Fortschr. Phys. 38 (1990) 6, 447–475

Randomness in the Quantum Description of Neutral Spin 1/2 Particles

PIOTR GARBACZEWSKI

Institute of Theoretical Physics, University of Wrocław,
PL-50 205 Wrocław, Poland

Summary

Recently the path integral quantization of the classical spin model was accomplished by Nielsen and Rohrlich. The configuration space of classical spin is the unit sphere with punctures at poles. Motivated by the fact that the Nelson's stochastic mechanics idea was to randomize the configuration space variable of the classical system, we give a review of the probabilistic approaches to the quantization of spin 1/2, with emphasis on the Dankel's problem of finding the probabilistic description of the Pauli equation in terms of stochastic diffusion processes. The original Dankel's analysis referred to the rigid top, and the problem of the point particle limit was left unsolved. Our observation is that if the spin stochastic process refers to the unit sphere with punctures, then Dankel's results provide a solution to the Pauli equation with no need of extra limiting procedures. The effects of the magnetic field can be successfully incorporated into the formalism.

1. Motivation

The notion of stochastic spin and stochastic spin space was introduced on intuitive grounds in the quantal description of spin systems [4, 16] see also [5] to deal with points on the Liouville sphere which although randomly distributed, are nevertheless concentrated about a certain fixed point with a given variance.

In particular the above mentioned randomness has been attributed [16] to spin fluctuations, which are understood as fluctuations of the random coordinate axis along which spin is measured. They are to occur around a certain mean direction which is identifiable in the classical frame of reference.

Any notion of stochasticity needs a specification of the underlying random variable and the dynamical rules (stochastic differential or difference equations) *which govern its time development*. The principal objective of the present paper is to give a complete description of the random variables inherent in the quantum spin 1/2 notion, in the framework of the stochastic diffusion processes. Since basically there are two types of the random variables of interest, the discrete (jump processes) and continuous (diffusions), we shall examine the applicability of the jump process decoding [33, 34] of the Pauli equation to find it problematic in the absence of magnetic field, but specifically in case of the homogeneous magnetic field where the interpretation in terms of random jumps is harmed by the presence of nodes. The problem appears more serious than in case of diffusion processes where Carlen's theorem [17] seems to offer the way out.

The notion of stochastic spin is introduced in Section 4, and its dynamics in the presence of homogeneous magnetic field is described in terms of the stochastic differential equation. In Section 5 we present the description of the Stern-Gerlach quantum propagation in terms of diffusion processes, which is based on the short time propagation solution of the corresponding inhomogeneous problem.

Although a byproduct with respect to our goal, we in fact give a solution to the Problem 11 stated by Nelson in Ref. [17].

2. Spatial orientation in the quantum mechanics of two-level systems

All self-adjoint operators in the two-dimensional Hilbert space can be represented by complex matrices of the form

$$A = aI + \vec{b}\vec{\sigma} \tag{2.1}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\vec{b} \in R^3, \qquad a \in R^1. \tag{2.2}$$

Let the Liouville sphere $\{\vec{n} = (n_1, n_2, n_3), |\vec{n}| = 1\}$ in R^3 be an index set for the family of selfadjoint operators in $\mathscr{H}_2$

$$P_n = \frac{1}{2}(I + \vec{n}\vec{\sigma}), \qquad |\vec{n}| = 1. \tag{2.3}$$

They are projections since $P_n{}^2 = P_n$ and [1] all projections in $\mathscr{H}_2$ have the form (2.3). The necessary and sufficient condition for two projections $\vec{n} \neq \vec{n}'$ to be orthogonal is $\vec{n} = -\vec{n}'$.

Eigenvalues of the matrix A (2.1) read $a \pm b$ and the corresponding spectral decomposition is given by

$$A = A_n = (a + |\vec{b}|)\, P_n + (a - |\vec{b}|)\, P_{-n}$$
$$\vec{n} = \vec{b}/|\vec{b}|. \tag{2.4}$$

Observable features of quantum systems are customarily attributed to projections (propositions [1]). Let $\mathscr{L}(\mathscr{H}_2)$ be the set of all projections in $\mathscr{H}_2$. Generally states of physical systems are identified with probability measures on the set of projections. However in two dimensions Gleason's theorem does not apply, thus leaving an open field to more or less educated guesses. The typical approach amounts to selecting a subset of acceptable probability measures by demanding that the expectation value of the projection P in a given state (interpreted as a probability with which the proposition P is detectable in the state D_α) is:

$$\alpha(P) = \mathrm{tr}\,(D_\alpha P) \tag{2.5}$$

for all $P \in \mathscr{L}(\mathscr{H}_2)$, where D_α is a trace class operator of unit trace called the density operator (matrix) of the state α.

The most general form of the density operator D_α reads [3, 4]

$$D_\alpha = \frac{1}{2}(1 + \vec{\alpha}\vec{\sigma}) \qquad \vec{\alpha} \in R^3, \qquad |\vec{\alpha}| \leq 1 \tag{2.6}$$

which in case of pure states ($|\alpha| = 1$) is a projection $D_\alpha = P_\alpha$, The probabilistic characterisation of the Hermitian operator $\vec{x}\vec{\sigma}$ in the pure state P_n can be here computed to give

$$\begin{aligned}\alpha(\vec{x}\vec{\sigma}) &= \mathrm{tr}\,(\vec{x}\vec{\sigma}P_n) = \vec{x}\vec{n} = E(\vec{x}\vec{\sigma}; P_n)\\ \alpha([\vec{x}(\vec{\sigma} - \vec{n}\mathrm{I})]^2) &= |\vec{x}|^2 - (\vec{x}\vec{n})^2 = \mathrm{Var}\,(\vec{x}\vec{\sigma}; P_n)\end{aligned} \tag{2.7}$$

where E stands for the expectation value while *Var* for the mean square deviation of $\vec{x}\vec{\sigma}$ from $\vec{n}\vec{x}$ in the state P_n.

In case of $|\vec{x}| = 1$ we find

$$E(\vec{x}\vec{\sigma}; P_n) = \cos\,(\widehat{x, n}); \qquad \mathrm{Var}\,(\vec{x}\vec{\sigma}; P_n) = \sin^2\,(\widehat{x, n}) \tag{2.8}$$

where $(\widehat{x, n})$ denotes an angle between $\vec{x}$ and $\vec{n}$.

If α is a pure probability measure on $\mathcal{L}(\mathcal{H}_2)$ then it takes the form

$$\alpha(P) = (P\varphi, \varphi) \tag{2.9}$$

for all P. Here (.,.) denotes a scalar product in $\mathcal{H}_2$ and φ is a normalised unit vector in $\mathcal{H}_2$: $\|\varphi\| = 1$.

Given a normalised unit vector $\vec{n}$ such that $\vec{n}\vec{\sigma}\varphi = \varphi$ holds true. Then

$$\begin{aligned}&P_{-n}\varphi = 0 \to\\ &\varphi = (P_n + P_{-n})\,\varphi = P_n\varphi\end{aligned} \tag{2.10}$$

and consequently for any $P_{\vec{x}}$, $\vec{x} \neq \vec{n}$ there holds

$$(\varphi, P_x\varphi) = \|P_x\varphi\|^2 = (P_nP_xP_n\varphi, \varphi) = \frac{1}{2}(1 + \vec{x}\vec{n}) = \cos^2 \frac{1}{2}\,(\widehat{x, n}) \tag{2.11}$$

which is supposed to give a probability with which a proposition P_x is detectable in the state φ for which P_n is realised with probability one. Another form of (2.11) is

$$(\varphi, P_x\varphi) = \mathrm{tr}\,(P_nP_x) \tag{2.12}$$

which is also called [4] a transition probability from the state P_n to the state P_x.

Although the inapplicability of the Gleason's theorem could be overcome, still quite a serious problem remains unsettled in connection with the above probabilistic interpretation. Namely, if the notion of probability is introduced, it is rather inevitable to identify the corresponding random variable. And there is an apparent discrepancy between the above probabilistic formalism and the familiar experimental understanding. Namely, from the point of view of the experimental analysis [7] the expectation value of a Hermitian operator is said to be the average over all eigenvalues which a considered property of the particle is assumed to take in a given state. A single measurement yields a single eigenvalue and a sufficiently large number of such single measurement repetitions generally shows up all possible eigenvalues. It is the average over them which is usually compared with the quantum mechanical expectation value. Then, in connection with spin 1/2 notion we can apparently say that whereas ± 1 eigenvalues of $\vec{n}\vec{\sigma}$ represent results of single measurements, it is the polarisation $P = \langle\vec{\sigma}\rangle$ which tells us about the average of such ± 1 outcomes for the spin particle ensemble $\langle\vec{n}\vec{\sigma}\rangle = \vec{n} \cdot \vec{P}$.

The problem is that while the random variable ± 1 is related to $\vec{n}\vec{\sigma}$ it is $\vec{\sigma}$ itself which is related to the notion of spin and it is not apparent at all what kind of randomness underlies the average $\langle\vec{\sigma}\rangle$. This problem we shall address later while introducing the stochastic spin notion.

The representation (2.2) of Pauli matrices implies that $\varphi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the eigenvector of P_z, $z = (0, 0, 1)$ while $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ this of P_{-z}. Since the group of rotations in R^3 is isomorphic with $SU(2)/Z_2$ we can exploit the standard parametrization in terms of Euler angles $0 \leqq \theta \leqq \pi$, $0 \leqq \varphi \leqq 2\pi$, $0 \leqq \psi \leqq 2\pi$ by attributing to each Euler rotation $R(\theta, \varphi, \psi)$ its double valued $\pm U(\theta, \varphi, \psi)$ $SU(2)$ group image. Accordingly [6–8]

$$\vec{u} \in R^3 \to \vec{u}' = R(\theta, \varphi, \psi)\, \vec{u}$$

$$R(\theta, \varphi, \psi) = \begin{pmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi, & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi, & \sin\psi\sin\theta \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi, & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi, & \cos\psi\sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi, & \cos\theta \end{pmatrix} \tag{2.13}$$

induces

$$\vec{u}\vec{\sigma} \to \vec{u}'\vec{\sigma} = U(\theta, \varphi, \psi)\, \vec{u}\vec{\sigma}\, U(\theta, \varphi, \psi)^*$$

$$U = \begin{pmatrix} \cos\frac{\theta}{2} \exp\frac{i}{2}(\psi + \varphi), & i \sin\frac{\theta}{2} \exp\frac{i}{2}(\psi - \varphi) \\ i \sin\frac{\theta}{2} \exp\frac{i}{2}(\varphi - \psi), & \cos\frac{\theta}{2} \exp\frac{i}{2}(-\psi - \varphi) \end{pmatrix}. \tag{2.14}$$

Setting $\vec{u} = (0, 0, 1)$ we observe that:

$$R(\theta, \varphi, \psi)\, \vec{u} = (\sin\psi\sin\theta, \cos\psi\sin\theta, \cos\theta)\,. \tag{2.15}$$

The eigenvalue problem $\vec{u}\vec{\sigma}\chi = \gamma\chi$, $\gamma = \pm 1$ is solved by spinors $\chi_u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively, while the analogous problem for $\vec{u}'\vec{\sigma} = U\vec{u}\vec{\sigma}U^*$ is solved by

$$\chi_u = U \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \exp\frac{i}{2}(\psi + \varphi) \\ i \sin\frac{\theta}{2} \exp\frac{i}{2}(\varphi - \psi) \end{pmatrix}$$

$$\chi_d = U \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} i \sin\frac{\theta}{2} \exp\frac{i}{2}(\psi - \varphi) \\ \cos\frac{\theta}{2} \exp\frac{i}{2}(-\psi - \varphi) \end{pmatrix} \tag{2.16}$$

respectively. Notice that $-\vec{u}$ is related to $\gamma = 1 \leftrightarrow \chi_u = -U\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\gamma = -1 \leftrightarrow \chi_d = -U\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

For further purposes let us observe that for $\chi = \begin{pmatrix} a \exp i\zeta \\ b \exp i\xi \end{pmatrix}$, $a^2 + b^2 = 1$ the expectation value of $\vec{\sigma}$ reads [9]

$$\vec{u} = (\chi, \vec{\sigma}\chi) = \big(2ab\cos(\xi - \zeta), 2ab\sin(\xi - \zeta), a^2 - b^2\big)\,. \tag{2.17}$$

The quantum mechanical description of a non-relativistic particle with spin is provided by the Pauli equation. We confine attention to neutral particles (neutrons), hence there is to hold [6]:

$$\left(-\frac{\hbar^2}{2m}\Delta + \mu\vec{\sigma}\vec{\mathscr{H}}\right)\Psi(\vec{r}, t) = i\hbar\,\partial_t\Psi(\vec{r}, t)$$
$$\mu = \frac{3.83}{4}\frac{|e|\,\hbar}{mc} \tag{2.18}$$

$\vec{\mathscr{H}}$ stands for the magnetic field.

