}\cdot c^*_{-k\downarrow} \tag{12}$$

and obviously cannot be literally viewed as bosons since:

$$[b_k, b^*_{k'}]_- = (1 - n_{k\uparrow} - n_{-k\downarrow})\delta_{kk'} \tag{13}$$

$$[b_k, b_{k'}]_- = 0$$

$$[b_k, b^*_{k'}]_+ = 2b_k b_{k'}(1-\delta_{kk'})$$

$$n_{k\sigma} = c^*_{k\sigma}\, c_{k\sigma}$$

Thus the Pauli principle is manifest in the two electron composite. It automatically precludes both the Bose-Einstein statistics for pairons and their Bose-Einstein condensation(64,65).

To avoid such problems, the correct mapping of fermion objects into their boson images must either involve some (Pauli principle saving) constraints in the Bose state space[65,66] or follow an old Jordan's route[53,59].

In the notation of[53] let us introduce Fermi operators in three space dimensions as follows:

$$[a_\Omega(n), a^*_{\Omega'}(m)]_+ = \delta_{\Omega\Omega'}\delta_{nm}$$

$$[a_\Omega(n), a_{\Omega'}(m)]_+ = 0 \qquad a_\Omega(-j) = a^*_\Omega(j) \tag{14}$$

$k=2\pi nL^{-1}$ is a momentum label in the direction fixed by the discrete spherical angle Ω , n,m, being integers. We introduce the operator:

$$\rho_\Omega(m) = \sum_{j=-\infty}^{+\infty}{}' \; a^*_\Omega(j)\, a_\Omega(j+m) \tag{15}$$

where summations Σ' are carried out with respect to the odd integers merely. Whenever $a_\Omega(0)$ or $a^*_\Omega(0)$ appears in the sum, we adopt the convention to replace it by $2^{-\frac{1}{2}}(a_\Omega(0)+a^*_\Omega(0))$. The following holds true:

$$\rho_\Omega(m) = 0 \qquad , \text{ m even}$$

$$[\rho_\Omega(m), \rho^*_{\Omega'}(m')]_- = \tfrac{1}{2}m\delta_{\Omega\Omega'}\delta_{mm'} \quad , \text{ m,m}' \text{ odd} \tag{16}$$

$$[\rho_\Omega(m), \rho_{\Omega'}(m')]_- = 0$$

Hence the conventional (fermionized) bosons are shown to live at the odd lattice points of the integral momentum space lattice:

$$b_k = \left(\frac{2}{m}\right)^{\frac{1}{2}} \rho_\Omega(m)$$

$$k = \hat{\Omega}\ 2\pi m L_\Omega^{-1} \qquad (17)$$

$\hat{\Omega}$ being the unit vector in the direction Ω.

The manifest boson-fermion unduality can be seen here in that the single component fermion is incapable of reproducing the whole of the phase space data of the single component boson. Even the compensation of this drawback by the increase of the internal degrees number does not remove this odd momentum difficulty. Indeed the Luther-Schotte combination of fermion operators relevant to the construction of photon operators is defined at odd momenta again:

$$b_{\Omega_+}(q) = q^{-\frac{1}{2}} \sum_p{}' [a^*_{1\Omega}(p) a_{1\Omega}(p+q) - c^*_{2\Omega}(p) c_{2\Omega}(p+q)]$$

$$b_{\Omega_-}(q) = q^{-\frac{1}{2}} \sum_p{}' [a^*_{2\Omega}(p) a_{2\Omega}(p+q) - c^*_{1\Omega}(p) c_{1\Omega}(p+q)] \qquad (18)$$

$$d_{\Omega_+}(q) = q^{-\frac{1}{2}} \sum_p{}' [a^*_{1\Omega}(p) a_{1\Omega}(p+q) + c^*_{2\Omega}(p) c_{2\Omega}(p+q)]$$

$$d_{\Omega_-}(q) = q^{-\frac{1}{2}} \sum_p{}' [a^*_{2\Omega}(p) a_{2\Omega}(p+q) + c^*_{1\Omega}(p) c_{1\Omega}(p+q)]$$

where the canonical anticommutation relations for a, a^*, c, c^* imply:

$$[b_{\Omega\lambda}(q), b^*_{\Omega'\lambda'}(q')]_- = \delta_{qq'}\delta_{\Omega\Omega'}\delta_{\lambda\lambda'} = [d_{\Omega\lambda}(q), d^*_{\Omega'\lambda'}(q')]_- \qquad (19)$$

other commutators vanishing. Let us mention that the need for odd q's appears also in another construction of[41]

Apart from the above odd momentum difficulty, the construction (18) clearly displays the need for the same number of boson and fermion internal degrees of freedom. This feature, which is rather unusual from the point of view of "colemanology" standards, has been demonstrated many times before[25,37,55], but remained unnoticed in[53]. The conclusion that the two independent photon type fields are necessary to establish the boson-fermion relationship in 1+3, was demonstrated in the bosonization studies of[37], and then related to the antisymmetric tensor field background in[67], see also[46].

The fermionization (16) can be reversed: fermions are then reconstructed from the odd momentum bosons. Let us add that in[41] bosons are assumed to live everywhere on the lattice, while the (bosonized) fermion is confined to the odd lattice points merely. In this respect the general construction of[25,26] is optional since no configuration or momentum space information is lost: both bosons and fermions live everywhere in space, either on a lattice or in continuum.

5. Boson-Fermion unduality on a lattice: joint Bose-Fermi spectral problems

As mentioned before, instead of confining oneself to the study of boson-fermion equivalences (e.g. $H_B=H_F$) one may relax the equivalence demand to admit the broader category of boson-fermion relationships. The so appearing boson-fermion unduality is clearly displayed in the formulas

(3). Following[56] we shall discuss an easy example of the lattice system in one space dimension which obeys (3). (We have also such an example in three space dimensions). Let us consider the fermion model:

$$H = -J \sum_k (c^*_{k+1} c_k + c^*_k c_{k+1}) + V(c^*, c)$$

$$[c_k, c^*_l]_+ = \delta_{kl} \qquad [c_k m c_l]_+ = 0 \tag{20}$$

where V is assumed to represent the density-density interaction e.g. $U\sum_k n_k n_{k+1}$ or $U\sum_k (n_k - n_{k+1})^2$, $n_k = c^*_k c_k$. The forgoing boson-fermion relationship will be established for the pure hopping term, but can be easily generalized to the problem (20). We assume the periodic boundary conditions, which implies that irrespective of the choice of (boson or fermion) statistics, the hopping Hamiltonian can be rewritten in the form:

$$H = -J \sum_{i,j=1}^{n} c^*_i W_{ij} c_j$$

$$W_{ij} = \delta_{ij-1} + \delta_{ij+1} \qquad i,j=1,\dots,n \tag{21}$$

so that upon introducing the vectors:

$$g_k = \frac{f_k}{\sqrt{n}} \qquad f_k = \{f_{k\alpha}\}_{\alpha=1,\dots,n} \tag{22}$$

$$f_{k1} = 1, \qquad f_{k2} = \phi^k, \ldots, f_{kn} = \phi^{(n-1)k}$$

$$\phi = \exp i \frac{2\pi}{n} \tag{22}$$

we observe that:

$$\frac{1}{n}(f_k, f_l) = \frac{1}{n}\sum_{q=0}^{n-1} \exp i \frac{2\pi}{n}(k-l)q = \delta_{kl} \tag{23}$$

Hence in the notation:

$$\eta_k = \sum_{\alpha=1}^{n} \bar{g}_{k\alpha} c_\alpha \qquad \eta_k^* = \sum_{\alpha=1}^{n} g_{k\alpha} c_\alpha^* \quad \Rightarrow$$

$$[\eta_k, \eta_l^*]_+ = \delta_{kl} \qquad [\eta_k, \eta_l]_+ = 0 \tag{24}$$

we get:

$$H = H_F = \sum_k (-2J \cos \frac{2\pi}{n} k)\eta_k^* \eta_k \tag{25}$$

But if instead of fermions to use bosons:

$$[a_k, a_l^*]_- = \delta_{kl}, \qquad [a_k, a_l]_- = 0 \tag{26}$$

the formulas (22)-(25) lead to the conclusion that:

$$H_B = -J \sum_{i,j=1}^{n} a_i^* W_{ij} a_j = \sum_k (-2J \cos\frac{2\pi}{n} k) \xi^*_k \xi_k \tag{27}$$

$$\xi_k = \sum_{\alpha=1}^{n} \bar{g}_{k\alpha} a_\alpha$$

The respective eigenvectors belong to the n-body Fock space of the Bose and Fermi chains:

$$|m_1,\ldots,m_n\rangle_B = \overset{*}{\xi}{}_1^{m_1} \ldots \overset{*}{\xi}{}_n^{m_n} |0\rangle_B$$

$$|p_1,\ldots,p_n\rangle_F = \overset{*}{\eta}{}_1^{p_1} \ldots \overset{*}{\eta}{}_n^{p_n} |0\rangle_F \tag{28}$$

$|0\rangle_B, |0\rangle_F$ being the Fock vacua.