The spatial dependence we shall incorporate to the previous formalism by demanding that at the initial time instant of the Pauli propagation the wave function has the form:

$$\Psi(\vec{r}, 0) = \psi(\vec{r}, 0)\cdot\chi(\theta, \varphi, \psi) \tag{2.19}$$

where $\psi(\vec{r}, 0)$ is the time $t = 0$ solution of the free Schrödinger equation

$$\psi(\vec{r}, t) = \frac{1}{(\pi\Delta^2)^{3/2}}\frac{1}{\left(1 + \frac{i\hbar t}{m\Delta^2}\right)^{3/2}}\exp\frac{i}{\hbar}\vec{p}\left[\vec{r} - \vec{q}(t) + \frac{\vec{p}}{2m}t\right]$$
$$\times\exp\left\{-\frac{1}{\left(2\Delta^2 + \frac{2i\hbar t}{m}\right)}(\vec{r} - \vec{q}(t))^2\right\}$$
$$\vec{q}(t) = \vec{q} + \frac{\vec{p}t}{m} \tag{2.20}$$

describing in fact the Pauli propagation in the absence of magnetic fields. In such case the wave function while initially peaked about the phase space point $(\vec{q}, \vec{p})$ with the spatial half-width Δ evolves freely in time to a new phase space location $(\vec{q}(t), \vec{p})$ while changing its shape (the corresponding probability distribution spreads out).

In below we shall mainly refer to the semiclassical propagation regimes. Then over the time scales of interest spreading effects can be disregarded and the frozen Gaussian evolution (11—13) applies:

$$\psi(\vec{r}, t) = \frac{1}{(\pi\Delta^2)^{3/2}}\exp\left\{-\frac{1}{2\Delta^2}(\vec{r} - \vec{q}(t))^2\right\}\exp\frac{i}{\hbar}\vec{p}\left(\vec{r} - \vec{q} - \frac{\vec{p}t}{2m}\right). \tag{2.21}$$

In fact for neutrons with the de Broglie wave-length ~ 2 Å the (mean) velocity is ~ 2000 m/s. The uncertainty relation $\Delta\cdot\Delta_v = \hbar/2m$ implies that the velocity dispersion $\Delta_v \sim 10$ m/s corresponds to the spatial dispersion $\Delta \sim 10^{-3}$ m. It in turn implies the validity of the classical description [10] on the time scales $T = m\Delta^2/\hbar \sim 10$ s.

For $v = 2000$ m/s neutrons, the mean transit time on a distance of 10 m is less than $5\cdot 10^{-3}$ while typical distances which appear in neutron experiments are much smaller.

Remark 1: By the particle beam (statistical ensemble notion) we understand the result of the operational recipe of injecting single neutrons into the experimental arrangement [14]. Their mean velocity and initial position-momentum dispersions are controlled in a reproducible series of single particle procedures (state preparation). It is then the totality of all single particle flights through the arrangement, which constitutes the particle beam. In particular for the mean velocity v neutrons, and the source-detector distance being L the initial wave function peak after the mean transit time $L/v = T$

approaches the detector location: it means that a substantial fraction of neutrons reaches the detector after about the mean transit time. If the detector active (neutron sensitive) area has a spatial extent exceeding the weave function half-width, then the operational localisation of the one-particle quantum mechanical state is possible with a confidence level [15, 16] close to 1, albeit never equal 1.

Remark 2: Although in the non-relativistic quantum mechanics the sharp localisation concept is mathematically well defined and the probability distribution $|\psi(\vec{r}, t)|^2$ in fact refers to the spatial distribution of point particle representatives of the beam, one should always keep in mind that we deal with a drastic oversimplification. The detection of a single particle is always spatially unsharp [5, 16]. One may here refer to the particle extension proper to each microparticle. On the other hand, the literal understanding of quantum mchanical wave functions is possible in the framework of Nelson's stochastic mechanics [17] in terms of stochastic trajectories (sample paths) of certain stochastic processes. Then the operational localisation of the one-particle state does automatically involve an element of stochasticity. Wave functions appear to govern the statistical dynamics of collections of individual stochastic trajectories. Although this dynamics is provided by the Schrödinger equation simultaneously for all of them, it is the particle beam notion which involves certain set of realized (among possible to follow scenarios) sample paths, of the individual particle motion. Hence it is quite natural to associate the stochastic extension notion with one-particle states. The Gaussian half-width is an example of the appropriate stochastic extension measure.

Let us however emphasise that the stochastic extension concept is very different from the individual particle extension, which we in fact disregard in further considerations (the point particle notion is used). Apart from the variety of proposals, take e.g. [15, 18—20] as a sample, the origin of the proper particle extension is as yet not deducible within the framework of quantum theory.

3. Randomness in spin space: Jump processes viewpoint

As long as the magnetic field is spatially homogeneous, the space and polarisation (Euler angles) variables remain uncoupled. It is thus rather easy to isolate the spin space stochasticity from this related to spatial motions [17, 13, 14].

As mentioned in Section 1 to have a proper description of stochastic features associated with the notion of spin, we must identify the random variable involved.

There is no general agreement in the literature on this issue. One may think [4] of the Liouville sphere to consist of stochastic points, which in a particular spin state are peaked about the polarisation $\vec{n}$ with the variance $\mathrm{Var}\,(\vec{u}\vec{\sigma})_n = \sin^2 (\widehat{n, u})$ referring to $E(\vec{u}\vec{\sigma})_n = \cos (\widehat{n, u})$. It obviously presupposes the classically inspired spin model: *classical spin vectors are subject to stochastic fluctuations of their direction.* An analysis of this problem within Nelson's stochastic mechanics was attempted in [3] see also [17, 22, 23]. The path integral derivation of the spin space propagator [24—26] follows the same idea, see also [27, 28].

However the above intuitions do not apparently fit to the probabilistic framework of Section 2.

At this point we may take an inspiration from the algebraic isomorphism between spinor rotations and the laser stimulation [29, 30] of transitions between two atomic or molecular states of any total angular momentum. In this case a concept of rotation is broadened to include not only the physical orientation alterations, but also the stimulated transitions between two selected states of the system. Another motive behind is obviously the ± 1 would be random variable of Section 2 and the fact that the path integration over discrete spin space is known [31, 32] and there is available a probabilistic analysis of the Pauli equation in terms of dichotomic (i.e. two valued) random variables [34, 19] see also [23, 35].

For neutral spin 1/2 particles (e.g. neutrons) the interaction Hamiltonian

$$H_{\text{spin}} = \mu\vec{\sigma}\vec{\mathcal{H}} \tag{3.1}$$

implies the Schrödinger evolution of the spin state

$$|\psi, t) = \exp\left(-iH_{\text{spin}}t/\hbar\right)|\psi, 0). \tag{3.2}$$

By denoting $\vec{\mathcal{H}} = \mathcal{H}\vec{n}$, $|\vec{n}| = 1$ and exploiting the diagonalisation formulas, we obtain:

$$\begin{aligned} H_{\text{spin}} &= \mu\mathcal{H}\vec{n}\vec{\sigma} = \hbar\omega\vec{n}\vec{\sigma} \\ \vec{n}\vec{\sigma} &= P_n - P_{-n} \end{aligned} \tag{3.3}$$

which obviously identifies Zeeman energies relative to the direction $\vec{n}$.

As a consequence, to implement the individual spin flip i.e. jump from one Zeeman state to another, we must allow the particle either to absorb or to emit the energy $\hbar\omega$. This effect is well known from the spin resonance experiments [36, 37] where the polarisation of the spin state becomes reversed, see also [29, 20, 28].

Let us consider the spin state dynamics in the magnetic field appropriate [39] for Rabi experiments $\vec{\mathcal{H}} = (\mathcal{H}_1 \cos \omega t, \mathcal{H}_1 \sin \omega t, \mathcal{H}_0)$:

$$\begin{aligned} &i\hbar\, \partial_t |\psi, t) = \mu(\vec{\mathcal{H}}\vec{\sigma})\,|\psi, t) \\ &|\psi, 0) = \begin{pmatrix}1\\0\end{pmatrix} \to |\psi, t) = \begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}(t) \end{aligned} \tag{3.4}$$

which implies:

$$\begin{aligned} \chi_+(t) &= \left[\cos \delta t + \frac{i}{2}\,\frac{\omega - \nu}{\delta}\sin \delta t\right]\exp\left(-i\,\frac{\omega t}{2}\right) \\ \chi_-(t) &= -i\,\frac{\nu\Delta}{\delta}\exp\left(i\,\frac{\omega t}{2}\right)\cdot \sin \delta t \\ \delta &= \frac{1}{2}\left(\omega^2 + q^2 + 2\omega q \cos\theta\right)^{1/2}, \qquad q = \nu(1 + 4\Delta^2)^{1/2} \\ \nu &= 2\mu\mathcal{H}_0/\hbar, \qquad \Delta = \mathcal{H}_1/2\mathcal{H}_0 \\ \tan\theta &= \frac{\mathcal{H}_1}{\mathcal{H}_0}. \end{aligned} \tag{3.5}$$

The smooth dependence on ω allows to set $\omega = 0$, when we are left with $\vec{\mathcal{H}} = (\mathcal{H}_1, 0, \mathcal{H}_0)$, $\delta = (\mu/\hbar)\,(\mathcal{H}_0^2 + \mathcal{H}_1^2)^{1/2}$

$$\begin{aligned} \chi_+(t) &= \cos \delta t - i\,\frac{\mu\mathcal{H}_0}{\hbar\delta}\sin \delta t \\ \chi_-(t) &= -i\,\frac{\mu}{\hbar\delta}\,\mathcal{H}_1 \sin \delta t. \end{aligned} \tag{3.6}$$

If we set $\mathscr{H}_0 = 0$ then

$$\chi_+(t) = \cos \omega_1 t \qquad \omega_1 = \frac{\mu}{\hbar} \mathscr{H}_1$$
$$\chi_-(t) = -i \sin \omega_1 t. \tag{3.7}$$

Our initial polarisation assumption was $|\psi, 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ so that $|\psi, 0)$ is the eigenvector of the projection P_z, $\vec{z} = (0, 0, 1)$. By $\vec{u}(t)$ we denote the polarisation of $|\psi, t)$. Let us employ the Dirac notation

$$P_z = |\uparrow)(\uparrow| \qquad P_{-z} = |\downarrow)(\downarrow| . \tag{3.8}$$

Then $|\uparrow) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\psi, 0)$, $|\downarrow) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the expectation value of Section 2 reads:

$$(\psi, t| P_z |\psi, t) = (\psi, t \mid \uparrow)(\uparrow \mid \psi, t) = |(\uparrow \mid \psi, t)|^2 = \cos^2 \frac{1}{2} \left(\widehat{z, u(t)}\right) = |\chi_+(t)|^2 \tag{3.9}$$

while for the general solution (3.5) we have:

$$(\psi, t| P_{-z} |\psi, t) = \sin^2 \frac{1}{2} \left(\widehat{z, u(t)}\right) = |\chi_-(t)|^2 = \frac{\nu^2 \Delta^2}{\delta^2} \sin^2 \delta t . \tag{3.10}$$

Both (3.9) and (3.10) are generally [38, 39] interpreted as probabilities for the spin flips to occur. This probabilistic interpretation retains its validity in the special case (3.7) as well, and introduces the notion of the discrete random variable for a stochastic analysis of the spin dynamics, even if null Zeeman energies are related to spin flips. However sin such case the physical background of the analogy between spinor rotations and stimulated transitions breaks down. We must then try to find out what is physical difference between the spin-up and down states when no energy splitting occurs. This problem cannot be solved within the jump process framework and we shall come back to it in Section 4.

A detailed jump process description in terms of random spin flips can be given following [33]. Let $\vec{n}$ be the initial polarisation of the spin state. The expectation value

$$(\psi, t| P_{\varkappa n} |\psi, t) = \frac{1}{2} \langle 1 + \gamma \vec{n}\vec{\sigma} \rangle_t = \varrho(t, \gamma), \qquad \gamma = \pm 1 \tag{3.11}$$

gives us a probability distribution of the observable $\gamma \vec{n}\vec{\sigma}$ at time t.

Let us absorb the coefficient $2\mu/\hbar$ in the definition of $\vec{\mathscr{H}}$. The Pauli equation in case of the homogeneous field implies the continuity equation:

$$\partial_t \varrho(t, \gamma) = \frac{\gamma}{2} \vec{n} \times \vec{\mathscr{H}} \cdot \langle \vec{\sigma} \rangle_t \tag{3.12}$$

with the convention $(\vec{n} \times \vec{\mathscr{H}})_i = \varepsilon_{ijk} n_j \mathscr{H}_k$ ($\vec{\mathscr{H}} \times \vec{n}$ was originally used in [33]).