Now in the boson case we compose the product of two-level projections:

$$P = \prod_k p_k \tag{29}$$

$$p_k = :\exp(-\xi_k^* \xi_k): + \xi_k^* : \exp(-\xi_k^* \xi_k): \xi_k$$

which has the following properties:

$$[H_B, P]_- = 0 \qquad H_B = PH_BP + (1-P)H_B(1-P)$$

$$P\xi_k^* P \equiv \sigma_k^+ \qquad P\xi_k P \equiv \sigma_k^- \tag{30}$$

$$[\sigma_k^-, \sigma_k^+]_+ = p_k, \qquad [\sigma_k^\#, \sigma_l^\#]_- = 0 \qquad k \neq l$$

$$P\overset{*}{\xi}{}_1^{k_1} \ldots \overset{*}{\xi}{}_n^{k_n} |0\rangle_B = (\sigma_1^+)^{k_1} \ldots (\sigma_n^+)^{k_n} |0\rangle_B$$

$$(\sigma_j^\pm)^k = 0, \qquad k > 1.$$

Since:

$$k_i \leq 1 \ \forall i \Rightarrow (\sigma_1^+)^{k_1} \ldots (\sigma_n^+)^{k_n}|0\rangle_B = \overset{*}{\xi}_1^{k_1} \ldots \overset{*}{\xi}_n^{k_n}|0\rangle_B \tag{31}$$

it is quite obvious that through defining fermion operators in terms of Pauli ones, by means of the Jordan-Wigner transformation:

$$\begin{aligned} \overset{*}{\eta}_k &= (\exp i\pi \sum_{j=1}^{k-1} \sigma_j^+ \sigma_j^-)\cdot\sigma_k^+ \\ \eta_k &= (\exp i\pi \sum_{j=1}^{k-1} \sigma_j^+ \sigma_j^-)\cdot\sigma_k^- \end{aligned} \tag{32}$$

we arrive at the following identities:

$$\overset{*}{\eta}_1^{P_1} \ldots \overset{*}{\eta}_n^{P_n} |0\rangle_B = (\sigma_1^+)^{P_1} \ldots (\sigma_n^+)^{P_n} |0\rangle_B = \overset{*}{\xi}_1^{P_1} \ldots \overset{*}{\xi}_n^{P_n}|0\rangle_B \tag{33}$$

in the state space of the Bose system. Hence:

$$H_B \overset{*}{\eta}_1^{P_1} \ldots \overset{*}{\eta}_n^{P_n} |0\rangle_B = PH_BP\overset{*}{\eta}_1^{P_1} \ldots \overset{*}{\eta}_n^{P_n}|0\rangle_B = H_F\overset{*}{\eta}_1^{P_1} \ldots \overset{*}{\eta}_n^{P_n}|0\rangle_B \tag{34}$$

The relevant information at this point is that the projection P though defined in terms of $\overset{*}{\xi}_k, \xi_k$, (and only through a^*_k, a_k expansions of $\overset{*}{\xi}_k, \xi_k$ in terms of the initial Bose operators) is nevertheless a projection on the state (sub)space $\mathcal{H}_F$ in $\mathcal{H}_B$, which includes all possible Fermi states of the Bose (CCR) algebra constructed about the Bose vacuum. In fact we have: $P(\xi^*,\xi)\mathcal{H}_B = P(a^*,a)\mathcal{H}_B = \mathcal{H}_F$.

6. Boson-fermion unduality in continuum: joint Bose-Fermi spectral problems for quantum fields in 1+1.

Let us consider the nonlinear Schroedinger field:

$$H = \tfrac{1}{2}\int \nabla\phi^*\nabla\phi dx + \tfrac{1}{2}\iint \phi^*(x)\phi^*(y)V(x-y)\phi(x)\phi(y)dxdy$$

$$[\phi(x),\phi^*(y)]_- = \delta(x-y) \qquad [\phi(x),\phi(y)]_- = 0 \tag{35}$$

$$V(x-y) = c\cdot\delta(x-y) \qquad c\geq 0$$

It is well known that the eigenvectors of H read as follows:

$$|k_1,\dots,k_n\rangle = \int[\prod_{j=1}^{n} dx_j \exp ik_jx_j]\times$$

$$\{\prod_{1\leq j<i\leq n} [\theta(x_j-x_i) + \theta(x_i-x_j)\frac{k_i-k_j-ic}{k_i-k_j+ic}]\}\phi^*(x_1)\dots\phi^*(x_n)|0\rangle \tag{36}$$

$$k_1< \dots <k_n \qquad \theta(x-y) = \begin{cases} 1 & x\geq y \\ 0 & x<y \end{cases}$$

$$\phi(x)|0\rangle = 0 \qquad \forall\ x\in\mathbb{R}\ .$$

The underlying Hilbert space in which (35),(36) make sense is the Fock space H_B for all $c\geq 0$. The particular $c=0$ limit of (36) reads:

$$|k_1,\dots,k_n\rangle_B = \int[\prod_{j=1}^{n} dx_j \exp ik_jx_j]\phi^*(x_1)\dots\phi^*(x_n)|0\rangle =$$
$$= (2\pi)^{n/2}\, a^*(k_1)\dots a^*(k_n)|0\rangle \tag{37}$$

$$[a(k),a(p)^*]_- = \delta(k-p)$$

which corresponds to the free Bose system.

The reverse limit of $c \to \infty$ allows to recover:

$$|k_1,\ldots,k_n>_F = \int dx_1 \ldots \int dx_n (\exp i \sum_{j=1}^{n} k_j x_j) \times$$
$$\times \sigma(x_1,\ldots,x_n)\phi^*(x_1)\ldots\phi^*(x_n)|0> \quad (38)$$

where:

$$\sigma(x_1,\ldots,x_n) = \prod_{1\le j<i\le n} [\Theta(x_i-x_j) - \Theta(x_j-x_i)] \quad (39)$$

satisfies:

$$\sigma_n^3 = \sigma_n \qquad \sigma_n^2(1-\sigma_n^2) = 0$$
$$\sigma(\ldots x_i \ldots x_j \ldots) = -\sigma(\ldots x_j \ldots x_i \ldots) \quad (40)$$

and is an example of the multiplicative alternation involved in the constructions of fermions from bosons presented in(25,26). Indeed, if to introduce the following bosonized Fermi operators:

$$b(x) = \sum_n (1+n)^{\frac{1}{2}} \int dy_1 \dots \int dy_n \, \sigma(y_1,\dots,y_n) \times$$

$$\times \bar{\sigma}(x,y_1,\dots,y_n)\phi^*(y_1)\dots\phi^*(y_n) : \exp[-\int dz \phi^*(z)\phi(z)] : \phi(x)\phi(y_1)\dots\phi(y_n)$$

$$[b(x),b(y)^*]_+ = \delta(x-y)\,1_F \tag{41}$$

where 1_F is a continuous generalization of the projection P of Section 5 (see e.g.[52]):

$$1_F = \sum_n \frac{1}{n!} \int dx_1 \dots \int dx_n \, \sigma^2(x_1,\dots,x_n)\phi^*(x_1)\dots\phi^*(x_n)$$
$$: \exp[-\int dy \phi^*(y)\phi(y)] : \phi(x_1) \dots \phi(x_n) \tag{42}$$

then, by inspection, it is easy to verify that:

$$|k_1,\dots,k_n>_F = \int dx_1 \dots \int dx_n (\exp i \sum_{j=1}^{n} k_j x_j) b^*(x_1)\dots b^*(x_n)|0> =$$

$$= (2\pi)^{n/2} \, b^*(k_1)\dots b^*(k_n)|0> =$$

$$= \sigma(k_1,\dots,k_n)\int dx_1 \dots \int dx_n (\exp i \sum_{j=1}^{n} k_j x_j) a^*(x_1)\dots a^*(x_n)|0> =$$

$$= (2\pi)^{n/2} \, \sigma(k_1,\dots,k_n) a^*(k_1) \dots a^*(k_n)|0> . \tag{43}$$

The free Fermi system thus arises in the $c \to \infty$ limit.

Here the continuous transition from $c=0$ to $c=\infty$ results in the contrac-

tion of the dynamically accessible state space of the Bose system from the Fock space $\mathcal{H}_B$ to its proper subspace $1_F\mathcal{H}_B=\mathcal{H}_F$.
The joint Bose-Fermi spectral problem relates here the free Bose and Fermi systems:

$$H_B^o = H_F^o + (1-1_F)H_B^o(1-1_F) \tag{44}$$

$$H_F^o = 1_F H_B^o 1_F$$

but one should realize that its extensions to the interacting systems in 1+1 were found: the examples of the massive Thirring and chiral invariant Gross-Neveu models satisfy (3), see(37,52,55).
Let us here mention that the existence of the Bose variant of the (conventionally considered as) Fermi model allows to establish a passage from the Fermi model to its well defined classical (non-Grassmann) partner, see(51,52).