Once having chosen the initial polarisation, we may think of (3.12) to represent the forward Kolmogorov equation for the Markov process $\sigma(t)$ with values in $Z_2 = (-1, 1)$

$$\frac{d\varrho(t, \gamma)}{dt} = -p(t, \gamma)\, \varrho(t, \gamma) + p(t, -\gamma)\, \varrho(t, -\gamma) \tag{3.13}$$

where $\varrho(t, \gamma)$ is the probabil
supposed to represent the ju
per unit time form the sta
identification [33]:

$$p(t, \gamma) = \frac{1}{2} \Big[|\vec{n} \times$$

Let us indicate that the abo
zero component perpendicu
the meaning of $p(t, \gamma)$ for
$\omega_1 = \mu/\hbar \mathscr{H}_1$. Since $\vec{n} = (0$

$$\varrho(t, +) = \cos^2 \omega_1 t$$
$$\frac{2\mu}{\hbar} (\vec{n} \times \vec{\mathscr{H}}) = 2\omega$$

which implies

$$p(t, +) = \omega_1 [|\tan$$
$$p(t, -) = \omega_1 [|\cot$$

The singular behaviour is
phasise that the original
the differential equations
densities) and the Pauli e
Although the nodal surfa
cesses framework, the si
worse: $p(t, +) \to \infty$ as ω_1
We view thus (3.14) as
can state the reverse.
As follows from (3.16) t
after $\omega_1 t = \pi$ it comes ba
ent spin flip scenarios.
down direction grows up
$p(t, -)$ in turn grows tow
spin-up and spin-down in
tradict the well founded in
way spin flips only in ea
As is well known the sp
phase effect is completely
known that this sign ch
distinguish atoms which
again to the original sta
46, 29, 30]. The analogo
although the jump proce
difference between the s
Zeeman energy splitting

where $\varrho(t, \gamma)$ is the probability distribution of the random variable $\sigma(t)$ while $p(t, \gamma)$ is supposed to represent the jump (spin flip along the polarisation direction n) probability per unit time form the state $\gamma \in Z_2$ to $-\gamma$. This interpretation is consistent upon the identification [33]:

$$p(t, \gamma) = \frac{1}{2}\left[|\vec{n} \times \vec{\mathcal{H}}| \left(\frac{\varrho(t, -\gamma)}{\varrho(t, \gamma)}\right)^{1/2} - \gamma \frac{\vec{n} \times \vec{\mathcal{H}}\langle\vec{\sigma}\rangle_t}{2\varrho(t, \gamma)}\right]. \tag{3.14}$$

Let us indicate that the above formula allows the spin flips to occur only if $\vec{\mathcal{H}}$ has a non-zero component perpendicular to $\vec{n}$. If we replace $\vec{\mathcal{H}}$ by $(2\mu/\hbar)\,\vec{\mathcal{H}}$ in (3.14) we can analyze the meaning of $p(t, \gamma)$ for a specific example (3.7). We set $\vec{\mathcal{H}} = (\mathcal{H}_1, 0, 0)$ and denote $\omega_1 = \mu/\hbar\mathcal{H}_1$. Since $\vec{n} = (0, 0, 1)$ we have:

$$\begin{gathered} \varrho(t, +) = \cos^2 \omega_1 t \qquad \varrho(t, -) = \sin^2 \omega_1 t \\ \frac{2\mu}{\hbar}(\vec{n} \times \vec{\mathcal{H}}) = 2\omega_1(0, 1, 0), \qquad \langle\vec{\sigma}\rangle_t = (0, -\sin 2\omega_1 t, \cos 2\omega_1 t) \end{gathered} \tag{3.15}$$

which implies

$$\begin{gathered} p(t, +) = \omega_1[|\tan \omega_1 t| + \tan \omega_1 t] \\ p(t, -) = \omega_1[|\cot \omega_1 t| - \cot \omega_1 t]. \end{gathered} \tag{3.16}$$

The singular behaviour is here induced by the nodes of $\varrho(t, \gamma)$. At this point let us emphasise that the original purpose of [33] was to show a complete equivalence between the differential equations related to stochastic variables (probabilities and probability densities) and the Pauli equation itself. It is a priori possible only if there are no nodes. Although the nodal surface problem has been partially overcome in the diffusion processes framework, the situation in the discrete case seems conceptually to be much worse: $p(t, +) \to \infty$ as $\omega_1 t \to n\pi/2$.

We view thus (3.14) as a possible consequence of the Pauli equation but by no means can state the reverse.

As follows from (3.16) the polarisation $\langle\vec{\sigma}\rangle_t$ after $\omega_1 t = \pi/2$ reverses its direction, while after $\omega_1 t = \pi$ it comes back to its initial direction. It is reflected in (2.17) by very different spin flip scenarios. Indeed for $0 \leqq \omega_1 t < \pi/2$ the spin flip probability from up to down direction grows up from 0 towards infinity, while $p(t, -) = 0$. For $\pi/2 \leqq \omega_1 t < \pi$, $p(t, -)$ in turn grows towards infinity while $p(t, +) = 0$. Thus there is *no coexistence of spin-up and spin-down individual flip possibilities at any time instant, which seems to contradict the well founded intuitions.* The stochastic process related to (3.7) involves one-way spin flips only in each $\pi/2$ interval run by $\omega_1 t$ beginning from $t = 0$.

As is well known the spinor wave function changes its sign under the 2π rotation. This phase effect is completely beyond the jump process scenario, although by now it is well known that this sign change has observable consequences [44]. In fact one is able to distinguish atoms which have undergone one transition to the second state and back again to the original state from atoms, which have undergone no transition at all [45, 46, 29, 30]. The analogous situation is known to arise in neutron interferometry, hence although the jump process description does not provide any hint, there is a detectable difference between the spin (polarisation) up and spin-down states even if there is no Zeeman energy splitting among them.

4. Randomness in spin space: Diffusion processes viewpoint

4.1 Stationary diffusions

For mass m particles mc^2 stands for their rest frame energy. Let us consider the wave function:

$$\Psi(\vec{r}, \theta, \varphi, \psi, t) = \exp \frac{i}{\hbar} (\vec{p}\vec{r} - \hbar\omega t)\, \Psi(\theta, \varphi, \psi)$$
$$\omega = \frac{\vec{p}^2}{2m} + mc^2 \tag{4.1}$$

with $\vec{p}$ fixed. Let us demand the following equation to hold:

$$i\hbar\, \partial_t \Psi(t) = \left(-\frac{\hbar^2}{2m} \Delta + \frac{1}{2I} \vec{M}^2\right) \psi(t)$$
$$\vec{M}^2 = -\hbar^2 \left\{\frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\psi^2} + \frac{\partial^2}{\partial\varphi^2}\right) - 2\frac{\cot\theta}{\sin\theta} \frac{\partial^2}{\partial\varphi\, \partial\psi}\right\}. \tag{4.2}$$

Setting

$$I = \frac{3}{8} \frac{\hbar^2}{mc^2} \tag{4.3}$$

we arrive at

$$\vec{M}^2 \Psi(\theta, \varphi, \psi) = \frac{3}{4} \hbar^2 \Psi(\theta, \varphi, \psi) \tag{4.4}$$

whose solutions in the Hilbert space $\mathscr{L}^2(SO(3))$ of the $SO(3)$ square integrable functions

$$\|\Psi\|^2 = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{2\pi} d\psi\, |\Psi(\theta, \varphi, \psi)|^2 \tag{4.5}$$

constitute the four-dimensional orthogonal system. If normalised, they read:

$$(2\pi^3)^{1/2} e_1 = i \cos\frac{\theta}{2} \exp\frac{i}{2}(\psi + \varphi)$$
$$(2\pi^3)^{1/2} e_2 = -i \sin\frac{\theta}{2} \exp\frac{i}{2}(\varphi - \psi)$$
$$(2\pi^3)^{1/2} e_3 = \sin\frac{\theta}{2} \exp\frac{i}{2}(-\varphi + \psi) \tag{4.6}$$
$$(2\pi^3)^{1/2} e_4 = \cos\frac{\theta}{2} \exp\frac{i}{2}(-\varphi - \psi).$$

Components of $\vec{M}$, $[M_i, M_j]_- = i\hbar\varepsilon_{ijk} M_k$ have the form

$$M_1 = i\hbar \left(\sin\varphi \frac{\partial}{\partial\theta} + \cot\theta \cos\varphi \frac{\partial}{\partial\varphi} - \frac{\cos\varphi}{\sin\theta} \frac{\partial}{\partial\psi}\right) \tag{4.7}$$

$$M_2 = i\hbar \left(-\cos\varphi \frac{\partial}{\partial\theta} + \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} - \frac{\sin\varphi}{\sin\theta} \frac{\partial}{\partial\psi} \right)$$

$$M_3 = -i\hbar \frac{\partial}{\partial\varphi}.$$

However one more angular momentum can be defined [22] on the linear span of (4.6)

$$[N_i, N_j]_- = i\hbar\varepsilon_{ijk}N_k, \qquad \vec{N}^2 = \vec{M}^2, \qquad [N_i, M_j]_- = 0$$

$$N_1 = i\hbar \left(\sin\psi \frac{\partial}{\partial\theta} + \cot\theta \cos\psi \frac{\partial}{\partial\psi} - \frac{\cos\psi}{\sin\theta} \frac{\partial}{\partial\varphi} \right) \tag{4.8}$$

$$N_2 = i\hbar \left(\cos\psi \frac{\partial}{\partial\theta} - \cot\theta \sin\psi \frac{\partial}{\partial\psi} + \frac{\sin\psi}{\sin\theta} \frac{\partial}{\partial\varphi} \right), \qquad N_3 = i\hbar \frac{\partial}{\partial\psi}.$$

Accordingly the eigenfunctions (4.6) can be classified by means of the maximal commuting set M^2, M_3, N_3 of operators. The wave equation (4.2) was originally exploited in [3, 22] to investigate the extended particle (rigid rotator) model of spin, and the corresponding diffusion on $SO(3)$ was completely analyzed in [3, 17].

Passing to point particles one needs a clear picture of random variables involved. Let us take a unit vector pointing out directions in space. It runs the Liouville sphere, hence is completely given in terms of two angles (cf. (2.15)). Let us puncture [26] the Liouville sphere to equip it with the north and south poles. Now, points of Liouville spheres whose poles have distinct orientations can be uniquely given in terms of Euler angles. Let us consider the unit vector $\vec{n}(t) = \vec{n}(\theta, \varphi, \psi)(t)$ as the *random variable*, which is allowed to run not only a given Liouville sphere, but all available punctured sphere orientations. We call *this random variable a stochastic spin vector as referring to random deflections* $\vec{n}(\theta, \varphi, \psi)(t)$ *on the surface of the punctured Liouville sphere, whose poles take orientation at random.*

Then $L^2(SO(3))$ can be called a stochastic spin space while the two-dimensional space of Section 2 is the standard spin space.

Given a wave function $\Psi(\theta, \varphi, \psi) = \exp(R + iS)$ which does not vanish in an open set. Accounting for the Riemann metric

$$g = \begin{bmatrix} \frac{1}{2I} & 0 & 0 \\ 0 & \frac{1}{2I\sin^2\theta} & -\frac{\cos\theta}{2I\sin^2\theta} \\ 0 & -\frac{\cos\theta}{2I\sin^2\theta} & \frac{1}{2I\sin^2\theta} \end{bmatrix} \tag{4.9}$$

we arrive at

$$u = 2\hbar dR = 2\hbar \frac{1}{R} \frac{\partial R}{\partial\theta} d\theta$$
$$v = 2\hbar dS = 2\hbar \left(\frac{\partial S}{\partial\varphi} d\varphi + \frac{\partial S}{\partial\psi} d\psi \right) \tag{4.10}$$

and after raising the coordinate indices

$$\vec{\omega}_u = g\vec{u} \qquad \vec{\omega}_v = g\vec{v} \tag{4.11}$$

defines the osmotic $\vec{\omega}_u$ and current $\vec{\omega}_v$ velocities related to the stochastic process of interest. Since we deal with angular velocities, the corresponding angular drift $\vec{\omega}_u + \vec{\omega}_v$

determines the angular momentum of the forward drift (a true random spin)

$$\vec{L} = \mathrm{I}(\vec{\omega}_u + \vec{\omega}_v). \tag{4.12}$$

The angular (osmotic and drift) velocities and expectation values of $\vec{L}$ and $\vec{L}^2$ were computed in [3] with the results $\langle\vec{L}\rangle_i^2 = 3/4\ \hbar^2$ for all i, and:

$$\begin{aligned}
&\langle\vec{L}\rangle_{i=1,2} = \frac{\hbar}{2}\vec{k} \\
&\vec{\omega}_v^{\,1} = \frac{1}{1+\cos\theta}\frac{\hbar}{2\mathrm{I}}(\vec{e}_\varphi + \vec{e}_\psi) \qquad \vec{\omega}_u^{\,1} = -\frac{\hbar}{2\mathrm{I}}\tan\frac{\theta}{2}\vec{e}_\theta \\
&\vec{\omega}_v^{\,2} = \frac{1}{1-\cos\theta}\frac{\hbar}{2\mathrm{I}}(\vec{e}_\varphi - \vec{e}_\psi) \qquad \vec{\omega}_u^{\,2} = \frac{\hbar}{2\mathrm{I}}\cot\frac{\theta}{2}\vec{e}_\theta \\
&\langle\vec{L}\rangle_{i=3,4} = -\frac{\hbar}{2}\vec{k} \\
&\omega_v^{\,3} = \frac{1}{1-\cos\theta}\frac{\hbar}{2\mathrm{I}}(-\vec{e}_\varphi + \vec{e}_\psi) \qquad \omega_v^{\,3} = \frac{\hbar}{2\mathrm{I}}\cot\frac{\theta}{2}\vec{e}_\theta \\
&\omega_v^{\,4} = \frac{1}{1+\cos\theta}\frac{\hbar}{2\mathrm{I}}(-\vec{e}_\varphi - \vec{e}_\psi) \qquad \omega_u^{\,4} = -\frac{\hbar}{2\mathrm{I}}\tan\frac{\theta}{2}\vec{e}_\theta
\end{aligned} \tag{4.13}$$

where $\{\vec{e}_\psi, \vec{e}_\varphi, \vec{e}_\theta\}$ is the basis system consisting of unit vectors about which the ψ, θ, φ Euler rotations are executed. Each of the basis functions (4.6) refers to its own *stochastic process, in the course of which the unit vector $\vec{n}$ is subject to random fluctuations about the (mean) direction $\vec{k}$ in case of e_1, e_2 and $-\vec{k}$ in case of e_3, e_4. Euler deviations are however defined relative to $\vec{k}$ in both cases.*

For a given function $e_i(\theta, \varphi, \psi)$ the mean value $(2/\hbar)\ \langle\vec{L}\rangle_i$ defines a reference direction. The random variable $\vec{n}(t) = \vec{n}(\theta, \varphi, \psi)\ (t)$ has assigned the probability distribution $|e_i(\theta, \varphi, \psi)|^2$. Thus one can justifiably tell about a stochastic spin state which is centered about k and $-\vec{k}$ directions respectively, see e.g. [4].