7. Mathematics of bosonization: representations of the CAR generated by representations of the CCR.

As mentioned before, the fermions-from-bosons construction which is general enough to account for any number of space time dimensions and is in fact model -independent, was invented in the years 1972-3(25,26) in the attempt to get rid of Grassmann algebras while quantizing spinor

fields via path integration, see also(51,52,55).
So far the straight applicability of this construction was unambigously verified for space dimensions lower than three. In case of 1+3 the partial results were obtained(37,57): the physical meaning of such (a priori realizable) bosonization is yet obscure in the Minkowski space. Compare e.g. the interpretational problems of(53).
Our model independent bosonization originates from the study of isomorphisms between certain subspaces of antisymmetric and symmetric wave functions in the Hilbert space. Since such wave functions are used in the so called Fock construction of domains for field operators, we can combine this study with the demand of the equal time (anti)commutation relations. The result is that each CCR algebra carries its Fermi partner(s). From the algebraic point of view bosons can be viewed as primary against fermions.
Since the construction of(25,26) is not broadly known, it is not useless to bring it into light again especially because it is the only universal CAR=CAR(CCR) embedding, and because its applicability in different areas of physics could be verified.
Let K be a complex Hilbert space (e.g. $K=L^2(R^N)$ or $K=\bigoplus_{i=1}^{n} h_i$, $h_i=L^2(R^N)$ $\forall i$). We denote $H_n=K^{\otimes n}=\bigotimes_1^n K$. Let E_n be a bounded operator acting on the n-th tensor product $\bigotimes_1^n K=H_n$ with properties:

$$E_n^3 = E_n \quad , \qquad E_n^* = E_n \quad , \quad P_{ik}E_n = -E_nP_{ik} \tag{45}$$

where P_{ik} is an operator of permutation of the i-th and k-th K entry in

$K^{\otimes n}$. Thus E_n^2 is a projection and induces the following decomposition of H_n:

$$H_n = \overset{1}{H}_n \oplus \overset{2}{H}_n$$
$$E_n^2 H_n = \overset{1}{H}_n , \qquad (1-E_n^2)H_n = \overset{2}{H}_n \tag{46}$$

One may prove (26) that E_n is an automorphism of $\overset{1}{H}_n$ consisting in particular of isomorphisms:

$$E_n: \ S_n \overset{1}{H}_n \longleftrightarrow A_n \overset{1}{H}_n \tag{47}$$

where S_n, A_n stand for symmetrizing and antisymmetrizing Young's operators in the n-th tensor product.
Since E_n is a homomorphism $H_n \rightarrow \overset{1}{H}_n$ with the kernel $\ker E_n = \overset{2}{H}_n$ it is quite desirable to demand that E_n allows for:

$$A_n \overset{2}{H}_n = 0 \tag{48}$$

We are interested in mappings between symmetric and antisymmetric wave functions which enter the so called Fock construction of state spaces for quantum fields:

$$F = \{f=\{f_n\}_{n=0,1,\dots},\ f_n \in H_n,\ \|f\| < \infty\} = \bigoplus_{n=0}^{\infty} H_n \tag{49}$$

where: $H_o = \mathbb{C}$,

$$\mathcal{F}_S = \overset{1}{\mathcal{F}}_S \oplus \overset{2}{\mathcal{F}}_S \qquad \mathcal{F}_A = \overset{1}{\mathcal{F}}_A \oplus \overset{2}{\mathcal{F}}_A$$
$$\overset{i}{\mathcal{F}}_{S(A)} = \bigoplus_0^\infty \overset{i}{\mathcal{H}}{}_n^{S(A)} \qquad \mathcal{H}_n^S = S_n \mathcal{H}_n , \quad \mathcal{H}_n^A = A_n \mathcal{H}_n \tag{50}$$

and:

$$\| f_S \|^2 = \sum_0^\infty \| f_n^S \|^2 < \infty$$
$$\| f_A \|^2 = \sum_0^\infty \| \overset{1}{f}{}_n^A \|^2 = \sum_0^\infty \| \overset{1}{f}{}_n^S \|^2 = \| \overset{1}{f}_S \|^2 \tag{51}$$

provided we make use of:

$$f_n^A = E_n f_n^S . \tag{52}$$

Examples of the operator E_n were given in [26] e.g.