The general form of the stochastic differential equation describing random trajectories drawn by $\vec{n}(t)$ is

$$d\vec{n}(t) = \vec{b}(\vec{n}(t), t)\ dt + d\vec{W}(t) \tag{4.14}$$

where $\vec{b}$ is the forward drift, while $d\vec{W}(t)$ refers to the random noise. Since in our case $n(t)$ is to label points of the punctured Liouville sphere, *$d\vec{n}(t)$ refers to the rate of change of the unit vector in R^3* under an infinitesimal rotation. Apparently

$$d\vec{n}(t) = \vec{n}(t) \times d\vec{\Omega}(\vec{n}(t), t) \tag{4.15}$$

where $\times$ denotes the vector product in R^3, and the rate of change of Euler angles follows from the stochastic differential equation

$$\begin{aligned}
&d\vec{\Omega}(\vec{n}(t), t) = (\vec{\omega}_u + \vec{\omega}_v)\ dt + \sqrt{2\nu}\ d\vec{\omega}(t) \\
&\nu = \frac{\hbar}{2\mathrm{I}} = \frac{4}{3}\frac{mc^2}{\hbar}.
\end{aligned} \tag{4.16}$$

Here ν is the diffusion coefficient, while $d\vec{\omega}(t)$ represents the Wiener noise. A conventional discussion of the rotational Brownian motion can be found in [64].

The probabilistic formulas in the above refer to all possible to realise rotations (sample paths of the stochastic process). The probability density formulas $|e_i|^2 (\theta, \varphi, \psi)$ show that while being randomly triggered, the random variable $\vec{n}(t)$ resides mostly about $\theta = 0$ in case of e_1, e_4 and $\theta = \pi$ in case of e_2, e_3. Large θ (i.e. $\theta \sim \pi$) deflections from the mean location are very improbable in the course of the process.

Formulas (4.13) show that once deflected to the angle θ the vector $\vec{n}(t)$ gets subject to the osmotic angular motion which tries to restore the mean orientation i.e. $\theta = 0$ or $\theta = \pi$ respectively. The current velocity takes its smallest value for the mean direction and grows indefinitely while passing to less probable directions. Moreover $\langle \vec{L} \rangle = \pm \hbar/2\, \vec{k}$ refers to $\pm \vec{e}_\varphi$ i.e. to clok-wise or anti-clock-wise φ rotations about $\vec{k}$ respectively, thus showing a clear affinity with the standard understanding of the spin-up (down) projections in the $\vec{k}$ direction which are intuitively related to the φ precession.

4.2. Homogeneous magnetic field effects

Let $\vec{\mathcal{H}}$ denote the homogeneous magnetic field. We shall modify the equation (4.2) by adding the interaction term $(2\mu/\hbar)\, \vec{\mathcal{H}}\, \vec{M}$ to the Hamiltonian. Since $\vec{N}$ commutes with $\vec{M}$ it is obvious that the action of $(2\mu/\hbar)\, \vec{\mathcal{H}} \vec{M}$ on $\mathcal{L}^2(SO(3))$ leaves invariant the two-dimensional eigenspaces of N_3 corresponding to $\pm\hbar/2$ respectively.

In our notation $e_1(\theta, \varphi, \psi)$ and $e_3(\theta, \varphi, \psi)$ belong to the $-\hbar/2$ subspace while $e_2(\theta, \varphi, \psi)$, $e_4(\theta, \varphi, \psi)$ to the $\hbar/2$ subspace of the N_3 decomposition of $\mathcal{L}^2(SO(3))$. Notice that $M_3 e_1 = (\hbar/2)\, e_1$, $M_3 e_3 = -(\hbar/2)\, e_3$, $M_3 e_2 = (\hbar/2)\, e_2$, $M_3 e_4 = -(\hbar/2)\, e_4$.

We can thus investigate the action of $(2\mu/\hbar)\, \vec{\mathcal{H}} \vec{M}$ in each of the invariant subspaces separately, which is most transparent if to look at the matrix form of the equation involved.

Although the matrix elements of $\vec{M}$ in the $\{e_j\}$ basis can be computed explicitly by departing from the definitions (4.6), a simpler route may here be adopted.

We denote $|1) = e_1$, $|2) = e_3$ and observe that

$$M_3 = [(M_3)_{ij}] = [(i|\, M_3\, |j)] = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.17}$$

and analogously for e_2 and e_4.

By recalling the standard knowledge about effects of rotations in the two-dimensional space, we realise that an arbitrary vector $\vec{\mathcal{H}}'$, $|\vec{\mathcal{H}}'| = \mathcal{H}$ can be obtained from the vector $(0, 0, \mathcal{H})$ by an appropriate rotation. Let $\bar{\theta}$, $\bar{\varphi}$, $\bar{\psi}$ be the corresponding Euler angles. Then

$$\vec{\mathcal{H}}' \vec{\sigma} = U \mathcal{H} \sigma_3 U^*, \qquad U = U(\bar{\theta}, \bar{\varphi}, \bar{\psi}) \tag{4.18}$$

with U given in Section 2. Hence the two-dimensional matrix $\vec{\mathcal{H}} \vec{M}$ is given in terms of Pauli matrices

$$\vec{\mathcal{H}} \vec{M} \equiv \frac{\hbar}{2}\, \vec{\mathcal{H}} \vec{\sigma} \tag{4.19}$$

both in the $M_3 = \hbar/2$ and $M_3 = -\hbar/2$ subspaces of $\mathcal{L}^2(SO(3))$. The matrix form of the equation

$$\begin{gathered} i\hbar\, \partial_t\, |\psi) = H\, |\psi) \\ H = \frac{1}{2I} \vec{M}^2 + \frac{2\mu}{\hbar}\, \vec{\mathcal{H}} \vec{M} \end{gathered} \tag{4.20}$$

comes from

$$|\psi) = \sum_k a_k |k) \to \sum_k (i\hbar\, \partial_t a_k) |k) = \sum_k a_k H |e_k) \tag{4.21}$$

where

$$\sum_k a_k H |e_k) = \sum_{ijk} a_k |i) (i| H |j) (j | k) = \sum_i \left(\sum_k H_{ik} a_k\right) |i) . \tag{4.22}$$

Hence we arrive at

$$\begin{gathered} i\hbar\, \partial_t \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = H \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ H = mc^2 I + \mu \vec{\mathscr{H}} \vec{\sigma}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{gathered} \tag{4.23}$$

The obvious ansatz

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \exp\left(-\frac{i}{\hbar}\, mc^2 t\right) \chi, \qquad \chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} \tag{4.24}$$

reduces (4.24) to the Pauli equation in the spin space of Section 2. Let us consider the special cases covered by equations of Section 3.

Case $\mathscr{H} = \mathscr{H}(0, 0, 1)$

The solution of the Pauli equation is

$$\chi_+(t) = \exp(-i\omega t), \qquad \chi_-(t) = 0, \qquad \omega = \frac{\mu}{\hbar} \mathscr{H} \tag{4.25}$$

and reveals the time dependence related to the initial condition $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence we have:

$$\Psi(\theta, \varphi, \psi, t) = \exp\left[-\frac{i}{\hbar} (mc^2 + \mu \mathscr{H})\, t\right] \cdot e(\theta, \varphi, \psi) \tag{4.26}$$

with e being either e_1 or e_3. Consequently we have the previously discussed stationary stochastic process related to

$$i\hbar\, \partial_t \Psi = \frac{1}{2I'} \vec{M}^2 \psi, \qquad I' = \frac{3}{8} \frac{\hbar^2}{mc^2 + \mu \mathscr{H}}. \tag{4.27}$$

Case $\vec{\mathscr{H}} = \mathscr{H}(1, 0, 0)$

The situation becomes here more complicated. Since (3.7) applies, we get

$$\begin{aligned} a_1 &= a_1(t) = \exp\left(-\frac{i}{\hbar}\, mc^2 t\right) \cos \omega t \\ a_2 &= a_2(t) = \exp\left(-\frac{i}{\hbar}\, mc^2 t\right) (-i \sin \omega t) \\ \omega &= \frac{\mu}{\hbar} \mathscr{H} \end{aligned} \tag{4.28}$$

so that the independent solutions of (4.23) read

$$\Psi_{\text{I}} = a_1 e_1 + a_2 e_3 \qquad \Psi_{\text{II}} = a_1 e_2 + a_3 e_4 \tag{4.29}$$

i.e. we deal with a superpositions of stochastic spin states. Before proceeding further we must make an interlude on the meaning of spinor rotations.

4.3. Rotations in the stochastic spin space

Once the eigenvalue problem

$$\vec{\mathscr{H}}\vec{\boldsymbol{M}}e_j = \alpha_j e_j \qquad \alpha_j = \pm\frac{\hbar}{2} \tag{4.30}$$

is solved for $\vec{\mathscr{H}} = \mathscr{H}(0, 0, 1)$ i.e. $\mathscr{H}M_3$ we know that

$$\left(R(\bar{\theta}, \bar{\varphi}, \bar{\psi})\ \vec{\mathscr{H}}\right) \vec{\boldsymbol{M}}e_j' = a_j e_j' \tag{4.31}$$

is solved by the rotated basis system, where

$$(R\vec{\mathscr{H}})\ \vec{\boldsymbol{M}} = \vec{\mathscr{H}}(R^T\vec{\boldsymbol{M}}) = \vec{\mathscr{H}}UMU^* \qquad e_j' = Ue_j \tag{4.32}$$

and the restriction of U to the linear span of e_1, e_3 is given by

$$\hat{U} = \sum_{ik} |i)\ U_{ik}(k| \to \hat{U}\ |j) = \sum_i U_{ji}^T\ |i) \tag{4.33}$$

where U^T is the transposed 2×2 rotation matrix with $\bar{\theta}$, $\bar{\varphi}$, $\bar{\psi}$ instead of θ, φ, ψ originally employed.

The spinor rotation U^T of the e_1, e_3 basis is here accompanied by the $SO(3)$ rotation

$$R^T(\bar{\theta}, \bar{\varphi}, \bar{\psi})\ \vec{\boldsymbol{M}} = \vec{\boldsymbol{M}}' \tag{4.34}$$

of the momentum operator $\vec{\boldsymbol{M}}$, as given in the Cartesian coordinates. Since

$$\vec{\boldsymbol{M}} = (M_1, M_2, M_3) = M_1\vec{\imath} + M_2\vec{\jmath} + M_3\vec{k} \tag{4.35}$$

the formula (4.34) amounts to the $R(\bar{\theta}, \bar{\varphi}, \bar{\psi})$ rotation of the $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ frame relative to which the stochastic process was constructed. The underlying rotation is $\vec{\imath}' = R\vec{\imath}'$, $\vec{\jmath}' = R\vec{\jmath}$, $\vec{k}' = R\vec{k}$ and affects the $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ dependent formulas (4.13).

Remark: The basis vectors $\vec{e}_\psi$, $\vec{e}_\theta$, $\vec{e}_\varphi$ in (4.13) are given by

$$\begin{aligned} \vec{e}_\varphi &= \vec{k} \\ \vec{e}_\theta &= G_\varphi{}^T\vec{\imath} = (\cos\varphi, \sin\varphi, 0) = \vec{\imath}\cos\varphi + \vec{\jmath}\sin\varphi \end{aligned} \tag{4.36}$$

where $G_\varphi{}^T$ is the transposed G_φ rotation matrix [6]. The ψ rotation is performed about the vector

$$\vec{e}_\psi = G_\varphi{}^T G_\theta{}^T \vec{k} = \vec{\imath}\sin\theta\sin\varphi - \vec{\jmath}\cos\varphi\sin\theta + \vec{k}\cos\theta\,. \tag{4.37}$$

Variables θ, φ, ψ refer to random angular deflections about the Cartesian vector $\vec{k}$.

The rotation $R(\bar{\theta}, \bar{\varphi}, \bar{\psi})$ transforms the $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ basis into a new basis system $\vec{\imath}'$, $\vec{\jmath}'$, $\vec{k}'$ and thus e.g. the expectation value $\langle\vec{L}\rangle = (\hbar/2)\,\vec{k}$ is taken over to $\langle\vec{L}\rangle' = (\hbar/2)\,\vec{k}'$. *It however tells us that the stochastic deflection values θ, φ, ψ no longer refer to the vector $\vec{k}$ but are defined relative to $\vec{k}'$.* We have

$$\langle\vec{L}\rangle = \pm\frac{\hbar}{2}\,\vec{k} \xrightarrow[R(\bar{\theta},\bar{\varphi},\bar{\psi})]{} \pm\frac{\hbar}{2}\,R\vec{k} = \pm\frac{\hbar}{2}\,\vec{k}' \tag{4.38}$$

where

$$\vec{k} = (0, 0, 1) \xrightarrow[R(\bar{\theta},\bar{\varphi},\bar{\psi})]{} R\vec{k} = \vec{k}' = (\sin\bar{\psi}\sin\bar{\theta}, \cos\bar{\psi}\sin\bar{\theta}, \cos\bar{\theta}) \tag{4.39}$$

which shows that $\vec{k}$ and $\vec{k}'$ are identical with the spin state polarisation vectors of Section 2. Let us however emphasise that in contrast to Section 2 we deal here with a genuine stochastic average over all possible angular fluctuations about $\vec{k}$ and $\vec{k}'$ respectively, which attributes a clear probabilistic meaning to the otherwise doubtful average $\langle\vec{\sigma}\rangle$.

By (4.33) we have

$$U^T e_1 = \frac{1}{\sqrt{2}}(e_1 - e_3) = e_1' \qquad U^T e_3 = \frac{1}{\sqrt{2}}(e_1 + e_3) = e_3'$$
$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{4.40}$$

which refers to the Euler angles

$$\bar{\theta} = \frac{\pi}{2}, \quad \bar{\varphi} = \frac{\pi}{2}, \quad \bar{\psi} = -\frac{\pi}{2}$$
$$\vec{k} = (0, 0, 1) \to \vec{k}' = R(\bar{\theta}, \bar{\varphi}, \bar{\psi})\, \vec{k} = (-1, 0, 0). \tag{4.41}$$

We obtain here

$$\langle\vec{L}\rangle_1 = \frac{\hbar}{2}\vec{k} \to \langle\vec{L}\rangle_1' = -\frac{\hbar}{2}\vec{\imath}, \quad \langle\vec{L}\rangle_3 = -\frac{\hbar}{2}\vec{k} \to \langle\vec{L}\rangle_3' = \frac{\hbar}{2}\vec{\imath}. \tag{4.42}$$

It tells us that the superposition of the two competing (opposite polarisations) stochastic flows gives rise to the x-polarised stochastic flow.

Visualising interference in terms of competing stochastic alternatives [17] makes it a priori possible to get an insight into what happens in the neutron interferometry experiments [36, 37] designed to verify the quantum spin state superposition law. Compare e.g. the general stochastic discussion of the neutron interference, given in Ref. [14]. Let us now investigate a superposition

$$\Psi = \cos\frac{\bar{\theta}}{2}\, e_1 - i \sin\frac{\bar{\theta}}{2}\, e_3 \tag{4.43}$$

which is apparently related to the $SU(2)$ rotation matrix

$$U = U^T = \begin{pmatrix} \cos\frac{\bar{\theta}}{2} & -i\sin\frac{\bar{\theta}}{2} \\ -i\sin\frac{\bar{\theta}}{2} & \cos\frac{\bar{\theta}}{2} \end{pmatrix} \tag{4.44}$$

i.e. to the Euler angles

$$\theta \to -\bar{\theta}, \qquad \psi \to \bar{\psi} = \pi, \qquad \varphi \to \bar{\varphi} = -\pi$$
$$\tilde{U} e_1 = \cos\frac{\bar{\theta}}{2}\, e_1 - i\sin\frac{\bar{\theta}}{2}\, e_3 = \psi_+ \tag{4.45}$$
$$\tilde{U} e_3 = -i\sin\frac{\bar{\theta}}{2}\, e_1 + \cos\frac{\bar{\theta}}{2}\, e_3 = \psi_-.$$

In the above the subscripts $\pm$ indicate that we deal with $\pm\hbar/2$ eigenvectors of $U\mathcal{H}M_3U^*$ $= (R\vec{\mathcal{H}})\,\vec{M} = \vec{\mathcal{H}}(R^T\vec{M})$.

The expectation value of $\vec{L}$ in states ψ_+, ψ_- respectively reads $\langle\vec{L}\rangle_+ = \hbar/2\,\vec{k}'$, $\langle\vec{L}\rangle_-$ $= -\hbar/2\,\vec{k}'$ so that

$$\begin{aligned} U\begin{pmatrix}1\\0\end{pmatrix} &= \begin{pmatrix}\cos\dfrac{\bar\theta}{2}\\ -i\sin\dfrac{\bar\theta}{2}\end{pmatrix} = \chi_{\text{up}} \to \vec{P}_{\text{up}} = (0, -\sin\bar\theta, \cos\bar\theta) = \vec{k}' \\ U\begin{pmatrix}0\\1\end{pmatrix} &= \begin{pmatrix}-i\sin\dfrac{\bar\theta}{2}\\ \cos\dfrac{\bar\theta}{2}\end{pmatrix} = \chi_{\text{down}} \to \vec{P}_{\text{down}} = (0, \sin\bar\theta, -\cos\bar\theta) = -\vec{k}'. \end{aligned} \tag{4.46}$$

In particular setting $\bar\theta = \pi$ we have

$$\begin{aligned} &U_\pi\begin{pmatrix}1\\0\end{pmatrix} = \exp\left(-i\,\frac{\pi}{2}\right)\begin{pmatrix}0\\1\end{pmatrix} \qquad U_\pi\begin{pmatrix}0\\1\end{pmatrix} = \exp\left(-i\,\frac{\pi}{2}\right)\begin{pmatrix}1\\0\end{pmatrix} \\ &\psi_+ = \exp\left(-\frac{i}{2}\,\pi\right)e_3 \qquad \psi_- = \exp\left(-\frac{i}{2}\,\pi\right)e_1 \\ &\vec{k}_+' = (0, 0, -1) \qquad \vec{k}_-' = (0, 0, 1) \end{aligned} \tag{4.47}$$

i.e. the polarisation reversal, which is accompanied by the phase correction $\exp(-i\pi/2)$ of the spin state.

The Euler angles θ, φ, ψ of $\psi(\theta, \varphi, \psi)$ refer to random rotations about the polarisation vector associated with ψ. But this vector itself should be determined before in the external (observer's) reference frame.

By looking at Fig. 4–6 of Ref. [6] and its turned upside down version, we realise that the (θ, φ, ψ) rotation defined relative to $\vec{k}_+' = -\vec{k} = (0, 0, -1)$ coincides with the deflection

$$\varphi' = \varphi + \pi, \qquad \theta' = \pi - \theta, \qquad \psi' = -\psi \tag{4.48}$$

defined relative to $\vec{k}_-' = \vec{k}$.

Accounting for the periodicity of the angular wave functions we obtain

$$\begin{aligned} e_1(\theta', \varphi', \psi') &= \exp\left(-\frac{i}{2}\,\pi\right)e_2(\theta, \varphi, \psi) \\ e_3(\theta', \varphi', \psi') &= \exp\left(-\frac{i}{2}\,\pi\right)e_4(\theta, \varphi, \psi) \end{aligned} \tag{4.49}$$

$$\begin{aligned} e_2(\theta', \varphi', \psi') &= \exp\left(-\frac{i}{2}\,\pi\right)e_1(\theta, \varphi, \psi) \\ e_4(\theta', \varphi', \psi') &= \exp\left(-\frac{i}{2}\,\pi\right)e_3(\theta, \varphi, \psi)\,. \end{aligned} \tag{4.50}$$

Hence by (4.47) we get

$$\begin{aligned}(\hat{U}_\pi{}^T e_1)(\theta, \varphi, \psi) &= \exp\left(-\frac{i}{2}\pi\right) e_3(\theta, \varphi, \psi) = e_4(\theta', \varphi', \psi')\\ (\hat{U}_\pi{}^T e_3)(\theta, \varphi, \psi) &= \exp\left(-\frac{i}{2}\pi\right) e_1(\theta, \varphi, \psi) = e_2(\theta', \varphi', \psi').\end{aligned} \tag{4.51}$$

The 2π rotation leads to

$$\begin{aligned}(\hat{U}_{2\pi}^T e_1)(\theta, \varphi, \psi) &= -e_1(\theta, \varphi, \psi) = e_2(\theta', \varphi', \psi' + \pi)\\ (\hat{U}_{2\pi}^T e_3)(\theta, \varphi, \psi) &= -e_3(\theta, \varphi, \psi) = e_4(\theta', \varphi', \psi' + \pi)\end{aligned} \tag{4.52}$$

while the 3π rotation implies

$$\begin{aligned}(\hat{U}_{3\pi}^T e_1)(\theta, \varphi, \psi) &= \exp\left(\frac{i}{2}\pi\right) e_3(\theta, \varphi, \psi) = e_4(\pi + \theta, \varphi, -\psi + \pi)\\ (\hat{U}_{3\pi}^T e_3)(\theta, \varphi, \psi) &= \exp\left(\frac{i}{2}\pi\right) e_1(\theta, \varphi, \psi) = e_2(\pi + \theta, \varphi, -\psi + \pi).\end{aligned} \tag{4.53}$$

The above formulas manifestly disclose a physical meaning of phase factors. Namely once we choose the $\vec{k}$ axis of the frame of reference to coincide with the polarisation of the stochastic flow $e_1(\theta, \varphi, \psi)$, each subsequent $\hat{U}_\pi$ rotation produces a new stochastic flow, which is different from all preceding ones, unless the 4π rotation is finally made. All randomly accessible states of rotation are now determined relative to the initially chosen vector $\vec{k}$ so that:

$$\begin{aligned}e_1(\theta, \varphi, \psi) &\underset{\pi}{\longrightarrow} e_4(\pi - \theta, \varphi + \pi, -\psi) \underset{\pi}{\longrightarrow} e_2(\pi - \theta, \varphi + \pi, -\psi + \pi)\\ &\underset{\pi}{\longrightarrow} e_4(\pi + \theta, \varphi, -\psi + \pi) \underset{\pi}{\longrightarrow} e_1(\theta, \varphi, \psi)\end{aligned} \tag{4.54}$$

and analogously (interchange $2 \leftrightarrow 4$ subscripts in the above) for e_3.

4.4. Non-stationary diffusions

Let us finally come back to the time dependent problem with

$$U(t) = U(t)^T = \begin{pmatrix} \cos \omega t & -i \sin \omega t \\ -i \sin \omega t & \cos \omega t \end{pmatrix} \tag{4.55}$$

which implements the precession of the polarisation vector

$$\vec{P}_{\mathrm{up}}(t) = (0, -\sin 2\omega t, \cos 2\omega t) \tag{4.56}$$

about the $\vec{i}$ axes with the frequency 2ω.

The corresponding rotation matrix $R(t)$ reads

$$\theta \to -2\omega t, \qquad \psi \to \pi, \qquad \varphi \to -\pi$$

$$R(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\omega t & -\sin 2\omega t \\ 0 & \sin 2\omega t & \cos 2\omega t \end{pmatrix}. \tag{4.57}$$

The stochastic motion in the presence of the $\vec{\mathcal{H}} = \mathcal{H}\,(1, 0, 0)$ homogeneous magnetic field is no longer stationary. To describe it in terms of the stochastic differential equation, we must account not only for the random fluctuations about the mean direction but also for the rotation of the reference frame $\vec{\imath}, \vec{\jmath}, \vec{k}$ i.e. the mean direction itself.

Since the infinitesimal rotation coming from (4.57) refers [6] to the angular velocity

$$\vec{\omega}_{\rm cl} = (-2\omega, 0, 0) = -\frac{2\mu}{\hbar}\,(\mathcal{H}, 0, 0) \tag{4.58}$$

we arrive at the apparent generalisation of (4.58)

$$\vec{\omega}_{\rm cl} = -\frac{2\mu}{\hbar}\,\vec{\mathcal{H}} \tag{4.59}$$

so that the resulting stochastic differential equation reads

$$d\vec{n}(t) = \vec{n}(t) \times \vec{\omega}_{\rm cl}\,dt + \vec{n}(t) \times d\vec{\Omega}\big(\vec{n}(t), t\big) \tag{4.60}$$

with $d\vec{\Omega}(t)$ given previously, but now relative to the time dependent frame with $\vec{\imath} = \vec{\imath}(t)$, $\vec{\jmath} = \vec{\jmath}(t)$, $\vec{k} = \vec{k}(t)$, $\vec{e}(t) = R(t)\,\vec{e}$. Consequently *in the presence of the homogeneous magnetic field we deal* (as expected) *with a classical precession executed with the angular velocity* $\vec{\omega}_{\rm cl}$ *on which random fluctuations are superimposed.*

5. Stochastic aspects of the Stern-Gerlach experiment

Letting neutral particles with spin pass through the magnetically inhomogeneous area is the standard way to detect the polarisation (i.e. orientability) features of the coherent particle beam [48—50]. Albeit satisfactory with respect to predicting experimental outcomes, the theoretical analysis of the Stern-Gerlach experiment employs the impulsive approximation to describe the splitting of the beam in the $\vec{k}$-direction [38, 48, 54, 16, 4, 55] and includes an assumption (usually hidden, while explicit in [54]) that non-zero forces exerted by the magnetic field in the $\vec{\imath}$ and $\vec{\jmath}$ directions are canceled on the average due to the large homogeneous z-component of the field. As a consequence one entirely disregards the details of what happens in the inhomogeneity volume.

From the stochastic point of view of Section 4 it is rather crucial to provide the analysis of this disregarded problem.

To answer what probabilistically happens in the spin measurement, we must have in hands a detailed solution of the Stern-Gerlach propagation problem. It appears that [57] is the only realistic attempt in this direction, while still incomplete with respect to our needs, since the center-of-mass wave function is given only and the initial wave packet is assumed to be so sharply peaked that the Dirac delta approximation is possible, which is far from the experimental situation.

We intend to analyze effects purely due to the magnetic field inhomogeneity, so the large homogeneous component will not be introduced at all.