$$E_n = \sum_{i_1 \dots i_n} e_{i_1} \otimes \dots \otimes e_{i_n} \varepsilon_{i_1 \dots i_n} \bar{e}_{i_1} \otimes \dots \otimes \bar{e}_{i_n} \tag{53}$$

which is defined by eigenfunctions $e_{i_1} \otimes \dots \otimes e_{i_n}$ and eigenvalues $\varepsilon_{i_1 \dots i_n}$ (the Levi-Civita tensor). Here $\{e_i\}$ forms the basis system in K. Another example is provided by the integral operator, whose kernel reads:

$$E_n(x_1,\dots,x_n;y_1,\dots,y_n) = \sigma(x_1,\dots,x_n)\delta(x_1-y_1)\dots\delta(x_n-y_n) . \tag{54}$$

Here $\sigma_n=\sigma(x_1,\dots,x_n)$ is the Friedrichs-Klauder totally antisymmetric symbol which equals 0 if any two variables coincide, ±1 otherwise depending on the (even or odd) permutations of variables. All variables may stand for either pure space or space plus discrete labels of internal degrees of freedom.

Now given a (Fock) representation of the CCR algebra generated by:

$$[a(f),a(g)^*]_- = (f,g)$$

$$[a(f),a(g)]_- = 0 \qquad (f,g) = \int dx\bar{f}(x)g(x) \tag{55}$$

$$a(f)|0\rangle = 0 \quad \forall f\in K \qquad a(f) = \int dx\bar{f}(x)a(x)$$

where in the presence of the discrete labels the integral should be viewed as a symbol for both integration and summation. The domain for (55) consists of normalizable vectors of the form:

$$|f\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int dx_1 \dots \int dx_n\, f(x_1,\dots,x_n)a^*(x_1) \dots a^*(x_n) \tag{56}$$

where $f(x_1,\dots,x_n)$ is a symmetric wave function and:

$$\sum_n \int dx_1 \dots \int dx_n\, |f(x_1,\dots,x_n)|^2 < \infty \tag{57}$$

The previous isomorphisms can be used to prove the CAR=CAR(CCR) representation theorem of [26]. Let $\forall n$ $E_n(\underline{x};\underline{y})=E_n(x_1,\dots,x_n;y_1,\dots,y_n)$ be an integral kernel of E_n in $K^{\otimes n}$. Then the operators $b(f)$, $b(g)^*$ given by:

$$b(f) = \sum_{n\,m} \frac{1}{\sqrt{n!\ m!}} \int dx_1 \ldots \int dx_n \int dy_1 \ldots \int dy_m$$

$$f_{nm}(\underline{x};\underline{y})\ a^*(x_1) \ldots a^*(x_n):\ \exp\{-\int dz a^*(z) a(z)\}: a(y_1) \ldots a(y_m) \qquad (58)$$

with:

$$f_{nm}(\underline{x};\underline{y}) = \sqrt{1+n}\ \delta_{m,1+n} \int dq \int dz_1 \ldots \int dz_n ,$$

$$E_n(\underline{x};\underline{z}) \bar{f}(q)\ E_{1+n}(q,\underline{z};\underline{y}) \qquad (59)$$

generate a Fock representation of the canonical anticommutation relations algebra (CAR):

$$[b(f), b(g)^*]_+ = (f,g) 1_F$$

$$[b(f), b(g)]_+ = 0 \qquad (60)$$

$$b(f)|0\rangle = 0 \qquad \forall\ f \in k$$

on the Fock space H_B (which via the Fock construction is isomorphic to F). The representation becomes irreducible on the proper subspace of H_B determined by the operator unit 1_F of the CAR algebra, which is a projection in H_B

$$\mathcal{H}_F = 1_F \mathcal{H}_B$$

$$1_F = \sum_n \frac{1}{n!} \int dx_1 \dots \int dx_n \int dy_1 \dots \int dy_n \int dz_1 \dots \int dz_n$$

$$E_n(\underline{x};\underline{y})E_n(\underline{y};\underline{z})a^*(x_1) \dots a^*(x_n) \qquad (61)$$

$$: \exp\{- \int dq\ a^*(q)a(q)\} : a(z_1) \dots a(z_n) \ ,$$

$$1_F|f> = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int dx_1 \dots \int dx_n (E_n^2 f)(x_1,\dots,x_n)a^*(x_1)\dots a^*(x_n)|0>.$$