Following the idea of [57] let us consider the Heisenberg picture dynamics generated by the Pauli Hamiltonian

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\,\Delta + \mu\vec{\mathcal{H}}\vec{\sigma} \\ \vec{\mathcal{H}} &= \vec{\mathcal{H}}(\vec{r}) = B\cdot(-x, 0, z) \to \operatorname{div}\vec{\mathcal{H}} = 0\,. \end{aligned} \tag{5.1}$$

The dynamical variables of the problem obey

$$\dot{r}_j = \frac{i}{\hbar} [H, r_j]_- = \frac{p_j}{m}$$

$$\dot{\sigma}_j = \frac{i}{\hbar} [H, \sigma_j]_- = -\mu(\vec{\sigma} \times \vec{\mathscr{H}})_j \to \dot{\sigma}_x = \frac{\mu}{\hbar} B\sigma_y z$$

$$\dot{\sigma}_y = -\frac{\mu}{\hbar} B(\sigma_x z + \sigma_z x) \qquad \dot{\sigma}_z = -\frac{\mu}{\hbar} B\sigma_y x \tag{5.2}$$

$$\dot{p}_j = \frac{i}{\hbar} [H, p_j]_- = \mu \sum_k \sigma_k \frac{\partial \mathscr{H}_k}{\partial r_j} \to \dot{p}_x = -\mu B \sigma_x$$

$$\dot{p}_y = 0 \qquad \dot{p}_z = \mu B \sigma_z .$$

Following the impulsive approximation idea (neutrons feel the magnetic shock along the relatively short time in their overall motion through the experimental arrangement) we select the leading contribution to the Taylor series

$$\vec{r}(t) = \vec{r}(0) + t\dot{\vec{r}}(0) + \frac{t^2}{2!} \ddot{\vec{r}}(0) + \cdots \sim \vec{r}(0) + \frac{t}{m} \vec{p}(0) + \frac{t^2}{2m} \dot{\vec{p}}(0) . \tag{5.3}$$

Denoting by $\hat{\vec{r}}(t)$ the 2×2 matrix operator, not to confuse it with the configuration variable, we get

$$\hat{x}(t) \sim x + \frac{t}{m} p_x - \frac{t^2}{2m} \mu B \sigma_x$$

$$\hat{y}(t) \sim y + \frac{t}{m} p_y \tag{5.4}$$

$$\hat{z}(t) \sim z + \frac{t}{m} p_2 + \frac{t^2}{2m} \mu B \sigma_2$$

so that a passage from the Heisenberg to the Schrödinger picture can be accomplished by invoking

$$\Psi(\vec{r}, t) = \int dr' G(\vec{r}, t; \vec{r}', 0)\, \Psi(\vec{r}', 0)$$

$$\hat{A}(t) = \exp\left(\frac{i}{\hbar} Ht\right) \hat{A}(0) \exp\left(-\frac{i}{\hbar} Ht\right) \tag{5.5}$$

$$|\psi, t) = \exp\left(-\frac{i}{\hbar} Ht\right) |\psi, 0)$$

i.e.

$$\hat{\vec{r}}(-t)\, G(\vec{r}, t; \vec{r}', 0) = \vec{r}' G(\vec{r}, t; \vec{r}', 0) \tag{5.6}$$

with $\hat{\vec{r}}(t)$ given by (5.4).

This equation is accompanied by

$$\hat{\vec{p}}(-t)\, G(\vec{r}, t; \vec{r}', 0) = \vec{p}' G(\vec{r}, t; \vec{r}', 0)$$

$$\hat{p}_x(-t) \sim -p_x + t\mu B \sigma_x \qquad \vec{p} = -i\hbar \nabla \tag{5.7}$$

$$\hat{p}_y(-t) \sim -p_y \qquad \vec{p}' = -i\hbar\nabla'$$
$$\hat{p}_z(-t) \sim -p_z + t\mu B\sigma_x$$

and the solution of (5.5), (5.7) which is consistent with the initial condition

$$G(\vec{r}, t; \vec{r}', 0) \xrightarrow[t\to 0]{} \delta(\vec{r} - \vec{r}') \tag{5.8}$$

reads

$$G(\vec{r}, t; \vec{r}', 0) = \left(\frac{m}{i\hbar t}\right)^{3/2} \exp\left\{\frac{im}{2\hbar t}\left[\left(x - x' + \frac{t^2}{2m}\mu B\sigma_x\right)^2 + (y - y')^2 + \left(z - z' - \frac{t^2}{2m}\mu B\sigma_z\right)^2\right] + \frac{i}{\hbar}[-xt\mu B\sigma_x + zt\mu B\sigma_z]\right\}. \tag{5.9}$$

In the short passage time approximation of ours it can be furthermore given in the form

$$G(\vec{r}, t; \vec{r}', 0) \sim \left(\frac{m}{i\hbar t}\right)^{3/2} \exp\left[\frac{im}{2\hbar t}(\vec{r} - \vec{r}')^2\right] \exp\frac{i\mu Bt}{2\hbar}[\sigma_z(z + z') - \sigma_z(x + x')] \tag{5.10}$$

where for short times the familiar free propagator

$$K(\vec{r}, t; \vec{r}', 0) = \left(\frac{m}{i\hbar t}\right)^{3/2} \exp\left[\frac{\mathrm{im}}{2\hbar t}(\vec{r} - \vec{r}')^2\right] \tag{5.11}$$

can be literally viewed as an approximate expression for the Dirac delta functional.

Consequently the propagator reads

$$G(\vec{r}, t; \vec{r}', 0) \sim \left(\frac{m}{i\hbar t}\right)^{3/2} \exp\left[\frac{im}{2\hbar t}(\vec{r} - \vec{r}')^2\right] \exp\frac{i\mu Bt}{\hbar}(\sigma_z z - \sigma_x x) \tag{5.12}$$

and its action on the initial spinor wave function

$$\begin{aligned} &\Psi(\vec{r}, 0) = \frac{1}{(\pi\Delta^2)^{3/2}} \exp\left\{-\frac{1}{2\Delta^2}(\vec{r} - \vec{q})^2\right\} \exp\frac{i}{\hbar}\vec{p}(\vec{r} - \vec{q})\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \\ &\vec{p} = (0, p, 0) \qquad \vec{q} = (0, q, 0) \\ &\Psi(\vec{r}, 0) = \psi(\vec{r}, 0)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \end{aligned} \tag{5.13}$$

is given by

$$\begin{aligned} &\Psi(\vec{r}, t) = \psi_t(\vec{r}, 0) \exp\frac{i\mu Bt}{\hbar}(\sigma_z z - \sigma_x x)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \\ &\psi_t(\vec{r}, 0) = \psi\left(x, y - \frac{p}{m}t, z; 0\right) \exp\left(-\frac{i}{\hbar}\frac{p^2}{2m}t\right). \end{aligned} \tag{5.14}$$

It is quite remarkable that the net effect of the above propagation, although implemented by the pure spinor rotation $\exp(i\mu t/\hbar\ \mathcal{H}\vec{\sigma})$ is equivalent to the accelerated motion of the involved wave packets. In fact the up and down components of the spinor $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ acquire phase factors whose exponents contribute to the mean wave packet momenta

$$\vec{p} \to \vec{p}_\pm = (-\mu Bt, p, \pm\mu Bt). \tag{5.15}$$

Armed with the analysis of Section 4, we can address the stochastic diffusions related to the short time Stern-Gerlach propagation.

The uniform motion of the Gaussian wave packet was analysed from the stochastic viewpoint by GUERRA [60], see also [17, 23, 61, 62]. In case of the translational motion of the wave packet centroid in the y-direction, the stochastic differential equation for the random variable $\vec{X}(t)$ with values in R^3 reads:

$$d\vec{X}(t) = \vec{b}(\vec{X}(t), t)\, dt + \left(\frac{\hbar}{2m}\right)^{1/2} d\vec{W}(t) \tag{5.16}$$

where $d\vec{W}(t)$ is the Wiener noise in R^3, while $\vec{b}$ comes from $\psi_t(\vec{r}, 0)$ via standard definitions [17] of the current and osmotic velocities:

$$\begin{aligned}
\vec{u} &= \frac{\hbar}{m} \nabla \ln |\psi_t| = -\frac{\hbar^2}{m\Delta^2} [\vec{r} - \vec{q}(t)] \\
\vec{q}(t) &= \left(0, q - \frac{p}{m}\, t, 0\right) \\
\vec{v} &= \frac{\hbar}{m} \nabla \arg \psi_t = \frac{\vec{p}}{m} = \frac{p}{m}\, (0, 1, 0)
\end{aligned} \tag{5.17}$$

so that

$$\begin{aligned}
\vec{b}(\vec{X}(t), t) &= (\vec{u} + \vec{v})\, (\vec{X}(t), t) \\
&= -\frac{\hbar}{m\Delta^2} \left(X(t), Y(t) - q + \frac{p}{m}\, t, Z(t)\right) + \frac{\vec{p}}{m}\, (0, 1, 0) \\
&= -\frac{\hbar}{m\Delta^2}\, \vec{X}(t) + \left(0, \frac{\hbar}{m\Delta^2} \left(q - \frac{p}{m}\, t\right) + \frac{p}{m}, 0\right).
\end{aligned} \tag{5.18}$$

This translational stochastic process in R^3 is accompanied by the stochastic spin process. By the arguments of Section 4, the Pauli spinor $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = U(\bar{\theta}, \bar{\varphi}, \bar{\psi}) \begin{pmatrix}1\\ 0\end{pmatrix}$ represents the superposition $(\alpha e_1 + \beta e_3)\, (\theta, \varphi, \psi)$ of stationary diffusion processes.

The polarisation of this new process is given by $(2/\hbar)\, \langle \vec{L} \rangle = R(\bar{\theta}, \bar{\varphi}, \bar{\psi})\, \vec{k}$.

Particles passing the Stern-Gerlach magnet can be visualised to draw stochastic trajectories in R^3. Each sample path [63] of the process $\vec{X}(t)$ represents a possible variant of the spatial motion to be executed in R^3 by a point particle, provided it follows the Schrödinger-Pauli evolution pattern.

However the spinor rotation

$$\exp\left(i\, \frac{\mu}{\hbar}\, t \vec{\sigma} \vec{\mathscr{H}}\right) \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \left\{\cos \frac{\mu\, |\vec{\mathscr{H}}|}{\hbar}\, t + i \vec{\sigma}\, \frac{\vec{\mathscr{H}}}{|\vec{\mathscr{H}}|} \sin\left(\frac{\mu}{\hbar}\, |\vec{\mathscr{H}}|\, t\right)\right\} \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \tag{5.19}$$

couples the spin and spatial motions, and for the proper description of the stochasticity involved we need a pair of no longer independent random variables:

$$\{\vec{X}(t), \vec{n}(t)\} \to \{\vec{X}(t), \vec{n}(\vec{X}(t), t)\}. \tag{5.20}$$

We observe that (5.19) represents a stochastic spin process associated with the point particle, which is instantaneously at rest at the location $\vec{r} \in R^3$. Then $\vec{n}(t)$ is attached to the point $\vec{r}$ and is sensitive to the magnetic field in (about) this point.

By arguments of Section 4, the rotation of the local (at $\vec{r}$) reference frame $\vec{\imath}, \vec{\jmath}, \vec{k}$ is related to the $\vec{r}$-dependent angular velocity

$$\vec{\omega}(\vec{r}) = \frac{2\mu}{\hbar} \vec{\mathscr{H}}(\vec{r}) = \frac{2\mu B}{\hbar} (-x, 0, z) . \tag{5.21}$$

Since $\vec{r}$ is the value to be taken by the spatial random variable $\vec{X}(t)$, we arrive at the following stochastic differential equation describing the spatially dependent angular random motion

$$\begin{aligned} d\vec{n}(t) &= d\vec{n}(\vec{X}(t), t) \\ &= \vec{n}(\vec{X}(t), t) \times \vec{\omega}(\vec{X}(t))\, dt + \vec{n}(\vec{X}(t), t) \times d\vec{\Omega}(\vec{n}(\vec{X}(t), t), t) . \end{aligned} \tag{5.22}$$

It apparently describes random spatial trajectories, at whose each point the randomly achieved spin direction is modified by the magnetic field about (at) this point. At each random location $\vec{r}$ the random particle orientation is given by $\vec{n}(\vec{r}, \theta, \varphi, \psi)$.

The short time propagation regime adopted, if combined with the not-too-large Δ assumption and $\vec{q} = (0, q, 0)$ data, justifies the following approximation of the solution of the Pauli equation we have derived

$$\begin{aligned} \Psi(\vec{r}, t) &= \psi_t(\vec{r}, 0) \exp\left(i \frac{\mu t}{\hbar} \vec{\sigma}\vec{\mathscr{H}}\right) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \psi_t(\vec{r}, 0) \begin{pmatrix} \left[\cos \frac{\mu |\vec{\mathscr{H}}|\, t}{\hbar} + i \frac{zB}{\hbar} \sin \frac{\mu |\vec{\mathscr{H}}|\, t}{\hbar}\right] \alpha \\ \left[\cos \frac{\mu |\vec{\mathscr{H}}|\, t}{\hbar} - i \frac{zB}{\hbar} \sin \frac{\mu |\vec{\mathscr{H}}|\, t}{\hbar}\right] \beta \end{pmatrix} \\ &\quad - \psi_t(\vec{r}, 0) \left[i \frac{xB}{|\vec{\mathscr{H}}|} \sin \frac{\mu |\vec{\mathscr{H}}|\, t}{\hbar}\right] \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &\sim \psi_t(\vec{r}, 0) \begin{pmatrix} \alpha \exp \frac{i\mu}{\hbar} Btz \\ \beta \exp\left(-\frac{i\mu}{\hbar} Btz\right) \end{pmatrix} . \end{aligned} \tag{5.23}$$

Let the Stern-Gerlach evolution be confined to the time interval $t \in [0, T]$ and let for times $t > T$ the free propagation pattern be followed. At $t = 0$ we had

$$\Psi(\vec{r}, 0) = \psi(\vec{r}, 0)\, (\alpha e_1 + \beta e_3)\, (\theta, \varphi, \psi) \tag{5.24}$$

which at time $t = T$ got transformed into

$$\begin{aligned} \Psi(\vec{r}, T) &= \alpha \psi_t(\vec{r}, 0) \exp\left(\frac{i\mu}{\hbar} Btz\right) e_1(\theta, \varphi, \psi) \\ &\quad + \beta \psi_t(\vec{r}, 0) \exp\left(-\frac{i\mu}{\hbar} Btz\right) e_3(\theta, \varphi, \psi) . \end{aligned} \tag{5.25}$$

It amounts to the modification $\varphi \to \varphi \pm 2\mu/\hbar\, BTz$ of the Euler phase φ of the stochastic spin process. However the resulting phase factor can be as well absorbed in the (then redefined) spatial wave packet $\psi_t(\vec{r}, 0)$, which affects (acceleration) the value of the wave packet mean momentum at the initial time instant $t = T$ of the free evolution.

The free spatial propagator sends the centroid of the Gaussian wave packet to the new location

$$T \to t \Rightarrow \vec{q} = \left(0, q - \frac{p}{m} T, 0\right) \xrightarrow[p_\pm]{} \left(0, q - \frac{p}{m} t, \mp \frac{\mu}{\hbar} BT(t - T)\right) = \vec{q}_\pm(t) \tag{5.26}$$

modifying at the same time both the time dependent phase factor and the wave packet shape (spreading effects).

Processes related to $e_1(\theta, \varphi, \psi)$ and $e_3(\theta, \varphi, \psi)$ follow the free spin evolution pattern of Section 4. However although the spatial and orientational randomness are not coupled, the joint description of both for $t > T$ i.e. well beyond the Stern-Gerlach magnet, is provided by the superposition

$$\Psi(\vec{r}, t) = \alpha\psi_+(\vec{r}, t)\, e_1(\theta, \varphi, \psi, t) + \beta\psi_-(\vec{r}, t)\, e_3(\theta, \varphi, \psi, t) \tag{5.27}$$

hence still an intricate relationship of both persists. The subscript $\pm$ refers to the mean momenta $p_\pm$ of the wave packets.

Remark: Formulas (5.24)—(5.27) with the accuracy up to notation adjustments are precisely the starting point of the fuzzy spin space analysis of Ref. [15]. The unsharpness (fuzzy-ness) of any spin measurement is inherently rooted in the stochastic phenomena involved. Thus e.g. in terms of stochastic trajectories the expression

$$P\Delta = |\Psi(\vec{r}, \theta, \varphi, \psi, t)|^2 \, \Delta V \Delta\Omega \tag{5.28}$$

is the probability that the fraction of particles which at time t reach the small spatia volume ΔV about $\vec{r}$, has their stochastic spins concentrated in the angular volume $\Delta\Omega$ about the (θ, φ, ψ) Euler location of the punctured Liouville sphere. In general both the polarisation $\vec{k}$ and $-\vec{k}$ processes do non-trivially contribute to (5.28), unless we are interested in the not too large surrounding of either the north $\theta = 0$ or the south $\theta = \pi$ pole of the sphere.

Coming back to the formula (5.27) let us stress that it is a superposition of two processes, where each constituent displays an independence of spatial and angular motion. By invoking spatial and angular formulas for the forward drift, we associate two independent stochastic equations with each term of the superposition. Since both of them have the form

$$\begin{aligned} \Psi_j &= \exp(R_j + iS_j) \qquad k = \pm \equiv 1, 2 \\ \Psi &= \exp(R + iS) = \Psi_+ + \Psi_- \end{aligned} \tag{5.29}$$

the osmotic $\vec{u}$ (respectively $\vec{\omega}_u$) and drift $\vec{v}$ (respectively $\vec{\omega}_v$) velocities of the superposition of the stochastic flows are given in terms of these characterising constituent flows

$$\begin{aligned} \vec{u} &= \frac{1}{2}(\vec{u}_1 + \vec{u}_2) + \frac{\sinh(R_1 - R_2)(\vec{u}_1 - \vec{u}_2) - \sin(S_1 - S_2)(\vec{v}_1 - \vec{v}_2)}{\cosh(R_1 - R_2) + \cos(S_1 - S_2)} \\ \vec{v} &= \frac{1}{2}(\vec{v}_1 + \vec{v}_2) + \frac{\sinh(R_1 - R_2)(\vec{v}_1 - \vec{v}_2) + \sin(S_1 - S_2)(\vec{u}_1 - \vec{u}_2)}{\cosh(R_1 - R_2) + \cos(S_1 - S_2)} \end{aligned} \tag{5.30}$$

which implies that we arrive at two stochastic differential equations whose drifts display the coupling of spatial and angular degrees of freedom

$$\begin{aligned} d\vec{X}(t) &= \vec{b}\big(\vec{X}(t), \vec{n}(t), t\big)\, dt + \left(\frac{\hbar}{2m}\right)^{1/2} d\vec{W}(t) \\ d\vec{n}(t) &= \vec{n}(t) \times d\vec{\Omega}\big(\vec{n}(t), \vec{X}(t), t\big). \end{aligned} \tag{5.31}$$

Appendix: Rudiments of the stochastic mechanics

Fenyes-Nelson's stochastic mechanics is one of very few attempts to reconcile the individual particle trajectory notion with the wave (Schrödinger) theory of quantum phenomena. To a given solution of the Schrödinger equation one can in principle attribute a stochastic diffusion process satisfying the Newton second law in the mean. The corresponding stochastic differential equation describes a propagation of a point particle through a non-dissipative random medium. Sample paths of the process can be approximately identified with the realistic configuration space paths of (perhaps) physical particles.

For a quantum particle in the conservative force field we have

$$i\hbar\, \partial_t \psi(\vec{x}, t) = -\frac{\hbar^2}{2m} \Delta\psi(\vec{x}, t) + V(\vec{x})\, \psi(\vec{x}, t) \tag{A 1}$$

which implies the continuity equation for $\varrho = |\psi|^2$

$$\partial_t \varrho = -\operatorname{div} \vec{\jmath} \qquad \vec{\jmath} = \frac{\hbar}{2m_i} (\overline{\psi}\, \nabla\psi - \psi\, \nabla\overline{\psi}). \tag{A 2}$$

In case of nowhere zero ψ (locally at least, there are existence proofs for singular diffusion) upon a standard substitution $\psi = \exp\,(R + iS)$ we get

$$\partial_t \varrho = \operatorname{div} \left\{-\frac{\hbar}{m} \nabla S \cdot \varrho\right\} = \frac{\hbar}{2m} \Delta\varrho - \operatorname{div} \varrho\vec{b} \tag{A 3}$$

where

$$\varrho = \exp 2R \qquad \vec{b} = \vec{u} + \vec{v}, \qquad \vec{u} = \frac{\hbar}{m} \nabla R, \qquad \vec{v} = \frac{\hbar}{m} \nabla S, \qquad \vec{b}_* = \vec{v} - \vec{u} \tag{A 4}$$

and one more equation

$$\partial_t \varrho = -\frac{\hbar}{2m} \Delta\varrho - \operatorname{div} \varrho\vec{b}_* \tag{A 5}$$

is obeyed by ϱ. It is identifiable ast he backward Fokker-Planck equation, while (A 3) is the forward one.

In the theory of stochastic processes such equations are known to determine the time development of the respectively forward and backward transition probability densities for the diffusion process. Setting $\nu = \hbar/2m$, $\vec{b} = \hbar/m\, \nabla(R + S)$ in the forward case, we have

$$\partial_t p(\vec{y}, 0, \vec{x}, t) = \operatorname{div}_x \{\nu\, \nabla_x p(\vec{y}, 0, \vec{x}, t) - \vec{b}(\vec{x}, t)\, p(\vec{y}, 0, \vec{x}, t)\}. \tag{A 6}$$

According to the rules of the Ito stochastic calculus, one can uniquely associate (A 6) with the stochastic differential equation

$$d\vec{x}(t) = \vec{b}\big(\vec{X}(t), t\big) + \sqrt{2\nu}\, d\vec{W}(t) \tag{A 7}$$

where $d\vec{W}(t)$ represents the normalised Wiener noise. $\vec{X}(t)$ takes values in R^3 as a continuous function of time, and with time passing draws a stochastic trajectory in the configuration space. Given $\varrho_0(\vec{x}) = \varrho(\vec{x}, 0)$ and $p(\vec{y}, 0, \vec{x}, t)$ solving (A 6). Apparently $\int d^3\vec{y}\, p(\vec{y}, 0, \vec{x}, t)\, \varrho_0(\vec{y})$ provides a solution of (A 6) with the initial condition $\varrho_0(\vec{x})$, hence by the uniqueness theorem for the Kolmogorov equation it equals $\varrho(\vec{x}, t)$.

The normalisation $\int d^3\vec{x}\varrho(\vec{x}, t) = 1$ is preserved by virtue of $\int d^3\vec{x}\, p(\vec{y}, 0, \vec{x}, t) = 1$. Let us emphasize that the knowledge of $p(\vec{y}, 0, \vec{x}, t)$ does not determine $\varrho(\vec{x}, t)$ unless $\varrho_0(\vec{x})$ is specified. Consequently, given (A 3) it is rather natural to demand the validity of this equation not only for $\varrho(\vec{x}, t)$ but also for the transition probability densities $p(\vec{y}, 0, \vec{x}, t)$ which automatically associates (A 7) with (A 1).

As is well known the Schrödinger equation can be equivalently rewritten as a coupled Jystem of equations, one of which is (A 3), while another has the familiar Hamilton-sacobi form

$$\partial_t S = \frac{\hbar}{2m} \{|\nabla R|^2 - |\nabla S|^2 + \nabla R\} = \frac{V}{\hbar}. \tag{A 8}$$

Let us define the conditional expectation value for the stochastic process $\vec{X}(t)$ solving (A 7)

$$E_t[f(\vec{X}(t'))] = E[f(\vec{X}(t')) \mid \vec{X}(t) = \vec{x}] = \int d^3\vec{y}\, p(\vec{x}, t, \vec{y}, t')\, f(\vec{y}) \quad t' \geqq t. \tag{A 9}$$

In terms of (A 9) the mean forward and backward derivatives D_+, D_- of the process can be introduced

$$\begin{aligned}(D_\pm f)\left(\vec{X}(t), t\right) &= \lim_{\Delta t \downarrow 0} E_t \left\{\pm \frac{1}{\Delta t} \left[f\left(\vec{X}(t \pm \Delta t), t \pm \Delta t\right) - f\left(\vec{X}(t), t\right)\right]\right\} \\ &= \left(\partial_t + \vec{b}_\pm \nabla \pm \frac{\hbar}{2m} \Delta\right) f\left(\vec{X}(t), t\right)\end{aligned} \tag{A 10}$$

such that

$$D_+\vec{X}(t) = \vec{b}_+ = \vec{b} \qquad D_-\vec{X}(t) = \vec{b}_- = \vec{b}_* \tag{A 11}$$

and there holds

$$\begin{aligned}\frac{m}{2}(D_+D_- + D_-D_+) &= \frac{m}{2}(D_+b_- + D_-b_+)\left(\vec{X}(t), t\right) \\ &= \hbar\nabla \left\{\partial_t S - \frac{\hbar}{2m}[|\nabla R|^2 - |\nabla S|^2 + \nabla R]\right\}\left(\vec{X}(t), t\right).\end{aligned} \tag{A 12}$$

By equating (which is a restriction on the process)

$$\frac{m}{2}(D_+D_- + D_-D_+)\, \vec{X}(t) = -\nabla V \tag{A 13}$$

the second Newton law of motion is obeyed in the stochastic mean. Apparently we deal here with the gradient form of (A 9). Since the osmotic $\vec{u}$ and current $\vec{v}$ velocities are gradients, it is convenient to rewrite (A 3) and (A 8) in terms of them only. Then

$$\begin{aligned}\partial_t \vec{u} &= -\frac{\hbar}{2m} \Delta \vec{v} - \nabla(\vec{v}\vec{u}) \\ \partial_t \vec{v} &= \frac{\hbar}{2m} \Delta \vec{u} + \frac{1}{2} \nabla(\vec{u}^2) - \frac{1}{2} \nabla(\vec{v}^2) - \frac{1}{m} \nabla V\end{aligned} \tag{A 14}$$

may be considered as the starting point for the stochastic analysis, once the initial velocity fields $\vec{u}(\vec{x}, t_0)$, $\vec{v}(\vec{x}, t_0)$ are chosen and the Cauchy problem (A 14) is solvable.

Equations (A 3), (A 8) provide us merely with another form of (A 1), while the equivalence of (A 14) with (A 7), (A 13) is more intricate. On the other hand, by taking the gradients of (A 3), (A 8) we recover (A 14), hence on the mathematical (at least) level a manifest link exists between Schrödinger wave functions and random (diffusive) motions of point particles.

The major problem of stochastic mechanics is then to reveal to which extent wave functions are derivable on purely probabilistic (diffusion processes) grounds.

Apparently is amounts to recovering the potentials upon an assumption that $\vec{u}(\vec{x}, t)$, $\vec{v}(\vec{x}, t)$ solving (A 14) are gradient fields. Let $\vec{u}$, $\vec{v}$ solve (A 14) with the initial data $\vec{u}_0(\vec{x}) = \vec{u}(\vec{x}, t_0)$, $\vec{v}_0(\vec{u})$. By introducing $\vec{b} = \vec{u} + \vec{v}$ we can pass to the stochastic differential equation (A 7) which in turn implies (A 6). Accordingly $\varrho(\vec{x}, t)$ is determined by the choice of $\varrho_0(\vec{x})$. Assuming that $\vec{u}_0(\vec{x})$ is the gradient field, we can locally reproduce the potential with the accuracy up to the additive constant (e.g. the Poincaré lemma). The normalization condition $\int d^3\vec{x}\varrho(\vec{x}, t_0) = 1$, $\exp 2R_0 = \varrho_0$ removes the arbitrariness, hence $\vec{u}_0(\vec{x})$ determines $\varrho_0(\vec{x})$ and by (A 6) $\varrho(\vec{x}, t)$.