Notice that if to insert into (58)-(61) an example (53) of the integral kernel (and constrain it to one space dimension), we arrive at the operator formulas (41)-(43) discussed in the context of the nonlinear Schroedinger model. One should be aware that the multiplicative alternation σ_n in space dimension higher than one has a symbolic meaning merely. It is a conventional function of space variables in 1+1 only[60].
It is easy to deduce the lattice analogs of the construction (58)-(60), see[37,55].
In the above the Fock state $|0>$ is common for the Bose and Fermi systems. In terms of the Schroedinger representation of the CCR algebra it means that the symmetrized ground state is allowed for the Fermi system as well. For further discussion of this issue see[56].
It should be emphasized that in (58)-(61) the overall number of internal degrees of freedom must be the same for bosons and fermions. As noticed in[37,67,46] it is however possible to impose constraints which

diminish the number of effective boson degrees, so that in principle the standard 1 boson - 2 fermions mapping can appear in 1+1. At this point it is quite instructive to recall the studies of[46] and[53] where the relationship of internal energies of the boson and fermion mode in thermal bath was established:

$$E_F = (1 - 2^{-d})E_B \tag{62}$$

d standing for space dimension. In 1+1 it amounts to the imaginative statement that each fermion mode is equivalent to $\frac{1}{2}$ of the boson one, while in 1+3 the respective fermion to boson ratio equals $\frac{7}{8}$. Consistently the duality assumption:

$$\begin{aligned} N_F E_F &= N_B E_B \quad \rightarrow \\ N_F(1 - 2^{-d}) &= N_B \end{aligned} \tag{63}$$

would need the number of two fermion components against the single boson one, while (optionally) 8 fermions against 7 bosons in 1+3. One has thus explicitly revealed how the duality assumption (63) is used to hide rather than to cure the apparent boson-fermion unduality (62). The most important problem which cannot as yet be adequately discussed, is the physical meaning of the bosonic constituents which are (non-locally) to compose fermions in 1+3. Some preliminary suggestions in this connection can be found in[37,55], see also at our discussion of formulas (4),(5).

The magnetic monopole or dyon alternative(33,34,47) cannot be excluded as well.

Acknowledgment:

I would like to thank Professor Pratul Bandyopadhyay for his interest in the topic, which has resulted both in my stay in Calcutta, as well as at the presentation of this talk at the International Conference organized 60 years after the quantum statistics discovery by Satyendra Nath Bose.

References:

1. L.de Broglie, C.R.Acad.Sci. (Paris), 195,(1932),862
2. P. Jordan, Z.Phys.93,(1935). 464, ibid 99, (1936), 109 ibid 102, (1936), 243, ibid 105, (1937), 229
3. K. Pryce, Proc.Roy.Soc. A 165, (1938), 247
4. T. Holstein, H. Primakoff, Phys.Rev. 58, (1940), 1098
5. F. Dyson, Phys.Rev. 102, (1956), 1217
6. R. Jost, The general theory of quantized fields, AMS, Providence, 1965
   It is not useless to mention that against the wide spread opinion the relation between the causal independence and local commutativity still remains a postulate (than a proved statement rather), see H. Ekstein, Phys.Rev. 184, (1969), 1315, B. De Facio, D.C. Taylor, Phys.Rev. D8, (1973), 2729
7. W. Heisenberg, Introduction to the unified theory of elementary particles, Wiley, Interscience, NY. 1966, see also Y. Nambu, G. Jona-Lasinio, Phys.Rev. 122, (1961), 345
8. T.H.R. Skyrme, Proc.Roy.Soc. A 247, (1958), 260, ibid A 262, (1961), 237
9. T. Marumori, Progr. Theor.Phys. 24, (1960), 331
10. K. Case, Phys.Rev. 106, (1957), 1316
11. D.K. Sen, Nuovo Cimm. 31, (1964), 660
12. B. Ferretti, G. Venturi, Nuovo Cimm. 35, (1964), 644
13. P. Bandyopadhyay, Nuovo Cimm. 38, (1965), 1912
14. W. Perkins, Phys.Rev. 137, (1965), B 1291, and Phys.Rev. D5 (1972), 1375
15. H. Sarkar, B. Bhattacharya, P. Bandyopadhyay, Phys.Rev. D11 (1975), 935
16. J. Mickelsson, The vector form of the neutrino equation and the photon-neutrino duality, RITP Helsinki preprint, 1984
17. L. Biedenharn, J. Louck, Angular momentum in quantum physics, Addison-Wesley, Reading, 1981

18. R. Penney, J.Math.Phys. 6, (1965), 1026, ibid 1031
19. R.F. Streater, I. Wilde, Nucl.Phys. B24, (1970), 561
20. K. Kademova, Int.Journ.Theor.Phys.3, (1070), 109
21. A.J. Kalnay, E. Mac Cotrina, K. Kademova, Int.Journ.Theor.Phys. 7, (1973),9
22. C. Gonzalez-Bernardo, A.J. Kalnay, Int.Journ.Theor.Phys.22, (1983) 1037
23. Y. Freundlich, Nucl.Phys.B36, (1972), 621
24. S. Okubo, Phys.Rev.C 10, (1974),2048
25. P. Garbaczewski, J. Rzewuski, Rep.Math.Phys.6, (1974), 424
26. P. Garbaczewski, Commun.Math.Phys.43, (1975),131
27. A. Luther, I. Peschel, Phys.Rev.B9 (1974),131
28. S. Coleman, Phys.Rev.D11, (1975),2088
29. S. Mandelstam, Phys.Rev.D11, (1975),3026
30. M. Halpern, Phys.Rev.D12, (1975),1684
31. M. Kaku, Phys.Rev.D12, (1975),2330
32. V. Baluni, Phys.Lett.90B,(1980), 407
33. R. Jackiw, C. Rebbi, Phys.Rev.Lett.36, (1976),1116
34. A.S. Goldhaber, Phys.Rev.Lett.36, (1976), 1122
35. N. Nakanishi, Progr.Theor.Phys.57,(1977),269 and Z.Phys.C4,(1980),17
36. Y. Nakowaki, Progr.Theor.Phys.72, (1984),134,ibid 152
37. P. Garbaczewski, J.Math.Phys.19,(1978),642 ibid 23,(1982), 442, ibid 24, (1983), 1806, ibid 25, (1984), 862 also Physics Rep. 36C, (1978), 65
38. V. Zhelnorovich, Sov.Phys.Dokl.24, (1979), 899
39. A. Luther, Phys.Rev.B19, (1979),320
40. I.B. Frenkel, Journ.Funct.Anal.44, (1981),259)
41. P. Goddard, D. Olive, Algebras, Lattices and strings, DAMPT Cambridge preprint, 1983
42. J. Dobaczewski, Nucl.Phys.A369, (1981), 219
43. V. Zhelnorovich, Theory of spinors with applications in physics and mechanics, (in Russian), Nauka, Moscow, 1982
44. Y. Takahashi, Phys.Rev.D26 (1982,2169

45. Y. Takahashi, K. Okuda, Fortschr.Physik, 31, (1983),511
46. H. Aratyn, Nucl.Phys.B227,(1983),172
47. R. Sorkin, Phys.Rev.D27,(1983),1787
48. J.L. Friedman, R. Sorkin, Commun.Math.Phys.73,(1980),161
49. M. Apostol, J.Phys.C16,(1983),5936
50. P. Garbaczewski, J.Math.Phys.24,(1983), 641
51. P. Garbaczewski, Nucl.Phys.B218,(1983),321
52. P. Garbaczewski, Ann.Phys.(NY),150,(1983)-22
53. A. Luther, K.D. Schotte, Nucl.Phys.B242,(1984),269
54. E. Witten, Commun.Math.Phys.92,(1984),455, see also Y.K.Ha, Phys. Rev.D29,(1984),1744
55. P. Garbaczewski, Classical and quantum field theory of exactly soluble nonlinear systems, World Scientific, Singapore, 1985(in press)
56. P. Garbaczewski, Boson approximants for lattice Fermi systems, J.Math.Phys. to appear
57. P. Garbaczewski, Fermi states of Bose systems in three space dimensions, J.Math.Phys.to appear
58. S.G. Rajeev, Phys.Rev.D30, (1984),...
59. R. Mirman, Nuovo Cim.18B, (1973),110
60. M.D. Girardeau, J.Math.Phys.10,(1969),1302
61. J.M. Leinaas, J.Myrheim, Nuovo Cim.37B, (1977),1
62. H.S. Green, Phys.Rev.90, (1953),270
63. J. Bardeen, L. Cooper, J. Schrieffer, Phys.Rev.108,(1957),1175
64. A.J. Coleman, Phys.Rev.Lett.13, (1964),406
65. M.D. Girardeau, J.Math.Phys.4, (1963),1096, ibid 11, (1970),681
66. H. Barentzen, Phys.Rev.B28, (1983),4143
67. H. Aratyn, Phys.Lett.113B, (1982),248