Having $\varrho(\vec{x}, t)$ established, we are finally left with the equation (A 13) whose integration amounts to solving the Cauchy problem

$$\begin{aligned} &\partial_t \mathcal{S} + H(\nabla \mathcal{S}, \vec{x}, t) = 0 \\ &\mathcal{S}(\vec{x}, t_0) = \mathcal{S}_0(\vec{x}), \qquad \nabla \mathcal{S}_0(\vec{x}) = m\vec{v}_0(\vec{x}) \\ &\mathcal{S} = \hbar S \end{aligned} \tag{A 15}$$

with

$$\begin{aligned} &H(\vec{p}, \vec{x}, t) = \frac{\vec{p}^2}{2m} + U(\vec{x}, t) \\ &U(\vec{x}, t) = V(\vec{x}, t) - \frac{\hbar^2}{2m} \frac{\nabla \varrho^{1/2}}{\varrho^{1/2}}. \end{aligned} \tag{A 16}$$

Indeed, if we have a solution $\mathcal{S}(\vec{x}, t)$ of (A 15) then $\nabla \mathcal{S}(\vec{x}, t)$ solves (A 13) hence (A 14). By the uniqueness argument for solutions of the Cauchy problem, $\nabla \mathcal{S}(\vec{x}, t) = m\vec{v}(\vec{x}, t)$ provides a solution of (A 14) with $\vec{v}_0(\vec{x}) = 1/m \, \nabla \mathcal{S}_0(\vec{x})$. The only non-uniqueness pertains to the initial data $\nabla \mathcal{S}_0(\vec{x}) = m\vec{v}_0(\vec{x})$ since in the contractible spatial area $\vec{v}_0(\vec{x})$ determines the corresponding potential up to the additive constant.

To see how this arbitrariness can be removed, let us consider the absolute expectation value of (A 15). Then $\langle \partial_t \mathcal{S} \rangle = -\langle H \rangle$ where (integrate by parts)

$$\begin{aligned} \langle H \rangle &= \int d^3\vec{x} \left[\frac{m}{2} (\vec{v}^2 - \vec{u}^2) + V(\vec{x}, t) = \frac{\hbar}{2} \operatorname{div} \vec{u}\right] \varrho(\vec{x}, t) \\ &= \int d^3\vec{x} \left[\frac{m}{2} (\vec{u}^2 + \vec{v}^2) + V(\vec{x}, t)\right] \varrho(\vec{x}, t) \end{aligned} \tag{A 17}$$

and the assumption of the localizability (e.g. $\langle H \rangle < \infty$) of the total (mean) energy of the diffusion process is necessary to have (A 15) uniquely solved on the basis of (A 14). The term $\int d^3\vec{x} m/2(\vec{u}^2 + \vec{v}^2) \varrho(\vec{x}, t)$ is known as the kinetic energy of the diffusion process.

By the continuity equation we have

$$\partial_t \langle \mathcal{S} \rangle = \int d^3\vec{x} (\partial_t \varrho) \mathcal{S} + \langle \partial_t \mathcal{S} \rangle = m \langle \vec{v}^2 \rangle + \langle 2_t \mathcal{S} \rangle \tag{A 18}$$

hence (A 15) implies

$$\partial_t \langle \mathscr{S} \rangle = m \langle \vec{v}^2 \rangle - \langle H \rangle \tag{A 19}$$

which admits a unique solution $\langle \mathscr{S} \rangle (t)$ for given initial data $\langle \mathscr{S} \rangle (t_0) = \langle \mathscr{S}_0 \rangle$.

By making the restriction

$$\langle \mathscr{S}_0 \rangle = 0 \tag{A 20}$$

we have a guarantee that $\langle \mathscr{S} \rangle (t)$ is determined in terms of $\vec{u}, \vec{v}$ only

$$\langle \mathscr{S} \rangle (t) = \int_{t_0}^{t} [m \langle v^2 \rangle - \langle H \rangle] \, dt \,. \tag{A 21}$$

Given an arbitrary integral $\mathscr{S}'(\vec{x}, t)$, $\langle \mathscr{S}_0' \rangle \neq 0$ of (A 15). Then

$$\mathscr{S}(\vec{x}, t) = \mathscr{S}'(\vec{x}, t) - \langle \mathscr{S}_0' \rangle \tag{A 22}$$

obeys both (A 20) and (A 15): *such Schrödinger wave functions are in a one-to-one correspondence with the diffusion process* (A 7), (A 13).

References

[1] E. BELTRAMETTI, G. CASSINELLI, in: Problems in the Foundations of Physics, LXXII Corso, Soc. Ital. di Fisica, Bologna, 1979.
[2] W. G. FARIS, Found. Phys. **12** (1092) 1.
[3] T. DANKEL, Jr., Arch. Rat. Mech. Anal. **37** (1970) 192.
[4] F. SCHROECK, Jr., Found. Phys. **12** (1982) 479.
[5] P. BUSCH, Phys. Rev. **D 33** (1986) 2253.
[6] H. GOLDSTEIN, Classical Mechanics, Addison.Wesley, Cambridge, 1958.
[7] J. KESSLER, Polarised Electrons, Springer, Berlin 1976.
[8] A. ZEILINGER, Z. Phys. **B 25** (1976) 97.
[9] J. KALCKAR, Lett. Nuovo Cim. **5** (1972) 645.
[10] G. PATSAKOS, Amer. J. Phys. **44** (10976) 158.
[11] E. J. HELLER, J. Chem. Phys. **62** (1975) 1544, and ibid **67** (1977) 3339.
[12] A. KOVNER, B. ROSENSTEIN, Phys. Rev. **D 32** (1985) 2622.
[13] P. GARBACZEWSKI, D. PROROK, in: Karpacz Winter School Proc., ed. R. Gielerak and W. Karwowski, World Scientific, Singapore, 1988.
[14] P. GARBACZEWSKI, Neutron Interferometry and Stochastic Mechanics, Wrocław Univ. Report, 1988, see also Phys. Lett. **A 143** (1990) 85.
[15] E. PRUGOVEČKI, Stochastic Quantum Mechanics and Quantum Spacetime, Reidel, Dordrecht, 1986.
[16] E. PRUGOVEČKI, J. Phys. A **10** (1977) 543, see also S. ALI, A. PRUGOVEČKI, Acta Appl. Math. **6** (1986) 19.
[17] E. NELSON, Quantum Fluctuations, Princeton Univ. Press, Princeton, 1985.
[18] R. JACKIW, Rev. Mod. Phys. **49** (1977) 682.
[19] P. GARBACZEWSKI, Classical and Quantum Field Theory of Exactly Soluble Nonlinear Systems, World Scientific, Singapore, 1985.
[20] M. B. GREEN, J. H. SCHWARZ, W. WITTEN, Superstring Theory, Cambridge, 1987.
[21] K. NAMSRAI, Nonlocal Quantum Field theory and Stochastic Quantum Mechanics, Reidel, Dodrecht, 1986.
[22] F. BOPP, R. HAAG, Z. Naturforsch. **5a** (1950) 644, see also J. P. DAHL, Kong. Dansk. Mat. Fyz. Med. **39** (1977) no 12.
[23] P. GARBACZEWSKI, J. Phys. **A 20** (1987) 4799.
[24] L. S. SCHULMAN, Phys. Rev. **176** (1968) 1558.
[25] L. S. SCHULMAN, Techniques and Applications of the Path Integration, Wiley, NY, 1981.

[26] H. B. NIELSEN
[27] A. O. BARUT, I
[28] A. O. BARUT, in
[29] M. P. SILVERMA
[30] M. P. SILVERMA
[31] J. F. HAMILTO
[32] T. ICHINOSE, H
[33] G. F. DE ANGE
[34] G. F. JONA-LAS
[35] P. GARBACZEW
[36] G. BADUREK, H
[37] G. BADUREK, H
1133.
[38] G. BAYM, Lect
[39] D. I. BLOKHIN
[40] G. EDER, A. ZI
[41] H. RAUCH, A.
54 A (1975) 42
[42] S. A. WERNER,
[43] A. G. KLEIN, G
[44] E. KLEMPT, Ph
[45] M. STOLL, A. J
[46] M. E. STOLL, E
[47] R. MIRMAN, Ph
[48] M. BLOOM, K.
[49] B. HAMELIN, N
[50] N. A. KUEBLE
737.
[51] T. JONES, W.
[52] J. SUMMHAMME
[53] A. ZEILINGER,
[54] A. BÖHM, Quan
[55] M. F. BARROS,
(1987) 285.
[56] C. DEWDNEY,
[57] M. SCULLY, A.
[58] A. O. BARUT, M
[59] P. GARBACZEW
Singapore, 198
[60] F. GUERRA, Ph
[61] K. YASUE, J. Z
[62] M. MC CLENDO
[63] K. ITO, H. P. M
1965.
[64] P. S. HUBBARD,

[26] H. B. Nielsen, D. Rohrlich, Nucl. Phys. **B 299** (1988) 471.
[27] A. O. Barut, P. Meystre, Phys. Lett. **105 A** (1984) 458.
[28] A. O. Barut, in: Quantum Mechanics versus Local Realism, ed. F. Selleri Plenum, NY, 1987.
[29] M. P. Silverman, European J. Phys. **1** (1980) 116.
[30] M. P. Silverman, J. Phys. **B 13** (1980) 2367.
[31] J. F. Hamilton, L. S. Schulman, J. Math. Phys. **12** (1971) 160.
[32] T. Ichinose, H. Tamura, J. Math. Phys. **29** (1988) 103.
[33] G. F. de Angelis, G. Jona-Lasinio, J. Phys. **A 15** (1982) 2053.
[34] G. F. Jona-Lasinio, G. F. De Angelis, M. Serva, N. Zanghi, J. Phys. **A 19** (1986) 865.
[35] P. Garbaczewski, J. Math. Phys. **22** (1981) 574.
[36] G. Badurek, H. Rauch, J. Summhammer, Phys. Rev. Lett. **51** (1983) 1015.
[37] G. Badurek, H. Rauch, J. Summhammer, U. Kischko, A. Zeilinger, J. Phys. **A 16** (1983) 1133.
[38] G. Baym, Lectures on Quantum Mechanics, Benjamin, Reading, 1969.
[39] D. I. Blokhintsev, Principles of Quantum Mechanics (in Russian), Nauka, Moscow, 1976.
[40] G. Eder, A. Zeilinger, Nuovo Cim. **34 B** (1976) 76.
[41] H. Rauch, A. Zeilinger, G. Badurek, A. Wilfing, W. Bauspiess, U. Bonse, Phys. Lett. **54 A** (1975) 425.
[42] S. A. Werner, R. Colella, A. W. Overhauser, C. F. Eagen, Phys. Rev. Lett. **35** (1975) 425.
[43] A. G. Klein, G. I. Opat, Phys. Rev. Lett. **37** (1976) 238.
[44] E. Klempt, Phys. Rev. **D 13** (1976) 3125.
[45] M. Stoll, A. J. Vega, R. W. Vaughan, Phys. Rev. **A 16** (1977) 1521.
[46] M. E. Stoll, E. K. Wolff, M. Mehring, Phys. Rev. **A 17** (1978) 1561.
[47] R. Mirman, Phys. Rev. **D 1** (1970) 3349.
[48] M. Bloom, K. Erdman, Canad. J. Phys. **40** (1962) 179.
[49] B. Hamelin, N. Xiromeritis, P. Liand, Nucl. Instr. and Meth. 125 (1975) 79.
[50] N. A. Kuebler, M. B. Robin, J. Yang, A. Gedanken, D. Herrick, Phys. Rev. **A 38** (1988) 737.
[51] T. Jones, W. Williams, J. Phys. **E 13** (1980) 227.
[52] J. Summhammer, private communication.
[53] A. Zeilinger, C. G. Shull, Phys. Rev. **B 19** (1979) 3957.
[54] A. Böhm, Quantum Mechanics, Springer, Berlin, 1979.
[55] M. F. Barros, J. Andrade, E. Silva, M. H. Andrade e Silva, Ann. Found. L. de Broglie **12** (1987) 285.
[56] C. Dewdney, P. Holland, A. Kyprianidis, Phys. Lett. **A 119** (1986) 259.
[57] M. Scully, A. Barut, W. E. Lamb, Jr., Found Phys. **17** (1986) 259.
[58] A. O. Barut, M. Božić, Z. Marič, H. Rauch, Z. Phys. **A 238** (1987) 1.
[59] P. Garbaczewski, in: Problems in Quantum Physics, ed. L. Kostro et al., World Scientific, Singapore, 1988.
[60] F. Guerra, Phys. Reports **77** (1981) 263.
[61] K. Yasue, J. Zambrini, Ann. Phys. (NY) **159** (1985) 99.
[62] M. Mc Clendon, H. Rabitz, Phys. Rev. **A 37** (1988) 3479.
[63] K. Ito, H. P. McKean Jr., Diffusion Processes and their Sample Paths, Academic Press, NY, 1965.
[64] P. S. Hubbard, Phys. Rev. **A 6** (1972) 2421.